

NONUNIQUE FIXED POINT THEOREMS ON b -METRIC SPACES VIA SIMULATION FUNCTIONS

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ABSTRACT. Based on the concepts of α -orbital admissibility given by Popescu in [29] and simulation functions introduced by Khojasteh in [25], we introduce in this paper different types of contractive mappings. We also provide some nonunique fixed point results for such contractive mappings in the class of orbital complete b -metric spaces. Some consequences on known results in literature are also given in support of our obtained results.

1. INTRODUCTION AND PRELIMINARIES

As a generalization of metric spaces, the concept of a b -metric was introduced by Bakhtin [12] and Bourbaki [19] (see also [21]).

Definition 1.1 (Czerwik [21]). Let X be a nonempty set and $d : X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions:

- (b1) $d(x, y) = 0$ if and only if $x = y$;
- (b2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$, where $s \geq 1$.

The function d is called a b -metric and the space (X, d) is called a b -metric space, in short, bMS.

2000 *Mathematics Subject Classification.* 46T99, 47H10, 54H25.

Key words and phrases. Simulation function, b -metric space, nonunique fixed point, self-mappings, orbital admissible mappings.

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Received: March 16, 2018

Accepted: Sept. 26, 2018 .

Note that for $s = 1$, the b -metric becomes a usual metric.

For (common) fixed point results on b -metric spaces, see [2, 3, 4, 6, 7, 9, 10, 11, 15, 16, 17, 18, 24, 26, 27]. On the other hand, the case of Ćirić type [20], Karapınar type [23], Achari type [1] and Pachpatte type [32] contractions are considered in variant (generalized) metric spaces where the existence of nonunique fixed points has been proved. The aim of this paper is to establish some nonunique fixed point results by generalizing above type contractions using the concepts of α -orbital admissibility and simulation functions in the class of orbital complete b -metric spaces.

Example 1.1. Let $X = \mathbb{R}$. Define

$$(1.1) \quad d(x, y) = |x - y|^p$$

for $p > 1$. Then d is a b -metric on \mathbb{R} . Clearly, the first two conditions hold. Since

$$|x - y|^p \leq 2^{p-1}[|x - z|^p + |z - y|^p],$$

the third condition holds with $s = 2^{p-1}$. Thus, (\mathbb{R}, d) is a b -metric space with a constant $s = 2^{p-1}$.

Example 1.2. For $p \in (0, 1)$, take

$$X = l_p(\mathbb{R}) = \left\{ x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Define

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}.$$

Then (X, d) is a b -metric space with $s = 2^{1/p}$.

Example 1.3. Let E be a Banach space and 0_E be the zero vector of E . Let P be a cone in E with $\text{int}(P) \neq \emptyset$ and \preceq be a partial ordering with respect to P . Let X be a non-empty set. Suppose the mapping $d : X \times X \rightarrow E$ satisfies:

- (M1) $0 \preceq d(x, y)$ for all $x, y \in X$;
- (M2) $d(x, y) = 0$ if and only if $x = y$;
- (M3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y \in X$;
- (M4) $d(x, y) = d(y, x)$ for all $x, y \in X$.

Then d is called a cone metric on X , and the pair (X, d) is called a cone metric space (CMS).

Let E be a Banach space and P be a normal cone in E with the coefficient of normality denoted by K . Let $D : X \times X \rightarrow [0, \infty)$ be defined by $D(x, y) = ||d(x, y)||$, where $d : X \times X \rightarrow E$ is a cone metric space. Then (X, D) is a b -metric space with a constant $s := K \geq 1$.

The notion of comparison functions is defined by Rus [31] and it has been extensively studied by a number of authors to get more general forms of contractive mappings.

Definition 1.2. [14, 31] A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called a comparison function if it is increasing and $\phi^n(t) \rightarrow 0$ as $n \rightarrow \infty$ for every $t \in [0, \infty)$, where ϕ^n is the n -th iterate of ϕ .

Properties and examples of comparison functions can be found in [14, 31]. An important property of comparison functions is given by the following Lemma.

Lemma 1.1. ([14, 31]) If $\phi : [0, \infty) \rightarrow [0, \infty)$ is a comparison function, then

- (1) each iterate ϕ^k of ϕ , $k \geq 1$ is also a comparison function;
- (2) ϕ is continuous at 0;
- (3) $\phi(t) < t$ for all $t > 0$.

Definition 1.3. [13] Let $s \geq 1$ be a real number. A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called a (b) -comparison function if

- (1) ϕ is increasing;
- (2) there exist $k_0 \in \mathbb{N}$, $a \in [0, 1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq as^k\phi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$.

