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COMPLEX FUZZY AND GENERALIZED COMPLEX FUZZY SUBPOLYGROUPS OF A POLYGROUP

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Abstract. The objective of this paper is to combine the innovative concept of complex fuzzy sets and polygroups. We introduce the concepts of complex fuzzy subpolyrgroups and generalized complex fuzzy subpolyrgroups of a polygroup. We provide some examples and properties of them.

1. Introduction

Algebraic hyperstructures represent a natural generalization of classical algebraic structures and they were introduced by Marty [9] in 1934 at the eighth Congress of Scandinavian Mathematicians. Where he generalized the notion of a group to that of a hypergroup. He defined a hypergrop as a set equipped with associative and reproductive hyperoperation. In a group, the composition of two elements is an element whereas in a hypergroup, the composition of two elements is a set. Since then, many different hyperstructures (hyperring, hyperalgebra, hyperrepresentation, ...) were widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc. (see [4]). Moreover, certain subclasses of hypergroups were introduced such as polygroups. The latter

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were introduced by Comer [2], where he emphasized their importance in connections to graphs, relations, Boolean and cylindric algebras.

On the other hand, the theory of fuzzy sets, introduced in 1965 after the publication of Zadeh [13] as an extension of the classical notion of set, has achieved great success in various fields. The fuzzy set theory is based on the principle called by Zadeh "the principle of incompatibility", that is "the closer a phenomenon is studied, the more indistinct its definition becomes". Fuzzy sets are sets whose elements have degrees of membership. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Rosenfeld [10] applied this concept to the theory of groups and introduced the concept of a fuzzy subgroup of a group. Since then, a host of mathematicians are engaged in fuzzifying various notions and results of abstract algebra. For example, fuzzy subpolygroups of a polygroup [14] and fuzzy subhypergroup of a hypergroup [5] were introduced and studied.

As an extension of fuzzy sets, Ramot et al. [11, 12] introduced the concept of complex fuzzy sets in which the codomain of membership function was the unit disc of the complex plane. They introduced different fuzzy complex operations and relations such as the intersection, union and complement of complex fuzzy subsets.

Using the notion of "belonging ($\in$) and "quasi-coincidence (q) of fuzzy points with fuzzy sets, Davvaz and Corsini [7] introduced the concept of an $(\in, \in \vee q)$-fuzzy subpolygroup and studied some of its fundamental properties. Full details on the traditional fuzzy (and generalized fuzzy) subpolygroups of a polygroup is found in the book of Davvaz et al (see [8]). Our paper extends the results of Davvaz and

Corsini [7] to complex fuzzy sets, and it is constructed as follows: after an Introduction, in Section 2 we present some definitions and results about polygroups and traditional fuzzy subpolygroups. In Section 3, we introduce the concept of complex fuzzy subpolygroups, construct examples and investigate their properties. In Section 4, we introduce the concept of generalized complex fuzzy subpolygroups, construct examples and investigate their properties.

2. Polygroups and traditional fuzzy subpolygroups

In this section, we present some definitions and theorems related to polygroups and fuzzy subpolygroups that are used throughout the paper. For full details about polygroups and fuzzy subpolygroups, we refer the reader to the books [6, 8].

Let P be a non-empty set. Then, a mapping $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$ is called a *binary hyperoperation* on P, where $\mathcal{P}^*(P)$ is the family of all non-empty subsets of P. The couple $(P, \circ)$ is called a *hypergroupoid*.

In the above definition, if A and B are two non-empty subsets of P and $x \in P$, then we define:

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b,\ \ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Definition 2.1. [2] *A polygroup is a system $< P, \circ, e,^{-1} >$, where $e \in P$, $^{-1} : P \rightarrow P$ is a unitary operation on P, "$\circ$" maps $P \times P$ into the non-empty subsets of P, and the following axioms hold for all $x, y, z \in P$:*

(1) $(x \circ y) \circ z = x \circ (y \circ z)$,

(2) $e \circ x = x \circ e = \{x\}$,

(3) $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ x$.

For simplicity, we write x instead of $\{x\}$ for all x in the polygroup $< P, \circ, e,^{-1} >$.

A polygroup $< P, \circ, e, ^{-1} >$ is said to be commutative if $x \circ y = y \circ x$ for all $x, y \in P$. Let $(P, \circ)$ be a polygroup and $K \subseteq P$. Then $(K, \circ)$ is a subpolygroup of $(P, \circ)$ if for all $a, b \in K$, we have that $a \circ b \subseteq K$ and $a^{-1} \in K$.

Let $(P, \circ)$ be a polygroup and $(K, \circ)$ be a subpolygroup of $(P, \circ)$. Then $(K, \circ)$ is a normal subpolygroup of $(P, \circ)$ if for all $a \in P$, we have that $a^{-1} \circ K \circ a \subseteq K$.

A fuzzy set, defined on a universe of discourse U, is characterized by a membership function $\mu_A(x)$ that assigns any element a grade of membership in A. The fuzzy set may be represented by the set of ordered pairs $A = \{(x, \mu_A(x)) : x \in U\}$, where $\mu_A(x) \in [0, 1]$.

The concept of fuzzy subpolygroups of a polygroup is introduced by Zahedi et al. [14].

