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ALGEBRAIC PROPERTIES OF λ-FUZZY SUBGROUPS

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Abstract. In this paper, we initiate the study of λ-fuzzy sets. We define the notion of λ-fuzzy subgroup and prove that every fuzzy subgroup is λ-fuzzy subgroup. We introduce the notion of λ-fuzzy cosets and establish their algebraic properties. We also initiate the study of λ-fuzzy normal subgroups and quotient group with respect to λ-fuzzy normal subgroup and prove some of their various group theoretic properties. We also investigate effect on the image and inverse image of λ-fuzzy subgroup (normal subgroup) under group homomorphism and establish an isomorphism between the quotient group with respect to λ-fuzzy normal subgroup and quotient group with respect to the normal subgroup G_{ρ^λ}.

1. INTRODUCTION

Zadeh initiated the study of fuzzy set in [12] and since then there has been a fabulous concentration in this particular branch of mathematics due to its various applications ranging from computer science and engineering to study of social and economic behaviors. Rosenfeld initiated the idea of fuzzy groups on fuzzy set and established many basic results of fuzzy groups in [8]. Indeed, fuzzy subgroups admit many algebraic properties of groups. For more details about the developments on fuzzy groups, we refer to [9] and [10]. The fuzzy subgroups were redefined in [2] by

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Anthony. Later on, Das modified the work of Zadeh and Rosenfeld by defining the level subgroups of a given group in [4]. Chakrabatty and Khare defined the concept of fuzzy homomorphism between two groups and studied its effect to the fuzzy subgroups in [3]. Moreover, the idea of containment of an ordinary kernel of a group homomorphism in fuzzy subgroups was proposed by Ajmal in [1]. The recent developments about the applications of fuzzy sets in different algebraic structures may be viewed in [11, 13] and [14]. Gupta established the theory of T-operators on fuzzy sets in [5].

In this paper, we define a fuzzy set with respect to an N_T-operator. We use this fuzzy subset to define a new version of fuzzy subgroup known as λ-fuzzy subgroup and investigate further theory of λ-fuzzy subgroups and obtain analogs of various basic results of group theory. We prove the homomorphic image (pre-image) of λ-fuzzy subgroup is λ-fuzzy subgroup by using the classical homomorphism. We also introduce the concepts of λ-fuzzy coset and λ-fuzzy normal subgroup and have established isomorphism between the quotient group with respect to λ-fuzzy normal subgroup and quotient group with respect to the normal subgroup G_{ρ^λ}.

2. PRELIMINARIES

We start some important fundamental characterizations of fuzzy groups which play a key role in the order to obtain the results about those aspects of fuzzy groups which are group theoretic in a natural way. We review some of these basic concepts which are very essential for our further discussion. A brief review of these concepts may be studied in [4, 6] and [7].

Definition 2.1. [6] Let S be a nonempty set. A mapping

$$\rho : S \to [0, 1]$$

is called a fuzzy subset of S.

Definition 2.2. [6] Let ρ and σ be fuzzy sets of a set S. Their intersection $\rho \cap \sigma$ and union $\rho \cup \sigma$ are fuzzy sets of S defined by

(1) $(\rho \cap \sigma)(a) = \min\{\rho(a), \sigma(a)\}, \quad \forall \quad a \in S.$

(2) $(\rho \cup \sigma)(a) = \max\{\rho(a), \sigma(a)\}, \quad \forall \quad a \in S.$

Definition 2.3. [6] Let ρ be a fuzzy set of a set S. For $\gamma \in [0, 1]$, the set

$$\rho_\gamma = \{a : a \in S, \rho(a) \geq \gamma\}$$

is called level subset of ρ.

Definition 2.4. [6] Let ρ be a fuzzy subset of a group G. Then ρ is called a fuzzy subgroup if

(1) $\rho(ab) \geq \min\{\rho(a), \rho(b)\}, \quad$ for all $a, b \in G$

(2) $\rho(a^{-1}) \geq \rho(a), \quad$ for all $a \in G$.

Definition 2.5. [6] Let $\rho : G \to [0, 1]$ be a fuzzy subgroup of a group G, then for all $a \in G$ the following statements hold

(1) $\rho(e) \geq \rho(a), \quad \forall a \in G$

(2) $\rho(a^{-1}) \geq \rho(a)$.

Theorem 2.1. [4] Let G be a group and ρ be a fuzzy subset of G, then ρ is fuzzy subgroup if and only if the level subset ρ_γ, for $\gamma \in [0, 1], \rho(e) \geq \gamma$, is subgroup of G, where e is an identity of G.

Definition 2.6. [7] A fuzzy subgroup ρ of a group G is called a fuzzy normal subgroup if

$$\rho(ab) = \rho(ba), \quad \text{for all } a, b \in G.$$

Definition 2.7. [7] Let ρ be a fuzzy subgroup of a group G. For any $a \in G$, define a map $\rho_a : G \to [0, 1]$

by

$\rho_a(g) = \rho(ga^{-1}) \quad$ for all $g \in G$.

Definition 2.8. [5] A map $T : [0,1] \times [0,1] \to [0,1]$ define by $(a,b) \longmapsto \max\{a+b-1, 0\}$ is a T-norm if and only if for all $a,b,c,d \in [0,1]$

(1) $T(a,b) = T(b,a)$
(2) $T(a,T(b,c)) = T(T(a,b),c)$
(3) $T(a,1) = T(1,a) = 1$
(4) If $a \leq c$ and $b \leq d$ then $T(a,b) \leq T(c,d)$.