The collection of all (b) -comparison functions will be denoted by Ψ . In the literature, a (b) -comparison function is called (c) -comparison functions when $s = 1$. It can be shown that a (c) -comparison function is a comparison function, but the converse is not true in general. Berinde [13] also proved the following important property of (b) -comparison functions.

Lemma 1.2. [13] Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a (b) -comparison function. Then

- (1) the series $\sum_{k=0}^{\infty} s^k \phi^k(t)$ converges for any $t \in [0, \infty)$;
- (2) the function $b_s : [0, \infty) \rightarrow [0, \infty)$ defined as $b_s = \sum_{k=0}^{\infty} s^k \phi^k(t)$ is increasing and is continuous at $t = 0$.

Remark 1. Any (b) -comparison function ϕ satisfies $\phi(t) < t$ and $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for each $t > 0$.

In order to unify several existing fixed point results in the literature, Khojasteh *et al.* [25] introduced the notion of *simulation functions* and investigate the existence and uniqueness of a fixed point for different types of contractive mappings.

Definition 1.4. A *simulation function* is a mapping $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

- (ζ_1) : $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_2) : if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$, then

$$(1.2) \quad \limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0.$$

In [25], the condition $\zeta(0, 0) = 0$ was added, but Argoubi et al. [8] dropped it. Let \mathcal{Z} denote the family of all simulation functions $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$, that is, verifying (ζ_1) and (ζ_2) .

Due to the axiom (ζ_1) , we have

$$(1.3) \quad \zeta(t, t) < 0 \text{ for all } t > 0.$$

The following example is derived from [5, 25, 30].

Example 1.4. Let $\phi_i : [0, \infty) \rightarrow [0, \infty)$ be continuous functions such that $\phi_i(t) = 0$ if and only if, $t = 0$. For $i = 1, 2, 3, 4, 5, 6$, we define the mappings $\zeta_i : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$, as follows

(i): $\zeta_1(t, s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in [0, \infty)$, where $\phi_1, \phi_2 : [0, \infty) \rightarrow [0, \infty)$ are two continuous functions such that $\phi_1(t) = \phi_2(t) = 0$ if and only if $t = 0$ and $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$.

(ii): $\zeta_2(t, s) = s - \frac{f(t, s)}{g(t, s)}t$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty)^2 \rightarrow (0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.

(iii): $\zeta_3(t, s) = s - \phi_3(s) - t$ for all $t, s \in [0, \infty)$.

(iv): $\zeta_4(t, s) = s\varphi(s) - t$ for all $s, t \in [0, \infty)$, where $\varphi : [0, \infty) \rightarrow [0, 1)$ is a function such that $\limsup_{t \rightarrow r^+} \varphi(t) < 1$ for all $r > 0$.

(v): $\zeta_5(t, s) = \eta(s) - t$ for all $s, t \in [0, \infty)$, where $\eta : [0, \infty) \rightarrow [0, \infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$.

(vi): $\zeta_6(t, s) = s - \int_0^t \phi(u)du$ for all $s, t \in [0, \infty)$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a function such that $\int_0^\varepsilon \phi(u)du$ exists and $\int_0^\varepsilon \phi(u)du > \varepsilon$, for each $\varepsilon > 0$.

It is clear that each function ζ_i ($i = 1, 2, 3, 4, 5, 6$) forms a simulation function.

Definition 1.5. [29] Let $T : X \rightarrow X$ be a mapping and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. We say that T is α -orbital admissible if

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

Definition 1.6. [28, 29] Let (X, d) be a bMS and $x \in X$. A self-mapping T on X is said to be orbital continuous if $\lim_{i \rightarrow \infty} T^{n_i}(x) = z$ implies that $\lim_{i \rightarrow \infty} T(T^{n_i}(x)) = Tz$. A bMS (X, d) is called T -orbitally complete if every Cauchy sequence of the form $\{T^{n_i}(x)\}_{i=1}^\infty$, converges in (X, d) .

Remark 2. It is evident that orbital continuity of T yields orbital continuity of T^m for any $m \in \mathbb{N}$.

In this paper, we establish some nonunique fixed point results for different type of contractions using the concepts of α -orbital admissibility and simulation functions in the class of orbital complete b -metric spaces.