Definition 2.2. [14] *Let $(P, \circ)$ be a polygroup and A be a fuzzy subset of P with membership function $\mu_A(x) \in [0, 1]$. Then A is a fuzzy subpolygroup of P if the following conditions hold:*

(1) $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$ *for all $x, y \in P$;*

(2) $\mu_A(x) \leq \mu_A(x^{-1})$ *for all $x \in P$.*

Notation . Let $(P, \circ)$ be a polygroup, $x \in P$, $t \in [0, 1]$ and A be a fuzzy subset of P with membership function $\mu_A(x)$. We say that:

(1) $x_t \in \mu_A$ if $\mu_A(x) \geq t$;

(2) $x_t \in q\mu_A$ if $\mu_A(x) + t > 1$;

(3) $x_t \in \vee q\mu_A$ if $x_t \in \mu_A$ or $x_t \in q\mu_A$. Otherwise, we say that $x_t \in \overline{\vee q\mu_A}$.

Definition 2.3. *([8], 61) Let $(P, \circ)$ be a polygroup and A be a fuzzy subset of P with membership function $\mu_A(x)$. Then A is an $(\in, \in \vee q)$ fuzzy subpolygroup of P if for all $t, s \in]0, 1]$ and $x, y \in P$, the following conditions hold:*

(1) $x_t, y_s \in \mu_A$ *implies* $z_{t \wedge s} \in \vee q\mu_A$ *for all* $z \in x \circ y$,

(2) $x_t \in \mu_A$ *implies* $(x^{-1})_t \in \vee q\mu_A$.

Theorem 2.1. *([8], 62) Let* $(P, \circ)$ *be a polygroup and* A *be a fuzzy subset of* P *with membership function* $\mu_A(x)$. *Then* A *is an* $(\in, \in \vee q)$ *fuzzy subpolygroup of* P *if and only if for all* $x, y \in P$, *the following conditions hold:*

(1) $\mu_A(x) \wedge \mu_A(y) \wedge 0.5 \leq \mu_A(z)$ *for all* $z \in x \circ y$,

(2) $\mu_A(x) \wedge 0.5 \leq \mu_A(x^{-1})$ *for all* $x \in P$.

3. Complex fuzzy subpolygroups

Inspired by the defintion of fuzzy subpolygroups of polygroups and using the concept of complex fuzzy subsets, we introduce the concept of complex fuzzy subpolygroups of a polygroup. And we investigate their properties.

Definition 3.1. [1] *Let* $A = \{(x, \mu_A(x)) : x \in U\}$ *be a fuzzy set. Then the set* $A_\pi = \{(x, 2\pi\mu_A(x)) : x \in U\}$ *is said to be a* π*-fuzzy set.*

Remark 3.1. *Let* $(P, \circ)$ *be a polygroup. A* π*-fuzzy set* A_π *is a* π*-fuzzy subpolygroup of* P *if it satisfies the conditions of Definition 2.2.*

Proposition 3.1. *Let* $(P, \circ)$ *be a polygroup. A* π*-fuzzy set* A_π *is a* π*-fuzzy subpolygroup of* P *if and only if* A *is a fuzzy subpolygroup of* P.

Proof. The proof is straightforward. □

Definition 3.2. [11] *A complex fuzzy set, defined on a universe of discourse* U, *is characterized by a membership function* $\mu_A(x)$ *that assigns any element a complex-valued grade of membership in* A. *The complex fuzzy set may be represented by the set of ordered pairs*

$$A = \{(x, \mu_A(x)) : x \in U\},$$

where $\mu_A(x) = r(x)e^{iw(x)}$, $i = \sqrt{-1}$, $r(x) \in [0, 1]$ *and* $w(x) \in [0, 2\pi]$.

Remark 3.2. *By setting $w(x) = 0$ in the above definition, we return back to the traditional fuzzy subset.*

Definition 3.3. [12] *Let $A = \{(x, \mu_A(x)) : x \in U\}$ and $B = \{(x, \mu_B(x)) : x \in U\}$ be two complex fuzzy sets of the same universe U with the membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, respectively. Then*

- $\mu_{A\cap B}(x) = r_{A\cap B}(x)e^{i\omega_{A\cap B}(x)} = \min\{r_A(x), r_B(x)\}e^{i\min\{\omega_A(x),\omega_B(x)\}}$;
- $\mu_{A\cup B}(x) = r_{A\cup B}(x)e^{i\omega_{A\cup B}(x)} = \max\{r_A(x), r_B(x)\}e^{i\max\{\omega_A(x),\omega_B(x)\}}$;
- $\mu_{A^c}(x) = (1 - r_A(x))e^{i(2\pi-\omega_A(x))}$, *where A^c denotes the complement of A.*

Definition 3.4. [1] *Let $A = \{(x, \mu_A(x)) : x \in P\}$ and $B = \{(x, \mu_B(x)) : x \in P\}$ be complex fuzzy subsets of a non-void set P with membership functions $\mu_A(x) = r_A(x)e^{iw_A(x)}$ and $\mu_B(x) = r_B(x)e^{iw_B(x)}$ respectively. Then*

(1) *A complex fuzzy subset A is said to be homogeneous if for all $x, y \in P$, we have*

$$r_A(x) \leq r_A(y) \text{ if and only if } w_A(x) \leq w_A(y).$$

(2) *A complex fuzzy subset A is said to be homogeneous with B if for all $x, y \in P$, we have*

$$r_A(x) \leq r_B(y) \text{ if and only if } w_A(x) \leq w_B(y).$$

Notation . Let $A = \{(x, \mu_A(x)) : x \in P\}$ and $B = \{(x, \mu_B(x)) : x \in P\}$ be complex fuzzy subsets of a non-void set P with membership functions $\mu_A(x) = r_A(x)e^{iw_A(x)}$ and $\mu_B(x) = r_B(x)e^{iw_B}(x)$ respectively. By $\mu_A(x) \leq \mu_B(x)$, we mean that $r_A(x) \leq r_B(x)$ and $w_A(x) \leq w_B(x)$.

