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AN EXISTENCE AND CONVERGENCE RESULTS FOR CAPUTO FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper demonstrates a study on some significant latest innovations in the approximated technique to find the approximate solutions of Caputo fractional Volterra integro-differential equations. To apply this, the study uses homotopy analysis method. A wider applicability of this technique is based on their reliability and reduction in the size of the computational work. This study provides analytical approximate to determine the behavior of the solution. It proves the existence results and convergence of the solutions. In addition, it brings some examples to examine the validity and applicability of the proposed technique.

1. Introduction

Recently, numerous papers were concentrated on the development of analytical and numerical methods for fractional integro-differential equations. In this paper, we consider Caputo fractional Volterra integro-differential equation of the form:

$$^{c}D^{\alpha}u(x) = a(x)u(x) + g(x) + \int_0^x K(x,t)F(u(t))dt, \tag{1.1}$$

with the initial condition

$$u(0) = u_0, \tag{1.2}$$

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where ${}^cD^\alpha$ is the Caputo's fractional derivative, $0 < \alpha \leq 1$ and $u : J \longrightarrow \mathbb{R}$, where $J = [0, 1]$ is the continuous function which has to be determined, $g : J \longrightarrow \mathbb{R}$ and $K : J \times J \longrightarrow \mathbb{R}$, are continuous functions. $F : \mathbb{R} \longrightarrow \mathbb{R}$, is Lipschitz continuous function. An application of fractional derivatives was first given in 1823 by Abel [1] who applied the fractional calculus in the solution of an integral equation that arises in the formulation of the Tautochrone problem. The fractional integro-differential equations have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory and viscoelasticity [2, 5, 6, 7, 8, 9, 16, 17, 19]. In recent years, many authors focus on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, we can remember the following works. Al-Samadi and Gumah [3] applied the homotopy analysis method for fractional SEIR epidemic model, Zurigat et al. [22] applied HAM for system of fractional integro-differential equations, Yang and Hou [19] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [17] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Ma and Huang [16] applied hybrid collocation method to study integro-differential equations of fractional order. Moreover, properties of the fractional integro-differential equations have been studied by several authors [3, 11, 20, 22]. The homotopy analysis method that was first proposed by Liao [15], is implemented to derive analytic approximate solutions of fractional integro-differential equations and convergence of HAM for this kind of equations is considered. Unlike all other analytical methods, HAM adjusts and controls the convergence region of the series solution via an auxiliary parameter $\hbar$.

The main objective of the present paper studies the behavior of the solution that can be formally determined by an analytical approximated method as the homotopy analysis technique. Moreover, we proved the existence and convergence of the solutions for Caputo fractional Volterra integro-differential equation.

The rest of the paper is organized as follows: In Section 2, some preliminaries and basic definitions related to fractional calculus are recalled. In Section 3, homotopy analysis method is constructed for solving Caputo fractional Volterra integro-differential equations. In Section 4, the existence and convergence of the solution have been proved. In Section 5, the analytical examples are presented to illustrate the accuracy of this method. Finally, we will give a report on our paper and a brief conclusion are given in Section 6.

2. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative [14, 18, 21].

Definition 2.1. (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x>0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{2.1}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. (Caputo fractional derivative). The fractional derivative of $f(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}^cD_x^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, & m-1 < \alpha < m, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{2.2}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

(1) $J^\alpha J^v f = J^{\alpha+v} f, \quad \alpha, v > 0.$
(2) $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha},$
(3) $D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0.$
(4) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \leq m.$

Definition 2.3. (Riemann-Liouville fractional derivative). The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^\alpha f(x) = D^m J^{m-\alpha} f(x), \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.3}$$

Theorem 2.1. [21] ***(Banach contraction principle).*** *Let (X, d) be a complete metric space, then each contraction mapping $T : X \longrightarrow X$ has a unique fixed point x of T in X i.e. $Tx = x$.*

Theorem 2.2. [13] ***(Schauder's fixed point theorem).*** *Let X be a Banach space and let A a convex, closed subset of X. If $T : A \longrightarrow A$ be the map such that the set $\{Tu : u \in A\}$ is relatively compact in X (or T is continuous and completely continouous). Then T has at least one fixed point $u^* \in A : Tu^* = u^*$.*