2. NONUNIQUE FIXED POINTS ON b -METRIC SPACES

2.1. Results on $(\alpha - \psi)$ -Ćirić type simulated mappings.

Definition 2.1. Let (X, d) be a bMS. The self-mapping $T : X \rightarrow X$ is called $(\alpha - \psi)$ -Ćirić type simulated if there exist $\psi \in \Psi$, $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \rightarrow [0, \infty)$ such that

$$(2.1) \quad \zeta(m_T(x, y), \psi(d(x, y))) \geq 0$$

for all $x, y \in X$, where

$$m_T(x, y) := \alpha(x, y) \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\}.$$

Lemma 2.1. Let X be a non-empty set. Suppose that $\alpha : X \times X \rightarrow [0, \infty)$ is a function and $T : X \rightarrow X$ is an α -orbital admissible mapping. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, and $x_n = Tx_{n-1}$ for $n = 1, 2, \dots$, then

$$(2.2) \quad \alpha(x_n, x_{n+1}) \geq 1, \text{ for each } n = 0, 1, \dots$$

Proof. On account of the assumptions of the theorem, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Owing to the fact that T is α -orbital admissible, we find

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By iterating the above inequality, we derive that

$$\alpha(x_n, x_{n+1}) = \alpha(Tx_{n-1}, Tx_n) \geq 1, \text{ for each } n = 1, 2, \dots$$

□

Theorem 2.1. *Let T be an orbitally continuous self-map on the T -orbitally complete $bMS(X, d)$. Assume that*

- (i) *T is an α -orbital admissible mapping;*
- (ii) *there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- (iii) *T is $(\alpha - \psi)$ -Ćirić type simulated.*

Then the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. By condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Consider the iterative sequence $\{x_n\}$ defined by

$$(2.3) \quad x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

We suppose that

$$(2.4) \quad x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

Indeed, if for some $n \in \mathbb{N}$ we have $x_n = Tx_{n-1} = x_{n-1}$, then the proof is completed.

By Lemma 2.1, we have

$$(2.5) \quad \alpha(x_n, x_{n+1}) \geq 1, \text{ for each } n = 0, 1, \dots$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.1), we derive that

$$0 \leq \zeta(m_T(x_{n-1}, x_n), \psi(d(x_{n-1}, x_n))) < \psi(d(x_{n-1}, x_n) - m_T(x_{n-1}, x_n))$$

where

$$\begin{aligned} m_T(x_{n-1}, x_n) := & \alpha(x_{n-1}, x_n) \min\{d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ & - \min\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\}. \end{aligned}$$

Using (2.5), we have

$$\begin{aligned} & \min\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \\ & - \min\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\} \\ (2.6) \quad & = \min\{d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ & - \min\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\} \\ & \leq \alpha(x_{n-1}, x_n) \min\{d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ & \quad - \min\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\} \\ & \leq \psi(d(x_{n-1}, x_n)). \end{aligned}$$

It implies that

$$(2.7) \quad \min\{d(x_n, x_{n+1}), d(x_n, x_{n-1})\} \leq \psi(d(x_{n-1}, x_n)).$$

Since $\psi(t) < t$ for all $t > 0$, the case $d(x_n, x_{n-1}) \leq \psi(d(x_{n-1}, x_n))$ is impossible. Thus, we have

$$(2.8) \quad d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)).$$

Applying (2.8) repeatedly, we have

$$(2.9) \quad d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^n(d(x_0, x_1)).$$

Since $\psi \in \Psi$, by Remark 1, we have $\lim_{n \rightarrow \infty} \psi^n(d(x_0, x_1)) = 0$, so

$$(2.10) \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

In what follow, we shall prove that the sequence $\{x_n\}$ is Cauchy. By using the triangle inequality (b3), we get

$$\begin{aligned}
 d(x_n, x_{n+k}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k})] \\
 &\leq sd(x_n, x_{n+1}) + s\{s[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+k})]\} \\
 &= sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_{n+k}) \\
 &\vdots \\
 (2.11) \quad &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \dots \\
 &+ s^{k-1}d(x_{n+k-2}, x_{n+k-1}) + s^{k-1}d(x_{n+k-1}, x_{n+k}) \\
 &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \dots \\
 &+ s^{k-1}d(x_{n+k-2}, x_{n+k-1}) + s^kd(x_{n+k-1}, x_{n+k}),
 \end{aligned}$$

since $s \geq 1$. Combining (2.9) and (2.11), we derive that

$$\begin{aligned}
 d(x_n, x_{n+k}) &\leq s\psi^n(d(x_0, x_1)) + s^2\psi^{n+1}d(x_0, x_1) + \dots \\
 (2.12) \quad &+ s^{k-1}\psi^{n+k-2}(d(x_0, x_1)) + s^k\psi^{n+k-1}(d(x_0, x_1)) \\
 &= \frac{1}{s^{n-1}}[s^n\psi^n(d(x_0, x_1)) + s^{n+1}\psi^{n+1}d(x_0, x_1) + \dots \\
 &+ s^{n+k-2}\psi^{n+k-2}(d(x_0, x_1)) + s^{n+k-1}\psi^{n+k-1}(d(x_0, x_1))].
 \end{aligned}$$

