Jordan Journal of Mathematics and Statistics (JJMS) 12(2), 2019, pp 103 - 114

ON FARTHEST POINTS IN FUZZY NORMED SPACES

M. AHMADI. BASERI [(1)] AND H. MAZAHERI [(2)]

Abstract. The main purpose of this paper is to find t-farthest points in fuzzy normed spaces. We introduce the concept of t-remotest fuzzy sets and give some interesting theorems. In particular, we study the set of all t-farthest points to an element from a set and discuss some properties of the this set.

1. Introduction

The theory of fuzzy sets was introduced by L. Zadeh [14] in 1965. The concept of a fuzzy norm on a linear space was initiated by Katsaras [9] in 1984. Later, some mathematicians defined notions for a fuzzy norm from different points of view. In particular, following [5], Bag and Samanta in [1] and [2], introduced and studied an idea of a fuzzy norm on a linear space in such a manner that its corresponding fuzzy metric is of Kramosil and Michalek type [10]. Since then, many mathematicians have studied fuzzy normed spaces from several angles [7, 8, 11]. The notion of the farthest points has many nice applications in the study of some geometrical properties of a normed linear space, see e.g. [3, 4].

In this paper, we use the notion of a fuzzy norm introduced in [13] to define the set of all t-farthest points on fuzzy normed spaces and investigate some interesting results.

1991 *Mathematics Subject Classification.* 41A65, 41A52, 46N10.

Key words and phrases. Fuzzy normed space, t-farthest point, t-remotest fuzzy set, Nearly compact set.

Received: Nov. 19, 2017 Accepted: March 7, 2019 .

In following, we recall some definitions and preliminaries that is need for main rsults.

Definition 1.1. [6] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfying conditions:
(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a; b; c; d \in [0,1]$.

Definition 1.2. [6] The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$,
(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1 \; for \; all \; t > 0 \; if \; and \; only \; if \; x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s > 0$,
(v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Definition 1.3. [6] Let $(X, M, *)$ be a fuzzy metric space. A subset X is called fuzzy bounded (f-bounded), if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in X$.

Definition 1.4. [6] Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said convergent to $x \in X$ if for each $0 < \epsilon < 1$ and $t \in (0, \infty)$ there exists $N_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \epsilon$ for each $n \geq N_0$.

Definition 1.5. [6] Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be compact if any sequence $\{x_n\}$ in A has a subsequence converging to an element of A.

Every compact subset A of a fuzzy metric space $(X, M, *)$ is f-bounded.

Proposition 1.1. [12] *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.*

Definition 1.6. [13] The 3-tuple $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $t, s > 0$,
(i) $N(x, t) > 0$,
(ii) $N(x, t) = 1 \Leftrightarrow x = 0$,
(iii) $N(\alpha x, t) = N(x, t/|\alpha|)$, for all $\alpha \neq 0$,
(iv) $N(x, t) * N(y, s) \leq N(x + y, t + s)$,
(v) $N(x, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous,
(vi) $\lim_{t\to\infty} N(x, t) = 1$.

Lemma 1.1. [13] *Let $(X, N, *)$ be a fuzzy normed space. Then*
(i) $N(x, t)$ is nondecreasing with respect to t for each $x \in X$,
(ii) $N(x - y, t) = N(y - x, t)$.

Remark 1. [13] As was shown in [13], every fuzzy normed space induces a fuzzy metric space on it and is therefore a topological space.

Lemma 1.2. [13] *Let $(X, N, *)$ be a fuzzy normed space. If we define*

$$M(x, y, t) = N(x - y, t),$$

then M is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N.

Definition 1.7. [1] Let $(X, N, *)$ be a fuzzy normed space. A subset B of X is said to be the closure of F if for any $x \in B$, there exists a sequence $\{x_n\}$ in F such that $\lim_{n\to\infty} N(x_n - x, t) = 1, \ \forall t > 0$. We denote the set B by $\overline{F}$.

Theorem 1.1. [1] *Let $(X, N, *)$ be a finite dimensional fuzzy normed space. Then a subset A is compact iff A is closed and bounded.*

2. Main Results

Definition 2.1. Let W be a nonempty f-bounded subset of a fuzzy normed space $(X, N, *)$. For $x \in X,\ t > 0$, let

$$\delta(W, x, t) = \inf_{w \in W} N(w - x, t).$$

An element $q_W^t(x) \in W$ is said to be a t-farthest point of x from W if

$$N(q_W^t(x) - x, t) = \delta(W, x, t).$$

Definition 2.2. Let W be a nonempty f-bounded subset of a fuzzy normed space $(X, N, *)$. For $x \in X,\ t > 0$, we shall denote the set of all elements of t-farthest points of x from W by $F_W^t(x)$, i.e.,

$$F_W^t(x) = \{w \in W :\ N(w - x, t) = \delta(W, x, t)\}.$$

If each $x \in X$ has at least one (one) t-farthest in W, then W is called a t-remotest fuzzy set (uniquely t-remotest fuzzy set).