3. Homotopy Analysis Method (HAM)

Consider, $N[u] = 0$, where N is a nonlinear operator, $u(x)$ is unknown function and x is an independent variable. Let $u_0(x)$ denote an initial guess of the exact solution $u(x), \hbar \neq 0$ an auxiliary parameter, $H_1(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[s(x)] = 0$ when $s(x) = 0$. Then using $q \in [0,1]$ as an embedding parameter, we can construct a homotopy when consider, $N[u] = 0$, as follows [4, 20]:

$$(1-q)L[\phi(x;q) - u_0(x)] - q\hbar H_1(x)N[\phi(x;q)] = \hat{H}[\phi(x;q); u_0(x), H_1(x), \hbar, q]. \tag{3.1}$$

It should be emphasized that we have great freedom to choose the initial guess $u_0(x)$, the auxiliary linear operator L, the non-zero auxiliary parameter $\hbar$, and the auxiliary function $H_1(x)$. Enforcing the homotopy Eq.(3.1) to be zero, i.e.,

$$\hat{H}_1[\phi(x;q); u_0(x), H_1(x), \hbar, q] = 0, \tag{3.2}$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x;q) - u_0(x)] = q\hbar H_1(x)N[\phi(x;q)], \tag{3.3}$$

when $q = 0$, the zero-order deformation Eq.(3.3) becomes

$$\phi(x;0) = u_0(x), \tag{3.4}$$

and when $q = 1$, since $\hbar \neq 0$ and $H_1(x) \neq 0$, the zero-order deformation Eq.(3.3) is equivalent to

$$\phi(x;1) = u(x). \tag{3.5}$$

Thus, according to Eqs.(3.4) and (3.5), as the embedding parameter q increases from 0 to 1, $\phi(x;q)$ varies continuously from the initial approximation $u_0(x)$ to the exact solution $u(x)$. Such a kind of continuous variation is called deformation in homotopy [22]. Due to Taylor's theorem, $\phi(x;q)$ can be expanded in a power series of q as follows [10].

$$\phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)q^m, \tag{3.6}$$

where,

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x;q)}{\partial q^m}|_{q=0}. \tag{3.7}$$

Let the initial guess $u_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter $\hbar$ and the auxiliary function $H_1(x)$ be properly chosen so that the power series (3.6) of $\phi(x;q)$ converges at $q=1$, then, we have under these assumptions the solution series

$$u(x) = \phi(x;1) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{3.8}$$

From Eq.(3.6), we can write Eq.(3.3) as follows:

$$\begin{aligned} (1-q)L[\phi(x;q) - u_0(x)] &= (1-q)L[\sum_{m=1}^{\infty} u_m(x)q^m] \\ &= q\hbar H_1(x)N[\phi(x;q)], \end{aligned} \tag{3.9}$$

then,

$$L[\sum_{m=1}^{\infty} u_m(x)q^m] - qL[\sum_{m=1}^{\infty} u_m(x)q^m] = q\hbar H_1(x)N[\phi(x;q)]. \tag{3.10}$$

By differentiating Eq.(3.10) m times with respect to q, we obtain

$$\begin{aligned} \{L[\sum_{m=1}^{\infty} u_m(x)q^m] - qL[\sum_{m=1}^{\infty} u_m(x)q^m]\}^{(m)} &= q\hbar H_1(x)N[\phi(x;q)]^{(m)} \\ &= m!L[u_m(x) - u_{m-1}(x)] \\ &= \hbar H_1(x)m\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0}. \end{aligned}$$

Therefore,

$$L[u_m(x) - \chi_m u_{m-1}(x)] = \hbar H_1(x)\Re_m(\overrightarrow{u_{m-1}}(x)), \tag{3.11}$$

where,

$$\Re_m(\overrightarrow{u_{m-1}}(x)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(x;q)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{3.12}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases}$$

Note that the high-order deformation Eq.(3.11) is governing the linear operator L, and the term $\Re_m(\overrightarrow{u_{m-1}}(x))$ can be expressed simply by Eq.(3.12) for any nonlinear operator N.