Consequently, we have

$$(2.13) \quad d(x_n, x_{n+k}) \leq \frac{1}{s^{n-1}} [P_{n+k-1} - P_{n-1}], \quad n \geq 1, k \geq 1,$$

where $P_n = \sum_{j=0}^n s^j \psi^j(d(x_0, x_1))$, $n \geq 1$. From Lemma 1.2, the series $\sum_{j=0}^{\infty} s^j \psi^j(d(x_0, x_1))$ is convergent and since $s \geq 1$, upon taking limit $n \rightarrow \infty$ in (2.13), we get

$$(2.14) \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+k}) \leq \lim_{n \rightarrow \infty} \frac{1}{s^{n-1}} [P_{n+k-1} - P_{n-1}] = 0.$$

We conclude that the sequence $\{x_n\}$ is Cauchy in (X, d) .

Owing to the construction $x_n = T^n x_0$ and the fact that (X, d) is T -orbitally complete, there exists $z \in X$ such that $x_n \rightarrow z$. Due to the orbital continuity of T , we conclude that $x_n \rightarrow Tz$. Hence $z = Tz$. \square

Now, we give some consequences in b -metric spaces.

Corollary 2.1. *Let T be an orbitally continuous self-map on the T -orbitally complete $bMS (X, d)$ and $\psi \in \Psi$. Suppose that*

$$(2.15) \quad \zeta(n_T(x, y), \psi(d(x, y))) \geq 0$$

for all $x, y \in X$, where

$$n_T(x, y) := \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\}.$$

Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. It is sufficient to take $\alpha(x, y) = 1$ in Theorem 2.1. \square

Corollary 2.2. *Let T be an orbitally continuous self-map on the T -orbitally complete $bMS (X, d)$. Suppose there exists $k \in [0, \frac{1}{s})$ such that*

$$(2.16) \quad \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y),$$

for all $x, y \in X$. Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. It is sufficient to take in Corollary 2.1, $\zeta(s, t) = \nu(s) - t$ with $\nu(t) = at$ and $\psi(s) = k's$ where $a \in [0, 1)$ and $k' \in [0, \frac{1}{s})$. Note that $k = sk' \in [0, \frac{1}{s})$. \square

If we take $s = 1$ in the previous corollary, we get the famous nonunique fixed point theorem of Ćirić [20].

Corollary 2.3. *[nonunique fixed point theorem of Ćirić [20]] Let T be an orbitally continuous self-map on the T -orbitally complete standard metric space (X, d) . Suppose there is $k \in [0, 1)$ such that*

$$\begin{aligned} \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} &- \min\{d(x, Ty), d(Tx, y)\} \\ &\leq kd(x, y), \end{aligned}$$

for all $x, y \in X$. Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Remark 3. Regarding Example 1.3, we deduce that the analog of Ćirić nonunique fixed point theorem in the setting of cone metric spaces with a normal cone is still valid (see [23]) (It corresponds to Corollary 2.3).

2.2. Results on $(\alpha - \psi)$ -Karapinar type simulated mappings.

Definition 2.2. Let (X, d) be a bMS. The mapping $T : X \rightarrow X$ is said $(\alpha - \psi)$ -Karapinar type simulated if there exist $\psi \in \Psi$, $\zeta \in \mathcal{Z}$, $\alpha : X \times X \rightarrow [0, \infty)$ and real numbers a_1, a_2, a_3, a_4, a_5 with

$$(2.17) \quad 0 \leq s \frac{a_4 - a_2}{a_1 + a_2} < 1, \quad a_1 + a_2 \neq 0, \quad a_1 + a_2 + a_3 > 0 \text{ and } 0 \leq a_3 - a_5$$

such that

$$(2.18) \quad \zeta(p(x, y), \psi(l(x, y))) \geq 0,$$

for all $x, y \in X$, where

$$p(x, y) = a_1 d(Tx, Ty) + a_2 [d(x, Tx) + d(y, Ty)] + a_3 [d(y, Tx) + d(x, Ty)],$$

and

$$l(x, y) = a_4 d(x, y) + a_5 d(x, T^2 x).$$

Theorem 2.2. Let T be an orbitally continuous self-map on the T -orbitally complete bMS (X, d) . Assume that

- (i) T is an α -orbital admissible mapping;
- (ii) there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (iii) T is $(\alpha - \psi)$ -Karapinar type simulated.