Remark 3.3. *Let μ be a complex fuzzy subset of a non-void set P and $x, y \in P$. If μ is homogeneous then either we have $\mu(x) \leq \mu(y)$ or we have $\mu(y) \leq \mu(x)$.*

Throughout this paper, all complex fuzzy sets are considered homogeneous.

Definition 3.5. *Let $(P, \circ)$ be a polygroup and A be a (homogeneous) complex fuzzy subset of P with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subpolygroup of P if the following conditions hold:*

(1) $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$ *for all* $x, y \in P$;

(2) $\mu_A(x) \leq \mu_A(x^{-1})$ *for all* $x \in P$.

Proposition 3.2. *Let $(P, \circ)$ be a polygroup and μ be a (homogeneous) complex fuzzy subset of P. If μ is a complex fuzzy subpolygroup of P then $\mu(x^{-1}) = \mu(x)$ and $\mu(x) \leq \mu(e)$ for all $x \in P$.*

Proof. For all $x \in P$, we have that $x^{-1} \in P$, $\mu(x^{-1}) \geq \mu(x)$ and $\mu(x) = \mu((x^{-1})^{-1}) \geq \mu(x^{-1})$. Thus, $\mu(x^{-1}) = \mu(x)$.
We have that $e \in x \circ x^{-1}$ for all $x \in P$. Since μ is a complex fuzzy subpolygroup of P, it follows that $\mu(x) = \min\{\mu(x), \mu(x^{-1})\} \leq \mu(e)$. □

Definition 3.6. *Let $(P, \circ)$ be a polygroup and A be a complex fuzzy subpolygroup of P with membership function $\mu_A(x)$. Then A is said to be a normal complex fuzzy subpolygroup of P if $\mu_A(z) = \mu_A(z')$ for all $x, y \in P$, $z \in x \circ y$ and $z' \in y \circ x$.*

Example 3.1. *Let $(P, \circ)$ be a polygroup and c be a complex number with $|c| \leq 1$. We define a complex fuzzy subset μ of P as follows:*

$$\mu(x) = c \text{ for all } x \in P.$$

Then μ is a normal complex fuzzy subpolygroup of P.

Remark 3.4. *Let $(P, \circ)$ be a commutative polygroup and μ be a complex fuzzy subpolygroup of P. Then μ is not necessary a normal complex fuzzy subpolygroup of P.*

We illustrate Remark 3.4 by the following example.

Example 3.2. *Let $P = \{e, a, b\}$ and define the commutative polygroup $(P, \circ)$ by the following table:*

$\circ$	e	a	b
e	e	a	b
a	a	$\{e, b\}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{e, a\}$

It is clear that e is the identity of $(P, \circ)$ and that $a^{-1} = b$ (and $b^{-1} = a$).
We define a (homogeneous) complex fuzzy subset μ of P as follows: $\mu(e) = 0.9e^{i\frac{3\pi}{2}}$, $\mu(a) = \mu(b) = 0.7e^{i\frac{\pi}{2}}$. Since $\mu(a) = \mu(b)$, it follows that $\mu(x) = \mu(x^{-1})$ for all $x \in P$. Moreover, we have

$$\{\inf \mu(z) : z \in x \circ y\} = \begin{cases} 0.9e^{i\frac{3\pi}{2}}, & \text{if } x = y = e; \\ 0.7e^{i\frac{\pi}{2}}, & \text{Otherwise.} \end{cases} \geq \min\{\mu(x), \mu(y)\}.$$

Thus, μ is a complex fuzzy subpolygroup of P. Moreover, μ is not a normal complex fuzzy subpolygroup of P as $e, b \in a \circ a$ and $\mu(e) \neq \mu(b)$.

Theorem 3.1. *Let $(P, \circ)$ be a polygroup and μ be a (homogeneous) complex fuzzy subset of P with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then μ_A is a complex fuzzy subpolygroup of P if and only if r_A is a fuzzy subpolygroup of P and w_A is a π-fuzzy subpolygroup of P.*

Proof. Suppose that A is a complex fuzzy subpolygroup of P. We need to prove that the conditions of Definition 2.2 are satisfied for r_A and w_A. For all $x, y \in P$, we have $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$ and that $\mu_A(x) \leq \mu_A(x^{-1})$. The latter and Notation imply that $\inf\{r_A(z) : z \in x \circ y\} \geq \min\{r_A(x), r_A(y)\}$, $\inf\{w_A(z) : z \in x \circ y\} \geq \min\{w_A(x), w_A(y)\}$, $r_A(x) \leq r_A(x^{-1})$ and $w_A(x) \leq w_A(x^{-1})$.