Homotopy analysis method applied to Caputo fractional Volterra integro-differential equation. We consider Caputo fractional Volterra integro-differential equation given by (1.1), with the initial condition (1.2).

We can define

$$N[\phi(x;q)] \;=\; {}^cD^\alpha\phi(x;q) - a(x)\phi(x;q) - g(x) - \int_0^x K(x,t)F(\phi(t;q))dt.$$

Now we construct the zero-order deformation equation

$$(1-q)^c D^\alpha[\phi(x;q) - u_0(x)] = q\hbar N[\phi(x;q)], \tag{3.13}$$

subject to the following initial conditions

$$u_0(x) = \phi(0;q) = u_0, \tag{3.14}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $u_0(x)$ is an initial guess of the solution $u(x)$ and $\phi(x;q)$ is an unknown function on the independent variables x and q. Also we suppose that

$$ {}^cD^\alpha(C) = 0, \tag{3.15} $$

where C is an integral constant. When the parameter q increases from 0 to 1, then the homotopy solution $\phi(x;q)$ varies from $u_0(x)$ to solution $u(x)$ of the original equation (1.1). Using the parameter q, $\phi(x;q)$ can be expanded in Taylor series as follows:

$$ \phi(x;q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)q^m, \tag{3.16} $$

where $u_m(x)$ define as (3.7).

Assuming that the auxiliary parameter $\hbar$ is properly selected so that the above series is convergent when $q = 1$, then the solution $u(x)$ can be given by

$$ u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{3.17} $$

Differentiating (3.13) and the initial condition (3.14) m times with respect to q, then setting $q = 0$, and finally dividing them by $m!$, we get the m^{th}-order deformation equation

$$ {}^cD^\alpha[u_m(x) - \chi_m u_{m-1}(x)] = \hbar\Re_m(\overrightarrow{u_{m-1}}(x)), \tag{3.18} $$

subject to the following initial conditions,

$$ u_m(0) = 0, \tag{3.19} $$

where,

(3.20)

$$ \begin{aligned} \Re_m(\overrightarrow{u_{m-1}}(x)) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0}, \\ &= {}^cD^\alpha u_{m-1}(x) - a(x)u_{m-1}(x) - \int_0^x K(x,t)F(u_{m-1}(t))dt - (1-\chi_m)g(x), \end{aligned} $$

and

$$\overrightarrow{u_m} = u_0, u_1, \cdots, u_m.$$

Applying the operator J^α to both sides of the linear m-order deformation (3.18)

$$\begin{aligned} u_m(x) &= (\chi_m + \hbar)u_{m-1}(x) - \hbar J^\alpha[a(x)u_{m-1}(x) \\ &\quad + \int_0^x K(x,t)F(u_{m-1}(t))dt + (1-\chi_m)g(x)]. \end{aligned}$$

4. Main Results

In this section, we shall give the existence and convergence results of Eq.(1.1), with the initial condition (1.2) and prove it.

Before starting and proving the main results, we introduce the following hypotheses:

(A1): There exists a function $K^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that $K^* = \sup\limits_{x,t\in[0,1]} \int_0^x |K(x,t)|\, dt < \infty.$

(A2): The two functions $a, g : J \to \mathbb{R}$ are continuous.

Lemma 4.1. *If $u_0(x) \in C(J, \mathbb{R})$, then $u(x) \in C(J, \mathbb{R})$ is a solution of the problem* $(1.1)-(1.2)$ *iff u satisfying*

$$\begin{aligned} u(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s K(s,\tau)F(u(\tau))d\tau\right)ds, \end{aligned}$$

for $x \in J$.

Our first result is based on the Schauder's fixed point Theorem.

Theorem 4.1. *Assume that F is continuous function and (A1), (A2) hold, If*

$$\frac{\|a\|_\infty}{\Gamma(\alpha+1)} < 1. \tag{4.1}$$

Then there exists at least a solution $u(x) \in C(J, \mathbb{R})$ to problem $(1.1)-(1.2)$.

Proof. Let the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$, is defined by

$$\begin{aligned}(Tu)(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} a(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s K(\tau,x)F(u(\tau))d\tau\right)ds,\end{aligned}$$

Firstly, we prove that the operator T is completely continuous.

(**1**) We show that T is continuous.