Example 2.1. *Let $X = \mathbb{R}^2$. For $a, b \in [0, 1]$, let $a * b = ab$. Define $N : \mathbb{R}^2 \times (0, \infty) \to [0, 1]$ by*

$$N((x_1, x_2), t) = (\exp \frac{\sqrt{x_1^2 + x_2^2}}{t})^{-1}.$$

Then $(X, N, *)$ is a fuzzy normed space. Let $W = \{(x_1, x_2) \in \mathbb{R}^2 :\ -1 \leq x_1 \leq 1,\ 0 \leq x_2 \leq x_1^2\}$. Clearly W is f-bounded. Then for $x = (x_0, y_0)$,

$$\delta(W, (x_0, y_0), t) = \inf\{N((x_1 - x_0, x_2 - y_0), t) :\ -1 \leq x_1 \leq 1,\ 0 \leq x_2 \leq x_1^2\}$$

$$= \inf\{(\exp \frac{\sqrt{(x_1 - x_0)^2 + (x_2 - y_0)^2}}{t})^{-1} :\ -1 \leq x_1 \leq 1,\ 0 \leq x_2 \leq x_1^2\}$$

Since

$$F_W^t(x_0, y_0) = \{(-1, 1) : \; x_0 > 0, y_0 \leq 0\}$$

$$F_W^t(x_0, y_0) = \{(-1, 0) : \; x_0 > 0, y_0 > 0, y_0 \neq 1/2\}$$

$$F_W^t(x_0, y_0) = \{(-1, 0), (-1, 1) : \; x_0 > 0, y_0 = 1/2\}$$

$$F_W^t(x_0, y_0) = \{(1, 1) : \; x_0 < 0, y_0 \leq 0\}$$

$$F_W^t(x_0, y_0) = \{(1, 0) : \; x_0 < 0, y_0 > 0, y_0 \neq 1/2\}$$

$$F_W^t(x_0, y_0) = \{(1, 0), (1, 1) : \; x_0 < 0, y_0 = 1/2\}$$

$$F_W^t(x_0, y_0) = \{(1, 0), (-1, 0) : \; x_0 = 0, y_0 > 0\}$$

$$F_W^t(x_0, y_0) = \{(1, 1), (-1, 1) : \; x_0 = 0, y_0 < 0\}$$

Therefore W is a t-remotest fuzzy set, but W is not a uniquely t-remotest fuzzy set.

Example 2.2. *Let $X = \mathbb{R}$. For $a, b \in [0, 1]$, let $a * b = ab$. Define $N : \mathbb{R} \times (0, \infty) \to [0, 1]$ by*

$$N(x, t) = \frac{t}{t + |x|}.$$

Then $(X, N, *)$ is a fuzzy normed space. Let $W = [0, 1]$. Then for every $x > 1$, 0 is a t-farthest points of x from W and for every $x < 0$, 1 is a t-farthest points of x from W. So for each $x \in X$, $F_W^t(x)$ is a singleton. Therefore W is a uniquely t-remotest fuzzy set.

Theorem 2.1. *Let W be a nonempty f-bounded subset of a fuzzy normed space $(X, N, *)$. Then:*

(i) $\delta(W + y, x + y, t) = \delta(W, x, t),$ *for every* $x, y \in X$ *and* $t > 0$,

(ii) $F_W^t(x + y) = F_W^t(x) + y,$ *for every* $x, y \in X$ *and* $t > 0$,

(iii) $\delta(\alpha W, \alpha x, t) = \delta(W, x, t/|\alpha|),$ *for every* $x \in X,$ $t > 0$ *and* $\alpha \in \mathbb{R} - \{0\}$,

(iv) $F_{\alpha W}^{|\alpha| t}(\alpha x) = \alpha F_W^t(x),$ *for every* $x \in X,$ $t > 0$ *and* $\alpha \in \mathbb{R} - \{0\}$,

(v) W is t-remotest fuzzy set (uniquely t-remotest fuzzy set.) if and only if $W + y$ is

t-remotest fuzzy set (uniquely t-remotest fuzzy set.) for any given $y \in X$,
(vi) W is t-remotest fuzzy set (uniquely t-remotest fuzzy set.) if and only if αW is $|\alpha|t$-remotest fuzzy set (uniquely $|\alpha|t$-remotest fuzzy set.) for any given $\alpha \in \mathbb{R} - \{0\}$.