Suppose that r_A is a fuzzy subpolygroup of P and w_A is a π-fuzzy subpolygroup of P. We need to prove that the conditions of Definition 3.5 are satisfied. For all $x, y \in P$, we have $\inf\{r_A(z) : z \in x \circ y\} \geq \min\{r_A(x), r_A(y)\}$ and $\inf\{w_A(z) : z \in x \circ$

$y\} \geq \min\{w_A(x), w_A(y)\}$. The latter and Notation imply that $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$. Let $x \in P$. Then $r_A(x) \leq r_A(x^{-1})$ and $w_A(x) \leq w_A(x^{-1})$. Notation implies that the condition 2 of Definition 3.5 is satisfied for μ_A. □

Example 3.3. *Let $(P, \circ)$ be the polygroup defined in Example 3.2 and define the complement of the defined fuzzy subset. i.e., $\mu^c(e) = 0.1e^{i\frac{\pi}{2}}$ and $\mu^c(a) = \mu^c(b) = 0.3e^{i\frac{3\pi}{2}}$. Then μ^c is not a complex fuzzy subpolygroup of P. This is clear, by applying Proposition 3.2, as $\mu^c(a) = 0.3e^{i\frac{3\pi}{2}} \not\leq \mu^c(e) = 0.1e^{i\frac{\pi}{2}}$.*

Next, we find a necessary and sufficient condition for having both of μ and μ^c complex fuzzy subpolygroups.

Proposition 3.3. *Let $(P, \circ)$ be a polygroup and μ be a (homogeneous) complex fuzzy subset of P. Then μ and the complement μ^c of μ are complex fuzzy subpolygroups of P if and only if there exist a constant number a with $|a| \leq 1$ such that $\mu(x) = a$ for all $x \in P$.*

Proof. The proof is straightforward. □

Proposition 3.4. *Let $(P, \circ)$ be a polygroup and A, B be (homogeneous) complex fuzzy subsets of P such that A is homogeneous with B. If A and B are complex fuzzy subpolygroups of P then $A \cap B$ is a homogeneous complex fuzzy subpolygroup of P.*

Proof. First, we need to show that $A \cap B$ is a homogeneous complex fuzzy subset of P. Let $\mu_{A\cap B}(x) = r_{A\cap B}(x)e^{iw_{A\cap B}(x)}$ and $r_{A\cap B}(x) = \min\{r_A(x), r_B(x)\} \leq r_{A\cap B}(y) = \min\{r_A(y), r_B(y)\}$. Without loss of generality, we set $r_A(x) = \min\{r_A(x), r_B(x)\}$. Then $r_A(x) \leq r_B(x)$, $r_A(x) \leq r_A(y)$ and $r_A(x) \leq r_B(y)$. Since A, B are homogeneous complex fuzzy subsets of P, it follows that $w_A(x) \leq w_B(x)$, $w_A(x) \leq w_A(y)$ and $w_A(x) \leq w_B(y)$. Thus, $w_{A\cap B}(x) = \min\{w_A(x), w_B(x)\} = w_A(x) \leq w_{A\cap B}(y) = \min\{w_A(y), w_B(y)\}$. Similarly, if $w_{A\cap B}(x) = \min\{w_A(x), w_B(x)\} \leq w_{A\cap B}(y) =$

$\min\{w_A(y), w_B(y)\}$ we get that $r_{A\cap B}(x) = \min\{r_A(x), r_B(x)\} \leq r_{A\cap B}(y) = \min\{r_A(y), r_B(y)\}$.

Let A and B be homogeneous complex fuzzy subpolygroups of P and $x, y \in P$. We need to show that the conditions of Definition 3.5 are satisfied. Since A and B are homogeneous, it suffices to show that

(1) $\min\{r_{A\cap B}(x), r_{A\cap B}(y)\} \leq \inf\{r_{A\cap B}(z) : z \in x \circ y\}$ for all $x, y \in P$;
(2) $r_{A\cap B}(x) \leq r_{A\cap B}(x^{-1})$ for all $x \in P$.

Having A and B complex fuzzy subpolygroups of P implies that

(1) $\min\{r_A(x), r_A(y)\} \leq \inf\{r_A(z) : z \in x \circ y\}$ for all $x, y \in P$;
(2) $r_A(x) \leq r_A(x^{-1})$ for all $x \in P$;
(3) $\min\{r_B(x), r_B(y)\} \leq \inf\{r_B(z) : z \in x \circ y\}$ for all $x, y \in P$;
(4) $r_B(x) \leq r_B(x^{-1})$ for all $x \in P$.