Let u_n be a sequence such that $u_n \to u$ in $C(J,\mathbb{R})$. Then for each $u_n, u \in C(J,\mathbb{R})$ and for any $x \in J$ we have

$$\begin{aligned}& |(Tu_n)(x) - (Tu)(x)| \\ \leq & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} |a(s)|\, |u_n(s) - u(s)|\, ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\, |F(u_n(\tau)) - F(u(\tau))|\, d\tau \right) ds \\ \leq & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \sup_{s\in J} |a(s)| \sup_{s\in J} |u_n(s) - u(s)|\, ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\sup_{s,\tau\in J} \int_0^\tau |K(s,\tau)| \sup_{\tau\in J} |F(u_n(\tau)) - F(u(\tau))|\, d\tau \right) ds \\ \leq & \|a\|_\infty \|u_n(.) - u(.)\|_\infty \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} ds \\ & + K^* \|F(u_n(.)) - F(u(.))\|_\infty \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} ds.\end{aligned}$$

Since $\int_0^x (x-s)^{\alpha-1} ds$ is bounded, $\lim_{n\to\infty} u_n(x) = u(x)$ and F is continuous function, we conclude that $\|Tu_n - Tu\|_\infty \to 0$ as $n \to \infty$, thus, T is continuous on $C(J,\mathbb{R})$.

(**2**) We verify that T maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Indeed, just we show that for any $\lambda > 0$ there exists a positive constant ℓ such that for each $u \in \mathbb{B}_\lambda = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq \lambda\}$, one has $\|Tu\|_\infty \leq \ell$.

Let $\mu = \sup_{(u)\in J\times[0,\lambda]} F(u(x)) + 1$, and for any $u \in \mathbb{B}_r$ and for each $x \in J$, we have

$$
\begin{aligned}
& |(Tu)(x)| \\
= \ & |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |a(s)| \, |u(s)| \, ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |g(s)| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s |K(s,\tau)| \, |F(u(\tau))| \, d\tau \right) ds \\
\leq \ & |u_0| + \|u\|_\infty \|a\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{K^* \mu x^\alpha}{\Gamma(\alpha+1)} \\
\leq \ & \left(|u_0| + \frac{\|a\|_\infty \lambda + \|g\|_\infty + K^* \mu}{\Gamma(\alpha+1)} \right) \\
: \ & = \ell.
\end{aligned}
$$

Therefore, $\|Tu\| \leq \ell$ for every $u \in \mathbb{B}_r$, which implies that $T\mathbb{B}_r \subset \mathbb{B}_\ell$.

(3) We examine that T maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.

Let $\mathbb{B}_\lambda$ is defined as in (**2**) and for each $u \in \mathbb{B}_\lambda$, $x_1, x_2 \in [0,1]$, with $x_1 < x_2$ we have

$$
\begin{aligned}
& |(Tu)(x_2) - (Tu)(x_1)| \\
\leq \ & \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2-s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_1-s)^{\alpha-1} a(s)u(s)ds \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2-s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_1-s)^{\alpha-1} g(s)ds \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{x_2} (x_2-s)^{\alpha-1} \left(\int_0^s K(s,\tau) F(u(\tau)) d\tau \right) ds \\ - \int_0^{x_1} (x_1-s)^{\alpha-1} \left(\int_0^s K(s,\tau) F(u(\tau)) d\tau \right) ds \end{array} \right| \\
= \ & \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{x_2} (x_2-s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_2-s)^{\alpha-1} a(s)u(s)ds \\ + \int_0^{x_1} (x_2-s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_1-s)^{\alpha-1} a(s)u(s)ds \end{array} \right| \\
& + \frac{1}{\Gamma(\alpha)} \left| \begin{array}{c} \int_0^{x_2} (x_2-s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_2-s)^{\alpha-1} g(s)ds \\ + \int_0^{x_1} (x_2-s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_1-s)^{\alpha-1} g(s)ds \end{array} \right|
\end{aligned}
$$

$$+\frac{1}{\Gamma(\alpha)}\left|\begin{array}{c} \int_0^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^s K(s,\tau)F(u(\tau))d\tau\right)ds \\ -\int_0^{x_1}(x_2-s)^{\alpha-1}\left(\int_0^s K(s,\tau)F(u(\tau))d\tau\right)ds \\ +\int_0^{x_1}(x_2-s)^{\alpha-1}\left(\int_0^s K(s,\tau)F(u(\tau))d\tau\right)ds \\ -\int_0^{x_1}(x_1-s)^{\alpha-1}\left(\int_0^s K(s,\tau)F(u(\tau))d\tau d\tau\right)ds. \end{array}\right|$$