3. λ-fuzzy subsets and their properties

Definition 3.1. An operator N_T is defined as follows $N_T : [0,1] \times [0,1] \to [0,1]$ by

$$N_T(a,b) = \min\{2-a-b, 1\} \text{for all } a,b, \in [0,1]$$

Infect the operator N_T admits following properties, for all $a,b,c,d \in [0,1]$

(1) $N_T(a,b) = N_T(b,a)$
(2) $N_T(a,1) = N_T(1,a) = 1-a$
(3) If$a \leq c$ and $b \leq d$ then $N_T(a,b) \geq N_T(c,d)$.

It is important to mention here that the operator N_T is non-associative.

Definition 3.2. Let $\rho : X \to [0,1]$ be a fuzzy subset of a set X and $\lambda \in [0,1]$. Then the fuzzy subset ρ^λ of X (w. r. t fuzzy set ρ) is called λ-fuzzy subset of X and is defined as $\rho^\lambda(a) = \min\{2-\rho(a)-\lambda, 1\}$, for all $a \in X$.

Remark 3.3. It is important to note that one can obtain the negation of classical fuzzy subset $\rho(a)$ by choosing the value of $\lambda = 1$ in above definition whereas the case become crisp for the choice of $\lambda = 0$. These algebraic facts lead to note that the case illustrates the λ-fuzzy version with respect to any fuzzy subset for the value of λ, when $\lambda \in (0,1)$.

Definition 3.4. Let φ be any mapping from a group G to a group G', and ρ and σ be λ-fuzzy subsets in G and G'respectively. Then $\varphi(\rho^\lambda)$and $\varphi^{-1}(\sigma^\lambda)$ are respectively the image of ρ^λ and inverse image of σ^λ, defined as

(1) $\varphi(\rho^\lambda)(y) = \begin{cases} \sup \rho^\lambda(x) : x \in \varphi^{-1}(y) & \text{if } \varphi^{-1}(y) \neq \emptyset \\ 0 & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$

(2) $\varphi^{-1}(\sigma^\lambda)(x) = \sigma^\lambda \varphi(x), \text{for } x \in G.$

Theorem 3.1. (1) *Let ρ and σ be any two fuzzy subsets of a set S then $(\rho \cap \sigma)^\lambda = \rho^\lambda \cap \sigma^\lambda$.*

(2) *Let $\varphi : P \to Q$ be a mapping and ρ and σ be two fuzzy subsets of a set P and Q respectively, then*
 (a) $\varphi(\rho^\lambda) = (\varphi(\rho))^\lambda$.
 (b) $\varphi^{-1}(\rho^\lambda) = (\varphi^{-1}(\rho))^\lambda$

Proof. (1) In view of definition (3.2) we have

$$(\rho \cap \sigma)^\lambda(a) = \min\{2 - (\rho \cap \sigma)(a) - \lambda, 1\}, \text{where } a \in S \text{ and } \lambda \in [0, 1].$$

$$= \min\{2 - \min\{\rho(a), \sigma(a)\} - \lambda, 1\}$$

$$= \min\{\rho^\lambda(a), \sigma^(\lambda)(a)\}$$

$$= (\rho^\lambda \cap \sigma^\lambda)(a), \text{for all } a \in S$$

Consequently, $(\rho^\lambda \cap \sigma^\lambda)(a) = \rho^\lambda \cap \sigma^\lambda$.

(2) (a)

$$\sigma(\rho^\lambda)(b) = \sup\{\rho^\lambda(a)\varphi(a) = b\} = \sup\{\min\{2 - \rho(a) - \lambda, 1\}\}$$

$$= \min\{\sup\{2 - \rho(a) - \lambda, 1\}\} = \min\{2 - \varphi(\rho)(b) - \lambda, 1\}$$

$$= (\varphi(\rho))^\lambda, \text{for all } b \in Q$$

Hence $\varphi(\rho^\lambda) = (\varphi(\rho))^\lambda$.

(b) In view of Definition (3.2) we have

$\varphi^{-1}(\rho^{\lambda})(a) = (\rho^{\lambda})\varphi(a) = \min\{2 - \rho(\varphi(a)) - \lambda, 1\}$

$= \min\{2 - \varphi^{-1}\rho(a) - \lambda, 1\}$

$= (\varphi^{-1}(\rho))^{\lambda}(a)$, for all $a \in P$

Hence $\varphi^{-1}(\rho^{\lambda}) = (\varphi^{-1}(\rho))^{\lambda}$

□

4. λ-FUZZY SUBGROUPS

In this section, we define the notion of λ-fuzzy subgroup and λ-fuzzy normal subgroups. We prove that every fuzzy subgroup (normal subgroup) is also λ-fuzzy subgroup (normal subgroup) but converse need not to be true. The notion of λ-fuzzy coset has also been defined and discussed deeply in this section. Moreover, in view of λ-fuzzy normal subgroup, we apply this idea to introduce the concept of quotient group with respect to this particular fuzzy normal subgroup. This leads us to establish a natural homomorphism from a group G to its quotient group with respect to λ-fuzzy normal subgroup. We also obtain the homomorphic image and pre-image of λ-fuzzy subgroup (normal subgroup). We conclude this section by establishing an isomorphism between the quotient group G/ρ^{λ} and $G/G_{\rho^{\lambda}}$.

Definition 4.1. Let $\rho : G \to [0, 1]$ be a fuzzy subset of a group G and let $\lambda \in [0, 1]$. Then ρ is called λ-fuzzy subgroup of G if ρ^{λ} is fuzzy subgroup of G. In other words, ρ is λ-fuzzy subgroup of G if ρ^{λ} admits the following properties for all $a, b \in G$

(1) $\rho^{\lambda}(ab) \geq \min\{\rho^{\lambda}(a), \rho^{\lambda}(b)\}$
(2) $\rho^{\lambda}(a^{-1}) = \rho^{\lambda}(a)$.