Then for such $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. Starting with the element x_0 , we shall construct an iterative sequence $\{x_n\}$ as follows:

$$(2.19) \quad x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

We suppose that

$$(2.20) \quad x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

Indeed, if for some $n \in \mathbb{N}$ we have $x_n = Tx_{n-1} = x_{n-1}$, then the proof is completed.

By Lemma 2.1, we have

$$(2.21) \quad \alpha(x_n, x_{n+1}) \geq 1, \text{ for each } n = 0, 1, \dots$$

By substituting $x = x_n$ and $y = x_{n+1}$ in (2.18), we derive that

$$(2.22) \quad 0 \leq \zeta(p(x_n, x_{n+1}), \psi(l(x_n, x_{n+1}))) < \psi(l(x_n, x_{n+1})) - p(x_n, x_{n+1}),$$

where

$$\begin{aligned} p(x_n, x_{n+1}) = & a_1 d(Tx_n, Tx_{n+1}) + a_2 [d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] \\ & + a_3 [d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1})], \end{aligned}$$

and

$$l(x_n, x_{n+1}) = a_4 d(x_n, x_{n+1}) + a_5 d(x_n, T^2 x_n).$$

Replacing these identities in (2.22), we get

$$\begin{aligned} (2.23) \quad & a_1 d(Tx_n, Tx_{n+1}) + a_2 [d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] + a_3 [d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1})] \\ & \leq \psi(a_4 d(x_n, x_{n+1}) + a_5 d(x_n, T^2 x_n)) \\ & \leq a_4 d(x_n, x_{n+1}) + a_5 d(x_n, T^2 x_n) \end{aligned}$$

for all a_1, a_2, a_3, a_4, a_5 satisfying (2.17). One writes,

$$(2.24) \quad \begin{aligned} & a_1 d(x_{n+1}, x_{n+2}) + a_2 [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + a_3 [d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})] \\ & \leq a_4 d(x_n, x_{n+1}) + a_5 d(x_n, x_{n+2}). \end{aligned}$$

By a simple calculation, one can get

$$(2.25) \quad (a_1 + a_2) d(x_{n+1}, x_{n+2}) + (a_3 - a_5) d(x_n, x_{n+2}) \leq (a_4 - a_2) d(x_n, x_{n+1})$$

which implies

$$(2.26) \quad d(x_{n+1}, x_{n+2}) \leq k d(x_n, x_{n+1})$$

where $k = \frac{a_4 - a_2}{a_1 + a_2}$. Due to (2.17), we have $0 \leq sk < 1$. Taking account of (2.26), we get inductively

$$(2.27) \quad d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \dots \leq k^n d(x_0, x_1).$$

We shall prove that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. We have

$$\begin{aligned} d(x_n, x_{n+p}) & \leq s \cdot d(x_n, x_{n+1}) + s^2 \cdot d(x_{n+1}, x_{n+2}) + \dots + s^{p-2} \cdot d(x_{n+p-3}, x_{n+p-2}) + \\ & \quad + s^{p-1} \cdot d(x_{n+p-2}, x_{n+p-1}) + s^p \cdot d(x_{n+p-1}, x_{n+p}) \\ & \leq s \cdot k^n \cdot d(x_0, x_1) + s^2 \cdot k^{n+1} \cdot d(x_0, x_1) + \dots + \\ & \quad + s^{p-2} \cdot k^{n+p-3} \cdot d(x_0, x_1) + s^{p-1} \cdot k^{n+p-2} \cdot d(x_0, x_1) + \\ & \quad + s^p \cdot k^{n+p-1} \cdot d(x_0, x_1) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{s^n \cdot k} \cdot [s^{n+1} \cdot k^{n+1} \cdot d(x_0, x_1) + \dots + s^{n+p-1} \cdot k^{n+p-1} \cdot d(x_0, x_1) + \\
&\quad + s^{n+p} \cdot k^{n+p} \cdot d(x_0, x_1)] \\
&\leq \frac{1}{s^n \cdot k} \cdot [s^{n+1} \cdot k^{n+1} \cdot d(x_0, x_1) + \dots + s^{n+p} \cdot k^{n+p} \cdot d(x_0, x_1)] \\
&= \frac{1}{s^n \cdot k} \cdot \sum_{i=n+1}^{n+p} s^i \cdot k^i \cdot d(x_0, x_1) \\
&< \frac{1}{s^n k} \cdot \sum_{i=n+1}^{\infty} s^i \cdot k^i \cdot d(x_0, x_1) \longrightarrow 0 \text{ as } n \longrightarrow \infty,
\end{aligned}$$

since $sk < 1$. Thus, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

As in the proof of previous theorem, regarding the construction $x_n = T^n x_0$ together with the fact that (X, d) is T -orbitally complete, there exists $z \in X$ such that $x_n \rightarrow z$. Again by the orbital continuity of T , we deduce that $x_n \rightarrow Tz$. Hence $z = Tz$. \square