Consequently,

$$\begin{aligned}
& |(Tu)(x_2)-(Tu)(x_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\begin{array}{c} \int_{x_1}^{x_2}(x_2-s)^{\alpha-1}|a(s)|\,|u(s)|\,ds \\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}|a(s)|\,|u(s)|\,ds \end{array}\right) \\
& +\frac{1}{\Gamma(\alpha)}\left(\begin{array}{c} \int_{x_1}^{x_2}(x_2-s)^{\alpha-1}|g(s)|\,ds \\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}|g(s)|\,ds \end{array}\right) \\
& +\frac{1}{\Gamma(\alpha)}\left(\begin{array}{c} \int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds \\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds \end{array}\right) \\
=\ & I_1+I_2+I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\alpha)}(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}|a(s)|\,|u(s)|\,ds \\
&\quad +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}|a(s)|\,|u(s)|\,ds) \\
&\leq \frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\|a\|_\infty\lambda+\frac{x_1^{\alpha}}{\Gamma(\alpha+1)}\|a\|_\infty\lambda+\frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\|a\|_\infty\lambda-\frac{x_2^{\alpha}}{\Gamma(\alpha+1)}\|a\|_\infty\lambda \\
&= \frac{\|a\|_\infty\lambda}{\Gamma(\alpha+1)}\left(2\,(x_2-x_1)^{\alpha}+(x_1^{\alpha}-x_2^{\alpha})\right)
\end{aligned}$$

$$(4.2)\leq\frac{\|a\|_\infty\lambda}{\Gamma(\alpha+1)}2\,(x_2-x_1)^{\alpha},$$

$$
\begin{aligned}
I_2 &= \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}|g(s)|\,ds+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}|g(s)|\,ds\right)\\
&\le \frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty+\frac{x_1^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty+\frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty-\frac{x_2^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty\\
&= \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(2\left(x_2-x_1\right)^\alpha+\left(x_1^\alpha-x_2^\alpha\right)\right)\\
&\le \frac{\|g\|_\infty}{\Gamma(\alpha+1)}2\left(x_2-x_1\right)^\alpha, \qquad (4.3)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \frac{1}{\Gamma(\alpha)}\left(\begin{array}{c}\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^s|K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds\\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left(\int_0^s|K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds\end{array}\right)\\
&\le \frac{(K^*\mu)}{\Gamma(\alpha+1)}\left(2\left(x_2-x_1\right)^\alpha+\left(x_1^\alpha-x_2^\alpha\right)\right)\\
&\le \frac{(K^*\mu_1)}{\Gamma(\alpha+1)}2\left(x_2-x_1\right)^\alpha, \qquad (4.4)
\end{aligned}
$$

we can conclude the right-hand side of (4.2), (4.3) and (4.4) is independently of $u\in\mathbb{B}_\lambda$ and tends to zero as $x_2-x_1\to 0$. This leads to $|(Tu)(x_2)-(Tu)(x_1)|\to 0$ as $x_2\to x_1$. i.e. the set $\{T\mathbb{B}_\lambda\}$ is equicontinuous.

From I_1 to I_3 together with the Arzela–Ascoli theorem, we can conclude that $T: C(J,\mathbb{R})\to C(J,\mathbb{R})$ is completely continuous.