proof. (i) For any $x, y \in X$ and $t > 0$,

$$\begin{aligned}\delta(W + y, x + y, t) &= \inf\{N((w + y) - (x + y), t) : \ w \in W\} \\ &= \inf\{N(w - x), t) : \ w \in W\} \\ &= \delta(W, x, t).\end{aligned}$$

(ii) Using (i), $w_0 \in F^t_{W+y}(x + y)$ if and only if $w_0 \in W + y$ and $\delta(W + y, x + y, t) = N(x + y - w_0, t)$ if and only if $w_0 - y \in W$ and $\delta(W, x, t) = N(x - (w_0 - y), t)$ if and only if $w_0 - y \in F^t_W(x)$; i.e., $w_0 \in F^t_W(x) + y$.
(iii) For every $x \in X,\ t > 0$ *and* $\alpha \in \mathbb{R} - \{0\}$,

$$\begin{aligned}\delta(\alpha W, \alpha x, t) &= \inf\{N(\alpha w - \alpha x, t) : \ w \in W\} \\ &= \inf\{N(w - x), t/|\alpha|) : \ w \in W\} \\ &= \delta(W, x, t/|\alpha|).\end{aligned}$$

(iv) By (iii), $w_0 \in F^{|\alpha|t}_{\alpha W}(\alpha x)$ if and only if $w_0 \in \alpha W$ and $\delta(\alpha W, \alpha x, |\alpha|t) = N(\alpha x - w_0, |\alpha|t)$ if and only if $w_0/\alpha \in W$ and $N(x - w_0/\alpha, t) = \delta(W, x, t)$. So $w_0/\alpha \in F^t_W(x)$; i.e., $w_0 \in \alpha F^t_W(x)$.
(v) follows from (ii).
(vi) is an immediate consequence of (iv). □

Definition 2.3. Let W be a nonempty f-bounded subset of a fuzzy normed space $(X, N, *)$. The set W is said to be nearly compact, if for each $x \in X$, the sequence $\{w_n\}$ in W satisfying $N(x - w_n, t) \to \delta(W, x, t)$ contains a subsequence converging to an element of W.

Remark 2. Every compact set in a metric space is nearly compact.

Theorem 2.2. *Let W be a nonempty compact subset of a fuzzy normed space $(X, N, *)$. Then W is t-remotest fuzzy set.*

proof. Since W is compact, hence W is nearly compact. So for each $x \in X$, the sequence $\{w_n\}$ in W satisfying $N(x - w_n, t) \to \delta(W, x, t)$. Let $\{w_{n_k}\}$ be the subsequence of $\{w_n\}$, then $\{w_{n_k}\}$ is convergent to $w_0 \in W$. By Lemma 1.2 and Proposition 1.1, N is a continuous function on $X \times (0, \infty)$. Hence $N(x - w_{n_k}, t) \to N(x - w_0, t)$. Then $N(x - w_0, t) = \delta(W, x, t)$. Therefore W is t-remotest fuzzy set. □

Corollary 2.1. *Let W be a nonempty f-bounded and closed subset of a finite dimensional fuzzy normed space $(X, N, *)$. Then W is t-remotest fuzzy set.*

proof. It follows directly from Theorem 1.1 and Theorem 2.2. □

Theorem 2.3. *Let W be a nonempty closed and f-bounded subset of a finite dimensional fuzzy normed space $(X, N, *)$. Then for every $x \in X$, $F_W^t(x)$ is closed.*

proof. By Corollary 2.1, W is t-remotest fuzzy set. Let sequence $\{w_n\}$ in $F_W^t(x)$ converges to w. So $w_n \in W$ and $\delta(W, x, t) = N(w_n - x, t)$. Since W is closed, hence $w \in W$. As N is a continuous function on $X \times (0, \infty)$, then $\lim_{n\to\infty} N(w_n - x, t) = N(w - x, t)$, and $\delta(W, x, t) = N(w - x, t)$. Therefore $w \in F_W^t(x)$. Hence $F_W^t(x)$ is closed. □

Example 2.3. *If W is a t-remotest fuzzy set, then for every $x \in X$, $F_W^t(x)$ need not be closed. Consider $X = \mathbb{R}^2$. For $a, b \in [0, 1]$, let $a * b = ab$. We define $N : \mathbb{R}^2 \times (0, \infty) \to [0, 1]$ by $N((x_1, x_2), t) = \frac{t}{t+\max\{|x_1|, |x_2|\}}$. Let $W = \{(w_1, w_2) : \ 0 \leq w_1 \leq$*