Proposition 4.1. *Let ρ be a λ-fuzzy subgroup of a group G. Then the following statements hold*

(1) $\rho^{\lambda}(a) \leq \rho^{\lambda}(e)$ *for all $a \in G$ and e is identity element of G.*

(2) $\rho^\lambda(ab^{-1}) = \rho^\lambda(e)$ *implies that* $\rho^\lambda(a) = \rho^\lambda(b)$ *for all* $a, b \in G$.

Proof. (1) Since $\rho^\lambda(aa^{-1}) = \rho^\lambda(e)$ and also $\rho^\lambda(aa^{-1}) = \min\{\rho^\lambda(a), \rho^\lambda(a^{-1})\}$
$= \min\{\rho^\lambda(a), \rho^\lambda(a)\} = \rho^\lambda(a)$.
This implies that $\rho^\lambda(a) \leq \rho^\lambda(e)$ for all $a \in G$.

(2) Since we have $\rho^\lambda(a) = \rho^\lambda(ab^{-1}b) \geq \min\{\rho^\lambda(ab^{-1}), \rho^\lambda(b)\}$.
Then by our assumption we have $\rho^\lambda(a) \geq \min\{\rho^\lambda(e), \rho^\lambda(b)\}$ which implies that $\rho^\lambda(a) \geq \rho^\lambda(b)$.
Similarly, $\rho^\lambda(b) = \rho^\lambda(ba^{-1}a) \geq \min\{\rho^\lambda(ba^{-1}), \rho^\lambda(a)\}$ then by our assumption we have $\rho^\lambda(b) \geq \min\{\rho^\lambda(e), \rho^\lambda(a)\} = \rho^\lambda(a)$, which implies that $\rho^\lambda(b) \geq \rho^\lambda(a)$.
Hence $\rho^\lambda(a) = \rho^\lambda(b)$.

The following result leads to note that every fuzzy subgroup of a group G is λ-fuzzy subgroup of G. □

Proposition 4.2. *Every fuzzy subgroup of a group* G *is also* λ*-fuzzy subgroup of* G.

Proof. Let a, b be any two elements of a group G. Consider

$$\begin{aligned}
\rho^\lambda(ab) &= \min\{2 - \rho(ab) - \lambda, 1\}, \text{where } \lambda \in [0, 1]. \\
&\geq \min\{2 - \min\{\rho(a), \rho(b)\} - \lambda, 1\} \\
&= \min\{\min\{2 - \rho(a) - \lambda, 1\}, \min\{2 - \rho(b) - \lambda, 1\}\} \\
&= \min\{\rho^\lambda(a), \rho^\lambda(b)\}
\end{aligned}$$

Thus, we have $\rho^\lambda(ab) \geq \min\{\rho^\lambda(a), \rho^\lambda(b)\}$.
Moreover, $\rho^\lambda(a^{-1}) = \min\{2 - \rho(a^{-1}) - \lambda, 1\} = \min\{2 - \rho(a) - \lambda, 1\} = \rho^\lambda(a)$.
This implies that ρ is λ-fuzzy subgroup of G. □

Remark 4.2. The converse of above proposition need not to be true.

Example 4.3. Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$

$\rho((1)) = 0.4, \rho((12)) = \rho((13)) = \rho((23)) = 0.5$ and $\rho((123)) = \rho((132)) = 0.6$

Take $\lambda = 0.2$

$\rho^\lambda(a) = \min\{2 - \rho(a) - \lambda, 1\} = 1$. So, we have $\rho^\lambda(a) = 1$ for all$a \in G$.

$\rho^\lambda(ab) \geq \min\{\rho^\lambda(a), \rho^\lambda(b)\} = \min\{\min\{2 - \rho(a) - \lambda, 1\}, \min\{2 - \rho(b) - \lambda, 1\}\} = 1$

So $\rho^\lambda(ab) \geq \min\{\rho^\lambda(a), \rho^\lambda(b)\}$.

$\rho^\lambda(a^{-1}) = \min\{2 - \rho(a^{-1}) - \lambda, 1\} = 1$ and $\rho^\lambda(a) = \min\{2 - \rho(a) - \lambda, 1\} = 1$

This implies that ρ is λ-fuzzy subgroup of G.

Now we observe that ρ is not fuzzy subgroup of G because all possible level subset $\rho_{0.4} = \{(1), (12), (13), (23)\}, \rho_{0.5} = \{(12), (13), (23)\}$ and $\rho_{0.6} = \{(123), (132)\}$ are not subgroups of G.

Proposition 4.3. *Let ρ and σ be any two λ-fuzzy subgroups of a group G. Then $\rho \cap \sigma$ is also λ-fuzzy subgroup of G.*

Proof. Let ρ and σ be any two λ-fuzzy subgroups of a group G and let $a, b \in G$. Since $(\rho \cap \sigma)^\lambda(a) = (\rho^\lambda \cap \sigma^\lambda)(a)$ hold, then we have $(\rho \cap \sigma)^\lambda(ab) = (\rho^\lambda \cap \sigma^\lambda)(ab)$. So,

$$\begin{aligned}
(\rho^\lambda \cap \sigma^\lambda)(ab) &= \min\{\rho^\lambda(ab), \sigma^\lambda(ab)\} \\
&\geq \min\{\min\{\rho^\lambda(a), \rho^\lambda(b)\}, \min\{\sigma^\lambda(a), \sigma^\lambda(b)\}\} \\
&= \min\{\min\{\rho^\lambda(a), \sigma^\lambda(a)\}, \min\{\rho^\lambda(b), \sigma^\lambda(b)\}\} \\
&= \min\{(\rho^\lambda \cap \sigma^\lambda)(a), (\rho^\lambda \cap \sigma^\lambda)(b)\}
\end{aligned}$$

This implies that $(\rho^\lambda \cap \sigma^\lambda)(ab) \geq \min\{(\rho^\lambda \cap \sigma^\lambda)(a), (\rho^\lambda \cap \sigma^\lambda)(b)\}$.