Finally, we need to investigate that there exists a closed convex bounded subset $\mathbb{B}_{\widetilde{\lambda}}=\{u\in C(J,\mathbb{R}):\|u\|_\infty\le\widetilde{\lambda}\}$ such that $T\mathbb{B}_{\widetilde{\lambda}}\subseteq\mathbb{B}_{\widetilde{\lambda}}$. For each positive integer $\widetilde{\lambda}$, then $\mathbb{B}_{\widetilde{\lambda}}$ is clearly closed, convex and bounded of $C(J,\mathbb{R})$. We claim that there exists a positive integer ϵ such that $T\mathbb{B}_\epsilon\subseteq\mathbb{B}_\epsilon$. If this property is false, then for every positive integer $\widetilde{\lambda}$, there exists $u_{\widetilde{\lambda}}\in\mathbb{B}_{\widetilde{\lambda}}$ such that $(Tu_{\widetilde{\lambda}})\notin T\mathbb{B}_{\widetilde{\lambda}}$, i.e. $\left\|Tu_{\widetilde{\lambda}}(t)\right\|_\infty>\widetilde{\lambda}$ for some $x_{\widetilde{\lambda}}\in J$ where $x_{\widetilde{\lambda}}$ denotes x depending on $\widetilde{\lambda}$. But by using the previous hypotheses we have

$$
|u_0|+\|u\|_\infty\|a\|_\infty\frac{t^\alpha}{\Gamma(\alpha+1)}+\|g\|_\infty\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{K^*\mu x^\alpha}{\Gamma(\alpha+1)} \le \left(|u_0|+\frac{\|a\|_\infty\lambda+\|g\|_\infty+K^*\mu}{\Gamma(\alpha+1)}\right)
$$

$$\begin{aligned}
\widetilde{\lambda} &< \left\|Tu_{\widetilde{\lambda}}\right\|_{\infty} \\
&= \sup_{x\in J}\left|(Tu_{\widetilde{\lambda}})(x)\right| \\
&\leq \sup_{x\in J}\left\{\begin{array}{c} |u_0| + \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)\,|u(s)|\,ds\right| + \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds\right| + \frac{1}{\Gamma(\alpha)} \\ \times \int_0^x (x-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds \end{array}\right\} ds
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{x\in J}\left\{|u_0| + \|u\|_{\infty}\|a\|_{\infty}\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \|g\|_{\infty}\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{K^*\mu x^{\alpha}}{\Gamma(\alpha+1)}\right\} \\
&\leq \sup_{x\in J}\left(|u_0| + \frac{\|a\|_{\infty}\widetilde{\lambda} + \|g\|_{\infty} + K^*\mu}{\Gamma(\alpha+1)}\right).
\end{aligned}$$

Dividing both sides by $\widetilde{\lambda}$ and taking the limit as $\widetilde{\lambda} \to +\infty$, we obtain

$$1 < \frac{\|a\|_{\infty}}{\Gamma(\alpha+1)},$$

which contradicts our assumption (4.1). Hence, for some positive integer $\widetilde{\lambda}$, we must have $T\mathbb{B}_{\widetilde{\lambda}} \subseteq \mathbb{B}_{\widetilde{\lambda}}$.

An application of Schauder's fixed point Theorem shows that there exists at least a fixed point u of T in $C(J,\mathbb{R})$. Then u is the solution to (1.1) – (1.2) on J, and the proof is completed. □

Theorem 4.2. *If the series solution $u(x) = \sum_{m=0}^{\infty} u_m(x)$ obtained by the m-order deformation is convergent, then it converges to the exact solution of the fractional Volterra integro-differential equation* (1.1) – (1.2).

Proof. We assume $\sum_{m=0}^{\infty} u_m(x)$ converge to $u(x)$ then

$$\lim_{m\to\infty} u_m(x) = 0.$$

We can write,

$$\begin{aligned}\sum_{m=1}^{n} {}^{c}D^{\alpha}[u_m(x)-\chi_m u_{m-1}(x)] &= {}^{c}D^{\alpha}u_1(x)+({}^{c}D^{\alpha}u_2(x)-{}^{c}D^{\alpha}u_1(x)) \\ &\quad +({}^{c}D^{\alpha}u_3(x)-{}^{c}D^{\alpha}u_2(x))+\dots \\ &\quad +({}^{c}D^{\alpha}u_n(x)-{}^{c}D^{\alpha}u_{n-1}(x)) \\ &= {}^{c}D^{\alpha}u_n(x). \qquad (4.5)\end{aligned}$$