Without loss of generality, let

$$r_A(x) = \min\{r_{A\cap B}(x), r_{A\cap B}(y)\} = \min\{r_A(x), r_B(x), r_A(y), r_B(y)\}.$$

For all $z \in x \circ y$, we have that $r_A(z) \geq \min\{r_A(x), r_A(y)\} \geq r_A(x)$ and $r_B(z) \geq \min\{r_B(x), r_B(y)\} \geq r_A(x)$. The latter implies that

$$r_{A\cap B}(z) \geq r_A(x) = \min\{r_{A\cap B}(x), r_{A\cap B}(y)\}.$$

Moreover, for all $x \in P$, we have

$$r_{A\cap B}(x) = \min\{r_A(x), r_B(x)\} \leq \min\{r_A(x^{-1}), r_B(x^{-1})\} = r_{A\cap B}(x^{-1}).$$

□

Corollary 3.1. *Let $(P, \circ)$ be a polygroup and A_i be a (homogeneous) complex fuzzy subset of P for all $i = 1, 2, \ldots, n$. If A_i is a complex fuzzy subpolygroup of P then $\bigcap_{i=1}^{n} A_i$ is a complex fuzzy subpolygroup of P.*

Proof. The proof results from using mathematical induction and Proposition 3.4. □

Definition 3.7. [3] *Let* $< A, \circ_1, e,^{-1} >, < B, \circ_2, e,^{-1} >$ *be two polygroups whose elements are rearranged so that* $A \cap B = \{e\}$. *A new system* $A[B] =< M, \star, e,^I >$ *called the extension of* A *by* B *is formed in the following way: Set* $M = A \cup B$ *and let* $e^I = e, x^I = x^{-1}, x \star e = e \star x = x$ *for all* $x \in M$, *and for all* $x, y \in M \setminus \{e\}$

$$x \star y = \begin{cases} x \circ_1 y, & \text{if } x, y \in A; \\ x, & \text{if } x \in B, y \in A; \\ y, & \text{if } x \in A, y \in B; \\ x \circ_2 y, & \text{if } x, y \in B \text{ and } x^{-1} \neq y; \\ x \circ_2 y \cup A, & \text{if } x, y \in B \text{ and } x^{-1} = y. \end{cases}$$

Theorem 3.2. [3] *Let* $< A, \circ_1, e,^{-1} >, < B, \circ_2, e,^{-1} >$ *be two polygroups whose elements are rearranged so that* $A \cap B = \{e\}$. *Then* $A[B] =< M, \star, e,^I >$, *the extension of* A *by* B, *is a polygroup.*

Proposition 3.5. *Let* A *and* B *be two polygroups such that* $A \cap B = \{e\}$. *Define the homogeneous complex fuzzy subpolygroups* μ_1, μ_2 *of* A *and* B *respectively. If* $\mu_1(x) \geq \mu_2(y)$ *for all* $(x, y) \in A \times B$ *then* μ *is a complex fuzzy subpolygroup of* $A[B]$. *Where, for all* $x \in A[B]$,

$$\mu(x) = \begin{cases} \mu_1(e), & \text{if } x = e; \\ \mu_1(x), & \text{if } x \neq e \in A; \\ \mu_2(x), & \text{if } x \neq e \in B. \end{cases}$$

Proof. It is easy to see that μ is a homogeneous complex fuzzy subset of $A[B]$. We need to prove that the conditions of Definition 3.5 are satisfied for μ. Let $x, y \neq e \in A[B]$. We have the following cases for x, y:

- Case $x, y \in A$. For all $z \in x \star y = x \circ_1 y$, we have $\mu(z) = \mu_1(z) \geq \min\{\mu_1(x), \mu_1(y)\} = \min\{\mu(x), \mu(y)\}$.
- Case $x, y \in B$ and $y \neq x^{-1}$. For all $z \in x \star y = x \circ_2 y$, we have $\mu(z) = \mu_2(z) \geq \min\{\mu_2(x), \mu_2(y)\} = \min\{\mu(x), \mu(y)\}$.

- Case $x, y \in B$ and $y = x^{-1}$. For all $z \in x \star y = x \circ_2 y \cup A$, we have $\mu(z) = \begin{cases} \mu_1(z), & \text{if } z \in A; \\ \mu_2(z), & \text{if } z \in x \circ_2 y. \end{cases}$
  Since $\mu_2(z) \geq \min\{\mu_2(x), \mu_2(y)\}$ and $\mu_1(z) \geq \mu_2(z')$ for all $z' \in B$, it follows that $\mu_2(z) \geq \min\{\mu_2(x), \mu_2(y)\} = \min\{\mu(x), \mu(y)\}$ if $z \in x \circ_2 y$ and that $\mu(z) = \mu_1(z) \geq \min\{\mu_2(x), \mu_2(y)\} = \min\{\mu(x), \mu(y)\}$ if $z \in A$.
- Case $x \in A$ and $y \in B$. For all $z \in x \star y = y$, we have $\mu(z) = \mu_2(y) \geq \min\{\mu_1(x), \mu_2(y)\} = \min\{\mu(x), \mu(y)\}$.
- Case $x \in B$ and $y \in A$. For all $z \in x \star y = x$, we have $\mu(z) = \mu_2(x) \geq \min\{\mu_2(x), \mu_1(y)\} = \min\{\mu(x), \mu(y)\}$.

For all $x \neq e \in A[B]$, $\mu(x^I) = \begin{cases} \mu_1(x^{-1}), & \text{if } x \in A; \\ \mu_2(x^{-1}), & \text{if } x \in B. \end{cases} \geq \mu(x)$. □

Example 3.4. *Let $(A, \circ_1)$ and $(B, \circ_2)$ be the polygroups defined as follows:*

$\circ_1$	0	1	2
0	0	1	2
1	1	$\{0,2\}$	$\{1,2\}$
2	2	$\{1,2\}$	$\{0,1\}$

$\circ_2$	0	a	b
0	0	a	b
a	a	$\{0,b\}$	$\{a,b\}$
b	b	$\{a,b\}$	$\{0,a\}$

Define the (homogeneous) complex fuzzy subsets μ_1, μ_2 on A, B respectively as follows: $\mu_1(0) = 0.9e^{i\frac{3\pi}{2}}$, $\mu_1(1) = \mu_1(2) = 0.7e^{i\frac{\pi}{2}}$, $\mu_2(0) = 0.6e^{i\frac{\pi}{2}}$ and $\mu_2(a) = \mu_2(b) = 0.4e^{i\frac{\pi}{6}}$.