Moreover, $(\rho \cap \sigma)^\lambda(a^{-1}) = (\rho^\lambda \cap \sigma^\lambda)(a^{-1}) = \min\{\rho^\lambda(a^{-1}), \sigma^\lambda(a^{-1})\} = \min\{\rho^\lambda(a), \sigma^\lambda(a)\}$.

We have $(\rho \cap \sigma)^\lambda(a^{-1}) = (\rho \cap \sigma)^\lambda(a)$

Consequently, $\rho \cap \sigma$ is λ-fuzzy subgroup of G.

□

Corollary 4.1. *The intersection of any finite number of λ-fuzzy subgroups of a group G is also λ-fuzzy subgroup of G.*

Proposition 4.4. *Let ρ and σ be any two λ-fuzzy subgroups of a group G. Then $\rho \cup \sigma$ need not to be λ-fuzzy subgroup of G.*

Example 4.4. Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Take two subgroups of G those are Let $H_1 = \{\pm 1, \pm i\}$and $H_2 = \{\pm 1, \pm j\}$.
Let ρ and σ be any two λ-fuzzy subgroups of a group G as

$$\rho(a) = \begin{cases} 0.2 & \text{if } a \in H_1 \\ 0.9 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma(a) \begin{cases} 0.3 & \text{if } a \in H_2 \\ 1 & \text{otherwise} \end{cases}$$

Since $\rho^\lambda(a) = \min\{2 - \rho(a) - \lambda, 1\}$.
Note that if $\lambda = 1$ then $\rho^1(a) = 1 - \rho(a),$ for all $a \in G$. Then we have ρ^1 and σ^1 as follows

$$\rho^1(x) = \begin{cases} 0.8 & \text{if } a \in H_1 \\ 0.1 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma^1(x) \begin{cases} 0.7 & \text{if } a \in H_2 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that ρ^1andσ^1 are 1-fuzzy subgroups of G.
Now we define $\rho^1 \cup \sigma^1$ as

$$(\rho^1 \cup \sigma^1)(a) = \max\{\rho^1(a), \sigma^1(a)\}$$

So, we have

$$(\rho^1 \cup \sigma^1)(a) = \begin{cases} 0.8 & \text{if } a \in H_1 \\ 0.7 & \text{if } a \in H_2 \setminus H_1, \text{for all } a \in G \\ 0.1 & \text{otherwise} \end{cases}$$

Let $a = i$ and $b = j$,
Observe that $(\rho^1 \cup \sigma^1)(i) = 0.8$and $(\rho^1 \cup \sigma^1)(j) = 0.7$,
So, $\min\{(\rho^1 \cup \sigma^1)(i), (\rho^1 \cup \sigma^1)(j)\} = 0.7$, but $\rho^1 \cup \sigma^1)(ij) = (\rho^1 \cup \sigma^1)(k) = 0.1$.

This implies that

$(\rho^1 \cup \sigma^1)(ij) \leq \min\{(\rho^1 \cup \sigma^1)(i), (\rho^1 \cup \sigma^1)(j)\}$.

Hence $\rho^1 \cup \sigma^1$ is not 1-fuzzy subgroup of G.

Example 4.5. Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Take a subgroup of G that is Let $H_1 = \{\pm 1, \pm i\}$.

Let ρ and σ be two λ-fuzzy subgroups of a group G as

$$\rho(a) = \begin{cases} 0.3 & \text{if } a \in H_1 \\ 0.9 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma(a) \begin{cases} 0.2 & \text{if } a \in H_2 \\ 1 & \text{otherwise} \end{cases}$$

Then we have ρ^1 and σ^1 as follows

$$\rho^1(x) = \begin{cases} 0.7 & \text{if } a \in H_1 \\ 0.1 & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma^1(x) \begin{cases} 0.8 & \text{if } a \in H_2 \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$(\rho^1 \cup \sigma^1)(a) = \begin{cases} 0.8 & \text{if } a \in H_1, \text{for all } a \in G \\ 0.1 & \text{otherwise} \end{cases}$$

It can be easily seen that $\rho^1 \cup \sigma^1$ is 1-fuzzy subgroup.

Definition 4.6. Let ρ be λ-fuzzy subgroup of a group G, for any $a \in G$ define λ-fuzzy left coset $a\rho^\lambda$ of ρ in G as follows

$a\rho^\lambda(x) = \min\{2 - \rho(a^{-1}x) - \lambda, 1\}$ for all $a, x \in G$.

Similarly, we define λ-fuzzy right coset $\rho^\lambda a$ of ρ in G as follows

$\rho^\lambda a(x) = \min\{2 - \rho(xa^{-1}) - \lambda, 1\}$ for all $a, x \in G$.

Definition 4.7. Let ρ be λ-fuzzy subgroup of a group G. Then ρ be λ-fuzzy normal subgroup G if and only if

$a\rho^\lambda(x) = \rho^\lambda a(x)$ for all $a \in G$.