Hence, from Eq.(4.5)

$$\lim_{n\to\infty} u_n(x)=0. \qquad (4.6)$$

So, using Eq.(4.6), we have

$$\sum_{m=1}^{\infty} {}^{c}D^{\alpha}[u_m(x)-\chi_m u_{m-1}(x)] = \sum_{m=1}^{\infty}[{}^{c}D^{\alpha}u_m(x)-\chi_m {}^{c}D^{\alpha}u_{m-1}(x)]=0.$$

Therefore from Eq.(4.6), we can obtain that

$$\sum_{m=1}^{\infty} {}^{c}D^{\alpha}[u_m(x)-\chi_m u_{m-1}(x)] = \hbar\sum_{m=1}^{\infty}\Re_{m-1}(\overrightarrow{u_{m-1}}(x))=0.$$

Since $\hbar \neq 0$,then

$$\sum_{m=1}^{\infty}\Re_{m-1}(\overrightarrow{u_{m-1}}(x))=0. \qquad (4.7)$$

By substituting $\Re_{m-1}(\overrightarrow{u_{m-1}}(x))$ into the relation (3.20) and simplifying it, we have

(4.8)

$$
\begin{aligned}
\Re_{m-1}(\overrightarrow{u_{m-1}}(x)) &= \sum_{m=1}^{\infty}\Big[{}^cD^{\alpha}u_{m-1}(x) - a(x)u_{m-1}(x) - \int_0^x K(x,t)F(u_{m-1}(t))dt \\
&\quad -(1-\chi_m)g(x)\Big], \\
&= {}^cD^{\alpha}\Big(\sum_{m=1}^{\infty} u_{m-1}(x)\Big) - a(x)\sum_{m=1}^{\infty} u_{m-1}(x) \\
&\quad - \int_0^x K(x,t)\Big[\sum_{m=1}^{\infty} F(u_{m-1}(t))\Big]dt - \sum_{m=1}^{\infty}(1-\chi_m)g(x), \\
&= {}^cD^{\alpha}u(x)) - a(x)u(x) - \int_0^x K(x,t)F(u(t))dt - g(x).
\end{aligned}
$$

From Eq.(4.7) and Eq.(4.8), we have

$$
{}^cD^{\alpha}u(x) = a(x)u(x) + g(x) + \int_0^x K(x,t)F(u(t))dt,
$$

therefore, $u(x)$ must be the exact solution of Eq.(1.1) and the proof is complete. □

5. Illustrative Examples

In this section, we present the analytical technique based on HAM to solve Caputo fractional Volterra integro-differential equations.

Example 1. Let us consider Caputo fractional Volterra integro-differential equation:

$$
{}^cD^{0.75}[u(x)] = \frac{6x^{2.25}}{\Gamma(3.25)} - \frac{x^2e^x}{5}u(x) + \int_0^x e^x t u(t)dt, \tag{5.1}
$$

with the initial condition

$$
u(0) = 0.
$$

From (3.13), (5.1) can be written as

$$
N[\phi(x;q)] = {}^cD^{0.75}\phi(x;q) - \frac{6x^{2.25}}{\Gamma(3.25)} + \frac{x^2e^x}{5}\phi(x;q) - \int_0^x e^x t\phi(t;q)dt.
$$

Now, using the m^{th}-order deformation equation (3.18) and initial conditions (3.19), and recursive equation (3.21) we can write

$$\begin{aligned} u_m(x) &= (\chi_m+\hbar)u_{m-1}(x)+\hbar J^{0.75}[\frac{x^2e^x}{5}u_{m-1}(x)-(1-\chi_m)\frac{6x^{2.25}}{\Gamma(3.25)} \\ &\quad -\int_0^x e^x t u_{m-1}(t)dt] \end{aligned}$$

Then,

$$\begin{aligned} u_0(x) &= 0, \\ u_1(x) &= -\hbar x^3, \\ u_2(x) &= -\hbar(1+\hbar)x^3, \\ u_3(x) &= -\hbar(1+\hbar)^2x^3, \\ u_n(x) &= -\hbar(1+\hbar)^{n-1}x^3, \\ &\;\;. \\ &\;\;. \\ &\;\;. \end{aligned}$$

thus the HAM series solution can be written as

$$u_m(x)=\sum_{n=0}^{m}u_n(x)=-\hbar[1+(1+\hbar)+(1+\hbar)^2+\cdots+(1+\hbar)^{m-1}]x^3.$$