$1,\ 0 < w_2 < 1\}$. *Clearly* W *is* f*-bounded. Then, for* $(2,-1)$,

$$\begin{aligned}\delta(W,(2,-1),t) &= \inf_{(w_1,w_2)\in W} N((w_1-2,w_2+1),t)\\ &= \inf_{(w_1,w_2)\in W} \frac{t}{t+\max\{|w_1-2|,|w_2+1|\}}\\ &= \frac{t}{t+2}.\end{aligned}$$

Since

$$\begin{aligned}F^t_W(2,-1) &= \{(w_1,w_2)\in W:\ N((w_1-2,w_2+1),t)=\delta(W,(2,-1),t)\}\\ &= \left\{(w_1,w_2)\in W:\ \frac{t}{t+\max\{|w_1-2|,|w_2+1|\}}=\frac{t}{t+2}\right\}\\ &= \{(w_1,w_2)\in W:\ w_1=0,\ 0<w_2<1\}.\end{aligned}$$

Hence $F^t_W(2,-1)$ *is not closed.*

Theorem 2.4. *Let* W *be a nonempty* f*-bounded and closed subset of a finite dimensional fuzzy normed space* $(X,N,*)$*. Then* $\overline{W}$ *(closure of* W*) is t-remotest fuzzy set.*

proof. Let sequence $\{x_n\}$ in W converges to x, then $x\in\overline{W}$. So $\overline{W}$ is closed set. Let $w\in\overline{W}$, there exists a sequence $\{w_n\}$ in W such that for $\forall\epsilon\in(0,1),\ \forall t>0$, $N(w_n-w,t)>1-\epsilon$. For $s=t+h$

$$\begin{aligned}N(w,s) &> N(w_n-w,t)*N(w_n,h)\\ &> (1-\epsilon)*N(w_n,h).\end{aligned}$$

Since W is f-bounded, there exists $0<r<1$ such that $N(w_n,h)>1-r$. Then

$$N(w,s)>(1-\epsilon)*(1-r).$$

Put $1-\delta=(1-\epsilon)*(1-r),\ (0<\delta<1)$. So

$$N(w,s)>1-\delta.$$

Therefore $\overline{W}$ is f-bounded. By Corollary 2.1, $\overline{W}$ is t-remotest fuzzy set. □

Now we consider a set some what similar to the set $F^t_W(x)$ and study some properties of this set.

Let W be a nonempty f-bounded subset of a fuzzy normed space $(X, N, *)$ and $x_0 \in X$. For each $z \in X$ we know that

$$\inf_{w \in W} N(w - z, t) \geq \inf_{w \in W} N(w - x_0, t_1) * N(x_0 - z, t_2),$$

for every $t_1, t_2 > 0$ *and* $t = t_1 + t_2$.

Let us define the set $F^t(W, x_0)$ as

$$F^t(W, x_0) = \{z \in X \ : \inf_{w \in W} N(w - z, t) = \inf_{w \in W} N(w - x_0, t_1) * N(x_0 - z, t_2)\},$$

for every $t_1, t_2 > 0$ *and* $t = t_1 + t_2$.

i.e.

$$F^t(W, x_0) = \{z \in X \ : \delta(W, z, t) = \delta(W, x_0, t_1) * N(x_0 - z, t_2)\},$$

for every $t_1, t_2 > 0$ *and* $t = t_1 + t_2$.

Then $F^t(W, x_0)$ is a nonempty (since $x_0 \in F^t(W, x_0)$) closed subset of X.

Proposition 2.1. *Let* $w_n \in W$ *be such that* $\delta(W, z, t) = \lim_{n \to \infty} N(w_n - z, t)$ *for each* $z \in F^t(W, x_0) - \{x_0\}$*. Then* $\delta(W, x_0, t_1) * N(x_0 - z, t_2) = \lim_{n \to \infty} N(w_n - x_0, t) * N(x_0 - z, t_2)$*, for every* $t_1, t_2 > 0$ *and* $t = t_1 + t_2$.

proof. Since $z \in F^t(W, x_0) - \{x_0\}$. Hence

$$\begin{aligned} \delta(W, z, t) &= \delta(W, x_0, t_1) * N(x_0 - z, t_2) \\ &\leq N(w_n - x_0, t_1) * N(x_0 - z, t_2), \end{aligned}$$

for every $t_1, t_2 > 0$ *and* $t = t_1 + t_2$. So

$$\lim_{n \to \infty} N(w_n - z, t) \leq \lim_{n \to \infty} N(w_n - x_0, t_1) * N(x_0 - z, t_2).$$