Then $(A[B], \star)$ *is given as follows:*

$\star$	0	1	2	a	b
0	0	1	2	a	b
1	1	$\{0,2\}$	$\{1,2\}$	a	b
2	2	$\{1,2\}$	$\{0,1\}$	a	b
a	a	a	a	$\{0,1,2,b\}$	$\{a,b\}$
b	b	b	b	$\{a,b\}$	$\{0,1,2,a\}$

and μ *is a complex fuzzy subpolygroup of* $A[B]$.
$\mu(0) = 0.9e^{i\frac{3\pi}{2}}$, $\mu(1) = \mu(2) = 0.7e^{i\frac{\pi}{2}}$ *and* $\mu(a) = \mu(b) = 0.4e^{i\frac{\pi}{6}}$.

Definition 3.8. *Let* $(P, \circ)$ *be a polygroup,* μ *be a (homogeneous) complex fuzzy subset of* P *and* $0e^{i0} \leq t \leq 1e^{i2\pi}$. *We define the following:*

(1) $\mu_t = \{x \in P : \mu(x) \geq t\}$, *the level set of* μ;

(2) $\mu_t^{>} = \{x \in P : \mu(x) > t\}$, *the strong level set of* μ.

Theorem 3.3. *Let* $(P, \circ)$ *be a polygroup,* μ *be a (homogeneous) complex fuzzy subset of* P. *Then* μ *is a complex fuzzy subpolygroup of* P *if and only if* μ_t $(\neq \emptyset)$ *is subpolygroup of* P *for all* $0e^{i0} \leq t \leq 1e^{i2\pi}$.

Proof. Let μ be a complex fuzzy subpolygroup of P and $x, y \in \mu_t \neq \emptyset$. We need to show that $x \circ y \subseteq \mu_t$ and $x^{-1} \in \mu_t$. For all $z \in x \circ y$, we have that $\mu(z) \geq \min\{\mu(x), \mu(y)\} \geq t$ and $\mu(x^{-1}) \geq \mu(x) \geq t$. Thus, $x \circ y \subseteq \mu_t$ and $x^{-1} \in \mu_t$.

Let μ_t $(\neq \emptyset)$ be a subpolygroup of P and $t = \min\{\mu(x), \mu(y)\}$. Since $t = \min\{\mu(x), \mu(y)\}$, it follows that $\mu(x), \mu(y) \geq \min\{\mu(x), \mu(y)\} = t$. Thus, $x, y \in \mu_t$. The latter and having μ_t $(\neq \emptyset)$ a subpolygroup of P imply that $x \circ y \subseteq \mu_t$ and hence, $\mu(z) \geq \min\{\mu(x), \mu(y)\} = t$ for all $z \in x \circ y$. To show that $\mu(x^{-1}) \geq \mu(x)$, we set $t_0 = \mu(x)$. Since $x \in \mu_{t_0}$ and μ_{t_0} is a subpolygroup of P, it follows that $x^{-1} \in \mu_{t_0}$. Consequently, $\mu(x^{-1}) \geq t_0 = \mu(x)$. □

Proposition 3.6. *Let $(P, \circ)$ be a polygroup, μ be a complex fuzzy subpolygroup of P. Then μ is a normal complex fuzzy subpolygroup of P if and only if $\mu(z) \geq \mu(y)$ for all $x, y \in P$ and for all $z \in x \circ y \circ x^{-1}$.*

Proof. The proof is same as that of the traditional case (see [8], 48). □

Theorem 3.4. *Let $(P, \circ)$ be a polygroup and μ be a (homogeneous) complex fuzzy subset of P. Then μ is a normal complex fuzzy subpolygroup of P if and only if $\mu_t \neq \emptyset$ is a normal subpolygroup of P, for all $0e^{i0} \leq t \leq 1e^{i2\pi}$.*

Proof. The proof follows from Proposition 3.6. □

Theorem 3.5. *Let $(P, \circ)$ be a polygroup and μ be a (homogeneous) complex fuzzy subset of P. Then μ is a complex fuzzy subpolygroup of P if and only if $\mu_t^{>}$ $(\neq \emptyset)$ is subpolygroup of P, for all $0e^{i0} \leq t \leq 1e^{i2\pi}$.*

Proof. Let μ be a complex fuzzy subpolygroup of P and $x, y \in \mu_t^{>} \neq \emptyset$. We need to show that $x \circ y \subseteq \mu_t^{>}$ and $x^{-1} \in \mu_t^{>}$. For all $z \in x \circ y$, we have that $\mu(z) \geq \min\{\mu(x), \mu(y) > t$ and $\mu(x^{-1}) \geq \mu(x) > t$.