Corollary 2.4. (See [22]) *Let T be an orbitally continuous self-map on the T -orbitally complete standard metric space (X, d) . Assume there exist real numbers a_1, a_2, a_3, a_4, a_5 such that*

$$(2.28) \quad 0 \leq \frac{a_4 - a_2}{a_1 + a_2} < 1, \quad a_1 + a_2 \neq 0, \quad a_1 + a_2 + a_3 > 0 \text{ and } 0 \leq a_3 - a_5.$$

Suppose that

$$(2.29) \quad a_1 d(Tx, Ty) + a_2 [d(x, Tx) + d(y, Ty)] + a_3 [d(y, Tx) + d(x, Ty)] \leq a_4 d(x, y) + a_5 d(x, T^2 x)$$

for all $x, y \in X$. Then T has at least one fixed point.

Remark 4. As we discussed in Remark 3, we obtain the analog of Theorem 2.2 in the context of cone metric spaces. More precisely, again taking Example 1.3 into account, one can derive that Corollary 2.4 is also still fulfilled in the setting of cone metric spaces with a normal cone ([22]).

2.3. Results on the $(\alpha - \psi)$ -Achari type mappings.

Definition 2.3. Let (X, d) be a bMS. The self-mapping $T : X \rightarrow X$ is said $(\alpha - \psi)$ -Achari type simulated if there exist $\psi \in \Psi$, $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \rightarrow [0, \infty)$ such that

$$(2.30) \quad \zeta\left(\frac{\alpha(x,y)P(x,y)-Q(x,y)}{R(x,y)}, \psi(d(x,y))\right) \geq 0,$$

for all $x, y \in X$ with $R(x, y) \neq 0$, where

$$\begin{aligned} P(x, y) &= \min\{d(Tx, Ty)d(x, y), d(x, Tx)d(y, Ty)\}, \\ Q(x, y) &= \min\{d(x, Tx)d(x, Ty), d(y, Ty)d(Tx, y)\}, \\ R(x, y) &= \min\{d(x, Tx), d(y, Ty)\}. \end{aligned}$$

Theorem 2.3. Let T be an orbitally continuous self-map on the T -orbitally complete bMS (X, d) . Assume that

- (i) T is an α -orbital admissible mapping;
- (ii) there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (iii) T is $(\alpha - \psi)$ -Achari type simulated.

Then for such $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. As in the proof of Theorem 2.1, we shall construct an iterative sequence $\{x_n\}$: for an arbitrary initial value $x \in X$, take

$$(2.31) \quad x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

Also, by condition (i) and as (2.5), we have $\alpha(x_{n-1}, x_n) \geq 1$ for all $n \geq 1$. As it is discussed in the proof of Theorem 2.1, from now on assume that

$$(2.32) \quad x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

By substituting $x = x_{n-1}$ and $y = x_n$ in the inequality (2.30), we derive that

$$(2.33) \quad \begin{aligned} 0 &\leq \zeta\left(\frac{\alpha(x_{n-1}, x_n)P(x_{n-1}, x_n) - Q(x_{n-1}, x_n)}{R(x_{n-1}, x_n)}, \psi(d(x_{n-1}, x_n))\right) \\ &< \psi(d(x_{n-1}, x_n)) - \frac{\alpha(x_{n-1}, x_n)P(x_{n-1}, x_n) - Q(x_{n-1}, x_n)}{R(x_{n-1}, x_n)}, \end{aligned}$$

where

$$\begin{aligned} P(x_{n-1}, x_n) &= \min\{d(Tx_{n-1}, Tx_n)d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)\}, \\ Q(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n), d(x_n, Tx_n)d(Tx_{n-1}, x_n)\}, \\ R(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}. \end{aligned}$$

By (2.33), we have

$$\frac{\alpha(x_{n-1}, x_n)P(x_{n-1}, x_n) - Q(x_{n-1}, x_n)}{R(x_{n-1}, x_n)} \leq \psi(d(x_{n-1}, x_n)),$$

for all $n \geq 1$. Using $\alpha(x_{n-1}, x_n) \geq 1$, we have

$$(2.34) \quad \frac{P(x_{n-1}, x_n) - Q(x_{n-1}, x_n)}{R(x_{n-1}, x_n)} \leq \frac{\alpha(x_{n-1}, x_n)P(x_{n-1}, x_n) - Q(x_{n-1}, x_n)}{R(x_{n-1}, x_n)} \leq \psi(d(x_{n-1}, x_n)),$$

for all $n \geq 1$. Note that

$$\begin{aligned} P(x_{n-1}, x_n) &= \min\{d(Tx_{n-1}, Tx_n)d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)\} \\ &= d(x_n, x_{n+1})d(x_{n-1}, x_n), \\ Q(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n), d(x_n, Tx_n)d(Tx_{n-1}, x_n)\} = 0, \\ R(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} = \min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