Note: $a\rho^\lambda(x) = \rho^\lambda(a^{-1}x)$ and $\rho^\lambda a(x) = \rho^\lambda(xa^{-1})$ for all $x \in G$.

Proposition 4.5. *Every fuzzy normal subgroup of a group G is also λ-fuzzy normal subgroup G.*

Proof. Suppose that ρ is fuzzy normal subgroup of a group G which implies that $a\rho = \rho a$
Then for any $x \in G$ we have $\rho(a^{-1}x) = \rho(xa^{-1})$. So, we have
$\min\{2 - \rho(a^{-1}x) - \lambda, 1\} = \min\{2 - \rho(xa^{-1}) - \lambda, 1\}$.
This implies that $a\rho^{\lambda}(x) = \rho^{\lambda}a(x)$ for all $x \in G$. Consequently, ρ is λ-fuzzy normal subgroup G.
Note that the converse of above result need not to be true. □

Example 4.8. Let $G = D_3 =< a, ba^3 = b^2 = e, ba = a^2b >$ be the dihedral group. Define a fuzzy subset ρ of G as follows

$$\rho(a) = \begin{cases} 0.3 & \text{if } a \in < b > \\ 0.1 & \text{otherwise} \end{cases}$$

Take $\lambda = 1$ then we have

$$\rho^{\lambda}(a) = \begin{cases} 0.7 & \text{if } a \in < b > \\ 0.9 & \text{otherwise} \end{cases}$$

$a\rho^{\lambda}(g) = \min\{2 - \rho(a^{-1}g) - \lambda, 1\} = 1 - \rho(a^{-1}g)$
$= 1 - \rho(ga^{-1}) = \min\{2 - \rho(ga^{-1}) - \lambda, 1\} = \rho^{\lambda}a(g)$.
Then $a\rho^{\lambda}(g) = \rho^{\lambda}a(g)$ which implies that ρ is λ-fuzzy normal subgroup G. But it can be seen that ρ is not fuzzy normal subgroup G. This is because
$\rho((a^2)(ab)) = 0.3$ and $\rho((ab)(a^2)) = 0.1$. i.e. $\rho(a^{-1}g) = \rho(ga^{-1})$ not hold.

Proposition 4.6. *Let ρ be λ-fuzzy normal subgroup of a group G. Then $\rho^{\lambda}(b^{-1}ab) = \rho^{\lambda}(a)$ or $\rho^{\lambda}(ab) = \rho^{\lambda}(ba)$ hold for all $a, b \in G$.*

Proof. Since we have ρ be λ-fuzzy normal subgroup of a group G then we have $a\rho^{\lambda} = \rho^{\lambda}a$ for all $a \in G$.
This implies that $a\rho^{\lambda}(b^{-1}) = \rho^{\lambda}a(b^{-1})$ for all $b^{-1} \in G$.
$= \min\{2-\rho(a^{-1}b^{-1})-\lambda, 1\} = \min\{2-\rho(b^{-1}a^{-1})-\lambda, 1\} = \rho^{\lambda}(a^{-1}b^{-1}) = \rho^{\lambda}(b^{-1}a^{-1})$.
Consequently, we have

$$\rho^{\lambda}((ba)^{-1}) = \rho^{\lambda}((ab)^{-1})$$

Hence $\rho^{\lambda}(ab) = \rho^{\lambda}(ba)$. □

Theorem 4.1. *Let ρ be λ-fuzzy subgroup of a group G. Then following statements are equivalent*

(1) $\rho^{\lambda}(ba) = \rho^{\lambda}(ab),$ *for all* $a, b \in G$
(2) $\rho^{\lambda}(aba^{-1}) = \rho^{\lambda}(b),$ *for all* $a, b \in G$
(3) $\rho^{\lambda}(aba^{-1}) \geq \rho^{\lambda}(b),$ *for all* $a, b \in G$
(4) $\rho^{\lambda}(aba^{-1}) \leq \rho^{\lambda}(b),$ *for all* $a, b \in G$.

Proposition 4.7. *Let ρ be λ-fuzzy normal subgroup of a group G. Then the set define as*
$G_{\rho^{\lambda}} = \{a \in G : \rho^{\lambda}(a) = \rho^{\lambda}(e)\}$ *is normal subgroup of G.*

Proof. Since $G_{\rho^{\lambda}}$ is nonempty because $e \in G$. Let $a, b \in G_{\rho^{\lambda}}$

$$\begin{aligned} \rho^{\lambda}(ab^{-1}) &\geq \min\{\rho^{\lambda}(a), \rho^{\lambda}(b^{-1})\} \text{ for all } a, b \in G. \\ &= \min\{\rho^{\lambda}(a), \rho^{\lambda}(b)\} \text{ for all } a, b \in G. \\ &= \min\{\rho^{\lambda}(e), \rho^{\lambda}(e)\} \text{ for all } e \in G. \\ &= \rho^{\lambda}(e) \end{aligned}$$

This implies that $\rho^{\lambda}(ab^{-1}) \geq \rho^{\lambda}(e)$.
Since ρ^{λ} is fuzzy subgroup which implies that $\rho^{\lambda}(ab^{-1}) \leq \rho^{\lambda}(e)$

. Hence $\rho^{\lambda}(ab^{-1}) = \rho^{\lambda}(e)$ implies that $G_{\rho^{\lambda}}$ is subgroup of G.