Let $\mu_t^{>} \neq \emptyset$ be a subpolygroup of P. If $\mu(x) = \min\{\mu(x), \mu(y)\} = 0$ then it is clear that $\mu(x^{-1}) \geq 0 = \mu(x)$ and that $\mu(z) \geq 0 = \min\{\mu(x), \mu(y)\}$ for all $z \in x \circ y$. Without loss of generality, we suppose that $\min\{\mu(x), \mu(y)\} \neq 0$. Then there exists a complex number t satisfying $t < \min\{\mu(x), \mu(y)\}$. Suppose that there exist no $a \in P$ satisfying $t < \mu(a) < \min\{\mu(x), \mu(y)\}$. Having $x, y \in \mu_t^{>}$ implies that $x \circ y \subseteq \mu_t^{>}$ and hence, $\mu(z) > t$ for all $z \in x \circ y$. Since there exist no $a \in P$ satisfying $t < \mu(a) < \min\{\mu(x), \mu(y)\}$, it follows that $\mu(z) \geq \min\{\mu(x), \mu(y)\}$. (We can assume the validity of the latter statement since we are dealing with homogeneous complex fuzzy subsets). Let t_0 be the largest complex number on the closed unit disc of the complex plane such that $t_0 < \mu(x)$ and there is no $a \in P$ satisfying $t_0 < \mu(a) < \mu(x)$. We have that $x \in \mu_{t_0}^{>}$. Having $\mu_{t_0}^{>}$ a subpolygroup of P implies that $x^{-1} \in \mu_{t_0}^{>}$. Consequently, $\mu(x^{-1}) > t_0$. Since there is no $a \in P$ satisfying $t_0 < \mu(a) < \mu(x)$, it follows that $\mu(x^{-1}) \geq \mu(x)$. □

Theorem 3.6. *Let $(P, \circ)$ be a polygroup and μ be a (homogeneous) complex fuzzy subset of P. Then μ is a complex fuzzy normal subpolygroup of P if and only if $\mu_t^{>} \neq \emptyset$ is a normal subpolygroup of P.*

Proof. Let μ be a normal complex fuzzy subpolygroup of P, $x \in P$ and $y \in \mu_t^{>} \neq \emptyset$. We need to show that $x \circ y \circ x^{-1} \subseteq \mu_t^{>}$. For all $z \in x \circ y \circ x^{-1}$, Proposition 3.6 asserts that $\mu(z) \geq \mu(y) > t$.

Let $\mu_t^{>} \neq \emptyset$ be a normal subpolygroup of P and $\mu(y) > t$. For all $x \in P$, we have $z \in x^{-1} \circ y \circ x \subseteq \mu_t^{>}$. The latter implies that $\mu(z) \geq \mu(y) > t$. Proposition 3.6 completes the proof. □

Theorem 3.7. *Let $(P, \circ)$ be a polygroup, μ be a (homogeneous) complex fuzzy subset of P. Then the following are equivalent:*

(1) *μ is a complex fuzzy subpolygroup of P,*

(2) *$\mu_t \neq \emptyset$ is subpolygroup of P,*

(3) *$\mu_t^{>} \neq \emptyset$ is subpolygroup of P.*

Proof. The proof results from Theorems 3.3 and 3.5. □

Theorem 3.8. *Let $(P, \circ)$ be a polygroup, μ be a (homogeneous) complex fuzzy subset of P. Then the following are equivalent:*

(1) *μ is a normal complex fuzzy subpolygroup of P,*

(2) *$\mu_t \neq \emptyset$ is a normal subpolygroup of P,*

(3) *$\mu_t^{>} \neq \emptyset$ is a normal subpolygroup of P.*

Proof. The proof results from Theorems 3.4 and 3.6. □

4. Generalized complex fuzzy subpolygroups

In this section, we define the generalized complex fuzzy subpolygroups and investigate their properties.

Remark 4.1. *Let $(P, \circ)$ be a polygroup. A π-fuzzy set A_π is an $(\in, \in \vee q)$ π-fuzzy subpolygroup of P if it satisfies the conditions of Definition 2.3.*

Proposition 4.1. *Let $(P, \circ)$ be a polygroup. A π-fuzzy set A_π is an $(\in, \in \vee q)$ π-fuzzy subpolygroup of P if and only if A is an $(\in, \in \vee q)$ fuzzy subpolygroup of P.*

Proof. The proof is straightforward. □

Remark 4.2. *A fuzzy subset μ is an $(\in, \in \vee q)$ π-fuzzy subpolygroup of P if and only if for all $x, y \in P$, the following conditions hold:*

(1) $\mu_A(x) \wedge \mu_A(y) \wedge \pi \leq \mu_A(z)$ *for all* $z \in x \circ y$,

(2) $\mu_A(x) \wedge \pi \leq \mu_A(x^{-1})$ *for all* $x \in P$.

Notation . Let $(P, \circ)$ be a polygroup, $x \in P$ and A be a fuzzy subset of P with membership function $\mu_A(x)$ such that $0e^{0i} < t = se^{i\theta} \leq 1e^{2\pi i}$. We say that:

(1) $x_t \in \mu_A$ if $r_A(x) \geq s$ and $w_A(x) \geq \theta$;

(2) $x_t \in q\mu_A$ if $r_A(x) + s > 1$ and $w_A(x) + \theta > 2\pi$;

(3) $x_t \in \vee q\mu_A$ if $x_t \in \mu_A$ or $x_t \in q\mu_A$. Otherwise, we say that $x_t \in \overline{\vee q\mu_A}$.