On the other hand

$$\lim_{n\to\infty} N(w_n - z, t) \geq \lim_{n\to\infty} N(w_n - x_0, t_1) * N(x_0 - z, t_2),$$

for every $t_1, t_2 > 0$ *and* $t = t_1 + t_2$. Then

$$\lim_{n\to\infty} N(w_n - z, t) = \lim_{n\to\infty} N(w_n - x_0, t_1) * N(x_0 - z, t_2).$$

Therefore

$$\delta(W, x_0, t_1) * N(x_0 - z, t_2) = \lim_{n\to\infty} N(w_n - x_0, t_1) * N(x_0 - z, t_2).$$

□

Proposition 2.2. *Let* $z \in F^t(W, x_0)$ *and* $y \in F^t(W, z)$ *then* $\delta(W, y, t) \leq \delta(W, x_0, t) * N(x_0 - y, 2t_2)$, *for every* $t_1, t_2 > 0$ *and* $t = t_1 + t_2$.

proof. Since $z \in F^t(W, x_0)$ and $y \in F^t(W, z)$. Hence

$$\begin{aligned} \delta(W, y, t) &= \delta(W, z, t_1) * N(y - z, t_2) \\ &\leq \delta(W, z, t) * N(y - z, t_2) \\ &= \delta(W, x_0, t_1) * N(x_0 - z, t_2) * N(y - z, t_2) \\ &\leq \delta(W, x_0, t_1) * N(x_0 - y, 2t_2) \\ &\leq \delta(W, x_0, t) * N(x_0 - y, 2t_2), \end{aligned}$$

for every $t_1, t_2 > 0$ *and* $t = t_1 + t_2$. □

Proposition 2.3. *Let* $W \subseteq W_1$, *and* $x_0 \in X$ *be such that*

$$\delta(W, x_0, t) = \delta(W_1, x_0, t),$$

then $F^t(W, x_0) \subseteq F^t(W_1, x_0)$.

proof. Let $z \in F^t(W, x_0)$. So

$$\begin{aligned} \delta(W, z, t) &= \delta(W, x_0, t_1) * N(x_0 - z, t_2) \\ &= \delta(W_1, x_0, t_1) * N(x_0 - z, t_2) \\ &= \delta(W_1, z, t), \end{aligned}$$

for every $t_1, t_2 > 0$ *and* $t = t_1 + t_2$. Which implies that $z \in F^t(W_1, x_0)$. □

References

[1] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (3) (2003), 687-706.

[2] T. Bag and S. K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems, **151** (2005), 513-547.

[3] M. Baronti and P. L. Papini, *Remotal sets revisited*, Taiwanese J. Math. **5** (2001), 367-373.

[4] S. Cobzas, *Geometric properties of Banach spaces and the existence of nearest and farthest points*, Abstr. Appl. Anal. **3** (2005), 259-285.

[5] S. C. Chang and J. N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Cal. Math. Soc. **86** (1994), 429-436.

[6] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, **64** (1994), 395-399.

[7] H. R. Goudarzi, *t-proximinal sets in fuzzy normed spaces*, Journal of Fuzzy Set Valued Analysis, **1** (2016), 19-27.

[8] S. A. Kamel Mirmostafaee, *Fuzzy farthest points in the space of fuzzy continuous functions*, Journal of Fuzzy Mathematics, **22** (2014), 761-768.

[9] A. K. Katsaras, *Fuzzy topological vector space II*, Fuzzy set and System, **12** (1984), 143-154.

[10] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, 11, **5** (1975), 336-344.

[11] M. Mirzavaziri and S. A. Kamel Mirmostafaee, *Closability of farthest point maps in fuzzy normed spaces*, Bulletin of Mathematical Analysis and Applications, **2** (2010), 140-145.

[12] J. Rodrguez-López and S. Romaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, **147** (2) (2004), 273-283.

[13] R. Saadati and S. M. Vaezpour, *Some results on fuzzy Banach spaces*, J. Appl. Math. Computing, **17** (1-2) (2005), 475-484.

[14] L. A. Zade, *Fuzzy sets*, Information and Control **8** (1965), 338-353.

(1,2) Department of Mathematics, Yazd University, Yazd, Iran.

E-mail address: (1) m.ahmadi@stu.yazd.ac.ir

E-mail address: (2) hmazaheri@yazd.ac.ir