Now we prove it is normal subgroup of G. Let $a \in G_{\rho^{\lambda}}$ and $b \in G$, then we have

$$\rho^{\lambda}(b^{-1}ab) = \rho^{\lambda}(a) = \rho^{\lambda}(e).$$

This implies that $b^{-1}ab \in G_{\rho^{\lambda}}$

Consequently, we have $G_{\rho^{\lambda}}$ is normal subgroup of G. □

Proposition 4.8. *Let ρ be λ-fuzzy normal subgroup of a group G. Then the following statements hold*

(1) $a\rho^{\lambda} = b\rho^{\lambda}$ *if and only if* $a^{-1}b \in G_{\rho^{\lambda}}$
(2) $\rho^{\lambda}a = \rho^{\lambda}b$ *if and only if* $ab^{-1} \in G_{\rho^{\lambda}}$

Proof. (1) Suppose that $a\rho^{\lambda} = b\rho^{\lambda}$ then we have

$\rho^{\lambda}(a^{-1}b) = \min\{2 - \rho(a^{-1}b) - \lambda, 1\} = a\rho^{\lambda}(b) = b\rho^{\lambda}(b)$
$= \min\{2 - \rho(b^{-1}b) - \lambda, 1\} = \min\{2 - \rho(e) - \lambda, 1\} = \rho^{\lambda}(e)$

Thus $\rho^{\lambda}(a^{-1}b) = \rho^{\lambda}(e)$ implies that $a^{-1}b \in G_{\rho^{\lambda}}$.

Conversely,
$a\rho^{\lambda}(c) = \min\{2 - \rho(a^{-1}c) - \lambda, 1\}$
So, $\rho^{\lambda}(a^{-1}c) = \rho^{\lambda}(a^{-1}b.b^{-1}c) \geq \min\{\rho^{\lambda}(a^{-1}b), \rho^{\lambda}(b^{-1}c)\} = \min\{\rho^{\lambda}(e), \rho^{\lambda}(b^{-1}c)\}$
$= \rho^{\lambda}(b^{-1}c) = b\rho^{\lambda}(c)$
By interchanging the a and b we have $a\rho^{\lambda}(c) = b\rho^{\lambda}(c)$ for all $c \in G$.
Hence $a\rho^{\lambda} = b\rho^{\lambda}$.

(2) Similar as above proof.

□

Proposition 4.9. *Let ρ be λ-fuzzy normal subgroup of a group G and let $a, b, x, y \in G$. If $a\rho^\lambda = x\rho^\lambda$ and $b\rho^\lambda = y\rho^\lambda$ then $ab\rho^\lambda = xy\rho^\lambda$.*

Proof. Given that $a\rho^\lambda = x\rho^\lambda$ and $b\rho^\lambda = y\rho^\lambda$, which implies that $a^{-1}x, b^{-1}y \in G_{\rho^\lambda}$.
Now $(ab)^{-1}xy = b^{-1}(a^{-1}x)y = b^{-1}(a^{-1}x)(bb^{-1})y$
$= [b^{-1}(a^{-1}x)b](b^{-1}y) \in G_{\rho^\lambda} since G_{\rho^\lambda}$ is normal subgroup of G.
This implies that $(ab)^{-1}xy \in G_{\rho^\lambda}$.
Hence $ab\rho^\lambda = xy\rho^\lambda$. □

Proposition 4.10. *Let G/ρ^λ be the collection of all λ-fuzzy cosets of a λ-fuzzy subgroup ρ of G. This form a group under the binary operation $\otimes$ define on the set G/ρ^λ as follows*
$\rho^\lambda a \otimes \rho^\lambda b = \rho^\lambda ab$, for all $a, b \in G$.

Proof. As we know that
$$G/\rho^\lambda = \{\rho^\lambda a : a \in G\}$$
Let $\rho^\lambda a = \rho^\lambda a'$ and $\rho^\lambda b = \rho^\lambda b'$ for all $a, a', b, b' \in G$. Let $g \in G$ then
$$\begin{aligned}
(\rho^\lambda a \otimes \rho^\lambda b)(g) = \rho^\lambda ab(g) &= \min\{2 - \rho(g((ab)^{-1}) - \lambda, 1\} \\
&= \min\{2 - \rho((gb^{-1})a^{-1}) - \lambda, 1\} \\
&= \rho^\lambda a(gb^{-1}) = \rho^\lambda a'(gb^{-1}) \\
&= \min\{2 - \rho((gb^{-1})a'^{-1}) - \lambda, 1\} \\
&= \min\{2 - \rho((a'^{-1}g)b^{-1}) - \lambda, 1\} \\
&= \rho^\lambda b(a'^{-1}g) = \rho^\lambda b'(a'^{-1}g) \\
&= \min\{2 - \rho((a'^{-1}g)b'^{-1}) - \lambda, 1\} \\
&= \min\{2 - \rho(b'^{-1}(a'^{-1}g)) - \lambda, 1\} \\
&= \min\{2 - \rho(b'^{-1}(a'^{-1})) - \lambda, 1\}
\end{aligned}$$

$$= \min\{2 - \rho((a'b')^{-1}g) - \lambda, 1\}$$

$$= \min\{2 - \rho(g(a'b')^{-1}) - \lambda, 1\} = \rho^\lambda a'b'$$

Hence $\otimes$ is well define operation on the set G/ρ^λ . The set G/ρ^λ under this binary operation admits the associative law. The element $\rho^\lambda e$ of G/ρ^λ is the identity element and the inverse of an element $\rho^\lambda a$ is $\rho^\lambda a^{-1}$. □

Definition 4.9. The group G/ρ^λ of λ-fuzzy cosets of a λ-fuzzy normal subgroup ρ of G is called the factor group or quotient group of G by ρ^λ.