Using these above identities in (2.34), we get

$$(2.35) \quad \frac{P(x_{n-1}, x_n) - Q(x_{n-1}, x_n)}{R(x_{n-1}, x_n)} = \frac{d(x_n, x_{n+1})d(x_{n-1}, x_n)}{\min\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}} \leq \psi(d(x_{n-1}, x_n)) \quad \text{for all } n \geq 1.$$

If for some n , $R(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then the inequality (2.35) turns into

$$(2.36) \quad d(x_{n-1}, x_n) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n),$$

which is a contradiction. Accordingly, we deduce that

$$(2.37) \quad d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)), \quad \text{for all } n \geq 1.$$

Using (2.37) repeatedly, we obtain

$$(2.38) \quad d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^n(d(x_0, x_1)).$$

By Lemma 1.2, we deduce that

$$(2.39) \quad \lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0.$$

The rest of the proof is a verbatim repetition of the related lines in the proof of Theorem 2.1. □

Corollary 2.5. *Let T be an orbitally continuous self-map on the T -orbitally complete $bMS (X, d)$. Suppose that*

$$(2.40) \quad \frac{P(x,y)-Q(x,y)}{R(x,y)} \leq \eta(\psi(d(x,y))),$$

for all $x, y \in X$ with $R(x, y) \neq 0$, where $\psi \in \Psi$ and $\eta : [0, \infty) \rightarrow [0, \infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$.

Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. It is sufficient to take $\zeta(t, s) = \eta(s) - t$ and $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.3. □

Corollary 2.6. *Let T be an orbitally continuous self-map on the T -orbitally complete $bMS (X, d)$. Suppose that there exists $k \in [0, \frac{1}{s})$ such that*

$$(2.41) \quad \frac{P(x,y)-Q(x,y)}{R(x,y)} \leq kd(x, y),$$

for all $x, y \in X$, where

$$\begin{aligned} P(x, y) &= \min\{d(Tx, Ty)d(x, y), d(x, Tx)d(y, Ty)\}, \\ Q(x, y) &= \min\{d(x, Tx)d(x, Ty), d(y, Ty)d(Tx, y)\}, \\ R(x, y) &= \min\{d(x, Tx), d(y, Ty)\}. \end{aligned}$$

with $R(x, y) \neq 0$. Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. We choose $\eta(t) = at$ and $\psi(t) = k't$ where $a \in [0, 1)$ and $k' \in [0, \frac{1}{s})$ in Corollary 2.5. We consider $k = ak'$. \square

Corollary 2.7 (Nonunique fixed point of Achari [1]). *Let T be an orbitally continuous self-map on the T -orbitally complete standard metric space (X, d) . Suppose that there exists $k \in [0, 1)$ such that*

$$(2.42) \quad \frac{P(x, y) - Q(x, y)}{R(x, y)} \leq kd(x, y),$$

for all $x, y \in X$ with $R(x, y) \neq 0$, where

$$P(x, y) = \min\{d(Tx, Ty)d(x, y), d(x, Tx)d(y, Ty)\},$$

$$Q(x, y) = \min\{d(x, Tx)d(x, Ty), d(y, Ty)d(Tx, y)\},$$

$$R(x, y) = \min\{d(x, Tx), d(y, Ty)\}.$$

Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

2.4. Results on $(\alpha - \psi)$ -Pachpatte type mappings.

Definition 2.4. Let (X, d) be a complete bMS. The mapping $T : X \rightarrow X$ is said $(\alpha - \psi)$ -Pachpatte type simulated if there exist $\psi \in \Psi$, $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \rightarrow [0, \infty)$ such that

$$(2.43) \quad \zeta(\alpha(x, y)m(x, y) - p(x, y), \psi(d(x, Tx)d(y, Ty))) \geq 0,$$

for all $x, y \in X$, where

$$m(x, y) = \min\{[d(Tx, Ty)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2\},$$

$$p(x, y) = \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\}.$$

Theorem 2.4. *Let T be an orbitally continuous self-map on the T -orbitally complete bMS (X, d) . Assume that*

- (i) T is an α -orbital admissible mapping;
- (ii) there exists an element $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (iii) T is $(\alpha - \psi)$ -Pachpatte type simulated;
- (iv) $\psi(ab) \leq \alpha\psi(b)$ for all $a, b > 0$.