The exact solution of (5.1) when $-2<\hbar<0$ is

$$u(x)=\sum_{n=0}^{\infty}u_n(x)=-\hbar[1+(1+\hbar)+(1+\hbar)^2+\cdots]x^3=-\hbar\left(\frac{1}{1-(1+\hbar)}\right)x^3=x^3.$$

Example 2. Let us consider Caputo fractional Volterra integro-differential equation:

$$ {}^cD^{0.5}[u(x)]=\frac{32-3\sqrt{\pi}}{12\sqrt{\pi}}x^{1.5}+\int_0^x\frac{t}{x^{2.5}}u(t)dt, \tag{5.2}$$

with the initial condition

$$u(0) = 0.$$

From (3.13), (5.2) can be written as

$$N[\phi(x;q)] = {}^{c}D^{0.5}\phi(x;q) - \frac{32 - 3\sqrt{\pi}}{12\sqrt{\pi}}x^{1.5} - \int_0^x \frac{t}{x^{2.5}}\phi(t;q)dt.$$

Now, using the m^{th}-order deformation equation (3.18) and initial conditions (3.19), and recursive equation (3.21) we can write

$$u_m(x) = (\chi_m + \hbar)u_{m-1}(x) - \hbar J^{0.5}[(1-\chi_m)\frac{32 - 3\sqrt{\pi}}{12\sqrt{\pi}}x^{1.5} + \int_0^x \frac{t}{x^{2.5}}u_{m-1}(t)dt].$$

Then,

$$\begin{aligned} u_0(x) &= 0, \\ u_1(x) &= \hbar(\frac{3\sqrt{\pi}}{32} - 1)x^2, \\ u_2(x) &= \hbar(1 - \hbar(\frac{3\sqrt{\pi}}{32} - 1))(\frac{3\sqrt{\pi}}{32} - 1)x^2, \\ u_3(x) &= \hbar(1 - \hbar(\frac{3\sqrt{\pi}}{32} - 1))^2(\frac{3\sqrt{\pi}}{32} - 1)x^2, \end{aligned}$$

$$\begin{aligned} u_n(x) &= \hbar(1 - \hbar(\frac{3\sqrt{\pi}}{32} - 1))^{n-1}(\frac{3\sqrt{\pi}}{32} - 1)x^2, \\ &. \\ &. \\ &. \end{aligned}$$

thus the HAM series solution can be written as

$$u_m(x) = \sum_{n=0}^{m} u_n(x) = \hbar(\frac{3\sqrt{\pi}}{32} - 1)[1 + (1 - \hbar(\frac{3\sqrt{\pi}}{32} - 1)) + \cdots + (1 - \hbar(\frac{3\sqrt{\pi}}{32} - 1))^{m-1}]x^2.$$

The exact solution of (5.2) when $\frac{64}{3\sqrt{\pi}-32} < \hbar < 0$ is

$$\begin{aligned}
u(x) &= \sum_{n=0}^{\infty} u_n(x) \\
&= \hbar(\frac{3\sqrt{\pi}}{32} - 1)\left[1 + (1 - \hbar(\frac{3\sqrt{\pi}}{32} - 1)) + (1 - \hbar(\frac{3\sqrt{\pi}}{32} - 1))^2 + \cdots\right] x^2 \\
&= \hbar(\frac{3\sqrt{\pi}}{32} - 1)\left(\frac{1}{1 - (1 - \hbar(\frac{3\sqrt{\pi}}{32} - 1))}\right) x^2 = x^2.
\end{aligned}$$

6. Conclusion

The paper concludes with significant findings. The study successfully applies the homotopy analysis method to find the approximate solutions of Caputo fractional Volterra integro-differential equation. The presentation in this study proves that this method has a wider applicability that is based on the reliability of the method and reduction in the size of the computational work. The method is effectual in the sense that it finds analytical along with numerical solutions for wide classes of linear and nonlinear fractional Volterra integro-differential equations. Importantly, the study proves the existence of the solution and convergence of the technique. The examples presented in this paper illustrates further and establishes the precision and efficiency of the proposed technique.

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