Theorem 4.2. *Let G be a group and G/ρ^λ be quotient group with respect to λ-fuzzy normal subgroup ρ of G. There exist a natural epimorphism from G to G/ρ^λ which is defined as $\varphi(a) = \rho^\lambda a$ with $\operatorname{Ker}\varphi = G_{\rho^\lambda}$.*

Proof. Let $a, b \in G$ be any elements. Then $\varphi(ab) = \rho^\lambda ab = \rho^\lambda a \rho^\lambda b = \varphi(a)\varphi(b)$.
Therefore, φ is homomorphism. For each $\rho^\lambda a \in G_{\rho^\lambda}$ we have $a \in G$ such that $\varphi(a) = \rho^\lambda a$.
This implies that φ is onto homomorphism.
Now

$$\begin{aligned}\operatorname{Ker}\varphi &= \{a \in G : \varphi(a) = \rho^\lambda e\}\\ &= \{a \in G : \rho^\lambda a = \rho^\lambda e\}\\ &= \{a \in G : ae^{-1} \in G_{\rho^\lambda}\}\\ &= \{a \in G : a \in G_{\rho^\lambda}\}\\ &= G_{\rho^\lambda}.\end{aligned}$$

□

5. HOMOMORPHISM OF λ-FUZZY SUBGROUPS

Theorem 5.1. *Let $\varphi : G \to G'$ be a bijective homomorphism of a group G into a group G'. Let ρ be λ-fuzzy subgroup of G. Then the homomorphic image $\varphi(\rho)$ is λ-fuzzy subgroup of G'.*

Proof. Given that ρ be λ-fuzzy subgroup of G'. Let $a'_1, a'_2 \in G'$ be any element then we have unique elements $a_1, a_2 \in G$, such that $\varphi(a_1) = a'_1$ and $\varphi(a_2) = a'_2$.
Further,

$$\begin{aligned}
(\varphi(\rho))^\lambda(a'_1a'_2) &= \min\{2 - \varphi(\rho)(a'_1a'_2) - \lambda, 1\} \\
&= \min\{2 - \varphi(\rho)(\varphi(a_1)\varphi(a_2)) - \lambda, 1\} \\
&= \min\{2 - \varphi(\rho)(\varphi(a_1a_2)) - \lambda, 1\} \\
&= \min\{2 - \varphi(\rho)(a_1a_2) - \lambda, 1\} \\
&= \rho^\lambda(a_1a_2) \\
&\geq \min\{\rho^\lambda(a_1), \rho^\lambda(a_2)\} \text{ for all } a_1, a_2 \in G \\
&= \min\{\varphi(\rho)^\lambda\varphi(a_1), \varphi(\rho)^\lambda\varphi(a_2)\} \\
&= \min\{\varphi(\rho)^\lambda(a'_1), \varphi(\rho)^\lambda(a'_2)\} \\
&= \min\{(\varphi(\rho))^\lambda(a'_1), (\varphi(\rho))^\lambda(a'_2)\}
\end{aligned}$$

Consequently,
$(\varphi(\rho))^\lambda(a'_1a'_2) \geq \min\{(\varphi(\rho))^\lambda(a'_1), (\varphi(\rho))^\lambda(a'_2)\}$.
Also, $(\varphi(\rho))^\lambda(a'^{-1}) = (\varphi(\rho))^\lambda(a'^{-1}) = \varphi(\rho^\lambda)(\varphi(a^{-1})) = \rho^\lambda(a^{-1}) = \rho^\lambda(a)$

$$= \varphi(\rho^\lambda)(\varphi(a)) = (\varphi(\rho))^\lambda(a')$$

Thus, $(\varphi(\rho))^\lambda(a'^{-1}) = (\varphi(\rho))^\lambda(a')$.
Consequently, $\varphi(\rho)$ is λ-fuzzy subgroup of G'. □

Theorem 5.2. *Let $\varphi : G \to G'$ be a bijective homomorphism of a group G into a group G'. Let ρ be λ-fuzzy normal subgroup of G. Then the homomorphic image $\varphi(\rho)$ is λ-fuzzy normal subgroup of G'.*

Proof. Given that ρ be λ-fuzzy normal subgroup of G. Let $a_1', a_2' \in G'$ be any element then we have unique elements $a_1, a_2 \in G$, such that $\varphi(a_1) = a_1'$ and $\varphi(a_2) = a_2'$.

$$\begin{aligned}
(\varphi(\rho))^{\lambda}(a_1'a_2') &= \min\{2 - \varphi(\rho)(a_1'a_2') - \lambda, 1\} \\
&= \min\{2 - \varphi(\rho)(\varphi(a_1)\varphi(a_2)) - \lambda, 1\} \\
&= \min\{2 - \varphi(\rho)(\varphi(a_1a_2)) - \lambda, 1\} \\
&= \min\{2 - \varphi(\rho)(\varphi(a_2a_1)) - \lambda, 1\} \\
&= \min\{2 - \varphi(\rho)(\varphi(a_2)\varphi(a_1)) - \lambda, 1\} \\
&= \min\{2 - \varphi(\rho)(a_2'a_1') - \lambda, 1\} \\
&= (\varphi(\rho))^{\lambda}(a_2'a_1').
\end{aligned}$$

Consequently, $\varphi(\rho)$ is λ-fuzzy normal subgroup of G'. □

Theorem 5.3. *Let $\varphi : G \to G'$ be a homomorphism of a group G into G'. Let σ be λ-fuzzy subgroup of G'. Then the pre-image $\varphi^{-1}(\sigma)$ is λ-fuzzy subgroup of G.*