Then for such $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. By following the lines in the proof of Theorem 2.1, we shall formulate an recursive sequence $\{x_n\}$, for an arbitrary initial value $x \in X$:

$$(2.44) \quad x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

We assume that

$$(2.45) \quad x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

By replacing $x = x_{n-1}$ and $y = x_n$ in the inequality (2.43), we observe that

$$(2.46) \quad \begin{aligned} 0 &\leq \zeta(\alpha(x_{n-1}, x_n)m(x_{n-1}, x_n) - p(x_{n-1}, x_n), \psi(d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n))) \\ &< \psi(d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)) - \alpha(x_{n-1}, x_n)m(x_{n-1}, x_n) - p(x_{n-1}, x_n), \end{aligned}$$

for all $n \geq 1$, where

$$\begin{aligned} m(x_{n-1}, x_n) &= \min\{[d(Tx_{n-1}, Tx_n)]^2, d(x_{n-1}, x_n)d(Tx_{n-1}, Tx_n), [d(x_n, Tx_n)]^2\}, \\ p(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n), d(x_{n-1}, Tx_n)d(x_n, Tx_{n-1})\} = 0. \end{aligned}$$

It yields that

$$(2.47) \quad m(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)m(x_{n-1}, x_n) \leq \psi(d(x_{n-1}, x_n)d(x_n, x_{n+1})), \quad \text{for all } n \geq 1.$$

Since $Tx_{n-1} = x_n$, we have

$$\begin{aligned} m(x_{n-1}, x_n) &= \min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)d(x_n, x_{n+1}), [d(x_n, x_{n+1})]^2\} \\ &= \min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)d(x_n, x_{n+1})\}. \end{aligned}$$

Using the above identity and the fact that $\psi(t) < t$ (for all $t > 0$) in (2.47), we obtain that

$$(2.48) \quad \min\{[d(x_n, x_{n+1})]^2, d(x_{n-1}, x_n)d(x_n, x_{n+1})\} \leq \psi(d(x_{n-1}, x_n)d(x_n, x_{n+1})) < d(x_{n-1}, x_n)d(x_n, x_{n+1}).$$

If for some n , $m(x_{n-1}, x_n) = d(x_{n-1}, x_n)d(x_n, x_{n+1})(> 0)$, we get a contradiction with respect to (2.48). We deduce that

$$(2.49) \quad 0 < [d(x_n, x_{n+1})]^2 \leq \psi(d(x_{n-1}, x_n)d(x_n, x_{n+1})).$$

By condition (iv), we get

$$(2.50) \quad d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)), \quad \text{for all } n \geq 1.$$

Recurrently, we find that

$$(2.51) \quad d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)).$$

The rest of the proof is a verbatim repetition of the related lines in the proof of Theorem 2.1. \square

Corollary 2.8. *Let T be an orbitally continuous self-map on the T -orbitally complete $bMS(X, d)$. Suppose that there exists $k \in [0, \frac{1}{s})$ such that*

$$(2.52) \quad m(x, y) - p(x, y) \leq kd(x, Tx)d(y, Ty),$$

for all $x, y \in X$, where

$$\begin{aligned} m(x, y) &= \min\{[d(Tx, Ty)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2\}, \\ p(x, y) &= \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\}. \end{aligned}$$

Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Proof. It is sufficient to take $\alpha(x, y) = 1$ for all $x, y \in X$ in Theorem 2.4. Also, we choose $\zeta(t, s) = \eta(s) - t$ with $\eta(t) = at$ and $\psi(t) = k't$ where $a \in [0, 1)$ and $k' \in [0, \frac{1}{s})$. We consider $k = ak'$. \square

Corollary 2.8 is still valid in the context of standard metric spaces.

Corollary 2.9 (Nonunique fixed point of Pachpatte [32]). *Let T be an orbitally continuous self-map on the T -orbitally complete standard metric space (X, d) . Suppose that there exists $k \in [0, 1)$ such that*

$$(2.53) \quad m(x, y) - p(x, y) \leq kd(x, Tx)d(y, Ty),$$

for all $x, y \in X$, where

$$\begin{aligned} m(x, y) &= \min\{[d(Tx, Ty)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2\}, \\ p(x, y) &= \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\}. \end{aligned}$$

Then for each $x_0 \in X$, the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T .

Remark 5. One can deduce the analog of Corollary 2.8 in the context of cone metric spaces as it mentioned in Remark 3.

Acknowledgement

We would like to thank the editor and the referees for good suggestions to improve the paper.

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