Proof. Given that σ be λ-fuzzy subgroup of G'. Let $a_1, a_2 \in G$ be any element then we have

$(\varphi^{-1}(\sigma))^{\lambda}(a_1a_2) = \varphi^{-1}(\sigma)^{\lambda}(a_1a_2) = \sigma^{\lambda}(\varphi(a_1a_2)) = \sigma^{\lambda}(\varphi(a_1)\varphi(a_2))$

$\geq \min\{\sigma^{\lambda}(\varphi(a_1)), (\varphi(a_2))\}$

$= \min\{\varphi^{-1}(\sigma)^{\lambda}(a_1), \varphi^{-1}(\sigma)^{\lambda}(a_2)\}$

Thus,

$(\varphi^{-1}(\sigma))^{\lambda}(a_1a_2) \geq \min\{(\varphi^{-1}(\sigma))^{\lambda}(a_1), (\varphi^{-1}(\sigma))^{\lambda}(a_2)\}$.

Also $(\varphi^{-1}(\sigma))^{\lambda}(a^{-1}) = \varphi^{-1}(\sigma)^{\lambda}(a^{-1}) = \sigma^{\lambda}(\varphi(a^{-1})) = \sigma^{\lambda}(\varphi(a)) = \varphi^{-1}(\sigma^{\lambda})(a)$.

Thus, $(\varphi^{-1}(\sigma))^{\lambda}(a^{-1}) = (\varphi^{-1}(\sigma))^{\lambda}(a)$.

Hence $\varphi^{-1}(\sigma)$ is λ-fuzzy subgroup of G. □

Theorem 5.4. *Let $\varphi : G \to G'$ be a homomorphism of a group G into G'. Let σ be λ-fuzzy normal subgroup of G'. Then the pre-image $\varphi^{-1}(\sigma)$ is λ-fuzzy normal subgroup of G.*

Proof. Given that σ be λ-fuzzy subgroup of G'. Let $a_1, a_2 \in G$ be any element then we have

$(\varphi^{-1}(\sigma))^{\lambda}(a_1a_2) = \varphi^{-1}(\sigma)^{\lambda}(a_1a_2) = \sigma^{\lambda}(\varphi(a_1a_2)) = \sigma^{\lambda}(\varphi(a_1)\varphi(a_2)) = \sigma^{\lambda}(\varphi(a_2a_1) = (\varphi^{-1}(\sigma))^{\lambda}(a_2a_1)$

Thus,

$$(\varphi^{-1}(\sigma))^{\lambda}(a_1a_2) = (\varphi^{-1}(\sigma))^{\lambda}(a_2a_1)$$

Hence $\varphi^{-1}(\sigma)$ is λ-fuzzy subgroup of G. □

Theorem 5.5. *Let ρ be a λ-fuzzy normal subgroup of a group G and $a, b \in G$ be any element. If $a\rho^{\lambda} = b\rho^{\lambda}$ then $\rho^{\lambda}(a) = \rho^{\lambda}(b)$.*

Proof. Suppose that $a\rho^{\lambda} = b\rho^{\lambda}$ then by Proposition (4.20) we have $a^{-1}b \in G_{\rho^{\lambda}}$ and $b^{-1}a \in G_{\rho^{\lambda}}$. Since λ-fuzzy normal subgroup of G, this implies that

$\rho^{\lambda}(a) = \rho^{\lambda}(b^{-1}ab) \geq \min\{\rho^{\lambda}(b^{-1}a), \rho^{\lambda}(b)\} = \min\{\rho^{\lambda}(e), \rho^{\lambda}(b)\} = \rho^{\lambda}(b)$.

Therefore, we have $\rho^{\lambda}(a) \geq \rho^{\lambda}(b)$. Similarly, we have $\rho^{\lambda}(a) \leq \rho^{\lambda}(b)$.

Hence $\rho^{\lambda}(a) = \rho^{\lambda}(b)$. □

Theorem 5.6. *Let ρ be a λ-fuzzy normal subgroup of a group G. Then $G/\rho^{\lambda} \cong G/G_{\rho^{\lambda}}$*

Proof. Define a map $\varphi : G/\rho^{\lambda} \to G/G_{\rho^{\lambda}}$ by the rule

$\varphi(a\rho^{\lambda}) = aG_{\rho^{\lambda}}$ for all $a \in G$. In view of Proposition (4.20)φ is well define.

The application of Proposition (4.20) leads to note that φ is injective. φ is obviously surjective.

Now consider, for $a_1\rho^\lambda, a_2\rho^\lambda \in G/\rho^\lambda$ we have $((a_1\rho^\lambda)(a_2\rho^\lambda)) = \varphi(a_1a_2\rho^\lambda) = a_1a_2G_{\rho^\lambda} = a_1G_{\rho^\lambda}.a_2G_{\rho^\lambda} = \varphi(a_1\rho^\lambda)\varphi(a_2\rho^\lambda)$.

So, φ is homomorphism. Since φ is a bijective mapping, which implies this is an isomorphism. Hence $G/\rho^\lambda \cong G/G_{\rho^\lambda}$. □

Conclusion

In this paper, we have introduced the concept of λ-fuzzy subgroup and λ-fuzzy coset of a given group and have used them to introduce the concept of λ-fuzzy normal subgroup and discuss various related properties. We have also studied effect on the image and inverse image of λ-fuzzy subgroup (normal subgroup) under group homomorphism.

In the next studies, we shall extend this idea to intuitionist fuzzy sets and will investigate its various algebraic properties.

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