Definition 4.1. *Let $(P, \circ)$ be a polygroup and A be a complex fuzzy subset of P with membership function $\mu_A(x)$. Then A is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P if for all $0e^{0i} \leq t, s < 1e^{2\pi i}$ and $x, y \in P$, the following conditions hold:*

(1) $x_t, y_s \in \mu$ *implies* $z_{t\wedge s} \in \vee q\mu$ *for all* $z \in x \circ y$,

(2) $x_t \in \mu$ *implies* $(x^{-1})_t \in \vee q\mu$.

Throughout this section, all complex fuzzy subsets are considered to be homogeneous and homogenous with the complex number "$0.5e^{i\pi}$". The condition of being homogenous with the complex number "$0.5e^{i\pi}$" is essential because we will be using $\mu_A(x) \wedge 0.5e^{i\pi} = \min\{\mu_A(x), 0.5e^{i\pi}\}$. If A is not homogeneous with "$0.5e^{i\pi}$", $\mu_A(x) \wedge 0.5e^{i\pi}$ will not be defined.

Theorem 4.1. *Let $(P, \circ)$ be a polygroup and A be a (homogeneous) complex fuzzy subset of P with membership function $\mu_A(x)$ that is homogenous with the complex number "$0.5e^{i\pi}$". Then A is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P if and only if for all $x, y \in P$, the following conditions hold:*

(1) $\mu_A(x) \wedge \mu_A(y) \wedge 0.5e^{i\pi} \leq \mu_A(z)$ *for all* $z \in x \circ y$,

(2) $\mu_A(x) \wedge 0.5e^{i\pi} \leq \mu_A(x^{-1})$ *for all* $x \in P$.

Proof. The proof is similar to the traditional case (see [8], 62) by applying suitable modifications. □

Proposition 4.2. *Let $(P, \circ)$ be a polygroup and A be a (homogeneous) complex fuzzy subset of P with membership function $\mu_A(x)$ that is homogenous with the complex number "$0.5e^{i\pi}$". If A is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P then $\mu_A(e) \geq \mu_A(x) \wedge 0.5e^{i\pi}$ for all $x \in P$.*

Proof. Since $e \in x \circ x^{-1}$, it follows by applying Condition 1 of Theorem 4.1 that $\mu_A(x) \wedge \mu_A(x^{-1}) \wedge 0.5e^{i\pi} \leq \mu_A(e)$. By applying to the latter Condition 2, we get that $\mu_A(x) \wedge 0.5e^{i\pi} = \mu_A(x) \wedge 0.5e^{i\pi} \wedge (\mu_A(x) \wedge 0.5e^{i\pi}) \leq \mu_A(x) \wedge 0.5e^{i\pi} \wedge \mu_A(x^{-1}) \leq \mu_A(e)$. □

Proposition 4.3. *Let $(P, \circ)$ be a polygroup and A be a homogeneous complex fuzzy subset of P with membership function $\mu_A(x)$ that is homogeneous with the complex number "$0.5e^{i\pi}$". If A is a complex fuzzy subpolygroup of P then A is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P.*

Proof. The proof is straightforward. □

Example 4.1. *Let $(P, \circ)$ be any polygroup, c be any constant in the closed unit disc of the complex plane that is homogeneous with the complex number "$0.5e^{i\pi}$". Define a complex fuzzy subset μ of P as follows:*

$$\mu(x) = c \text{ for all } x \in P.$$

Then, using Example 3.1 and Proposition 4.3, we get that μ is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P.

Remark 4.3. *The converse of Proposition 4.3 is not always true.*

We illustrate Remark 4.3 by the following example.

Example 4.2. *Let $(P, \circ)$ be the polygroup given by the following table:*

$\circ$	0	a	b
0	0	a	b
a	a	$\{0, b\}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{0, a\}$

Define μ on P as follows: $\mu(0) = 0.7e^{i\frac{3\pi}{2}}$, $\mu(a) = 0.8e^{i\frac{7\pi}{4}}$, $\mu(b) = 0.5e^{i\frac{\pi}{2}}$. It is easy to see that μ is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P but not a complex fuzzy subpolygroup of P. The latter is easily shown as the level set $\mu_{0.8e^{i\frac{7\pi}{4}}} = \{a\}$ is not a subpolygroup of P ($a \in \mu_{0.8e^{i\frac{7\pi}{4}}} \not\supseteq \{0, b\} = a \circ a$).

Theorem 4.2. *Let $(P, \circ)$ be a polygroup and A be a (homogeneous) complex fuzzy subset of P with membership function $\mu_A(x)$ that is homogeneous with the complex number "$0.5e^{i\pi}$". Then A is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P if and only if for all $0e^{0i} < t \leq 0.5e^{\pi i}$, $\mu_t = \{x \in P : \mu_A(x) \geq t\} \neq \emptyset$ is a subpolygroup of P.*

Proof. The proof is similar to the traditional case (see [8], 63) by applying suitable modifications. ∎

Theorem 4.3. *$(P, \circ)$ be a polygroup and A be a (homogeneous) complex fuzzy subset of P with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$ that is homogeneous with the complex number "$0.5e^{i\pi}$". Then A is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P if and only if r_A is an $(\in, \in \vee q)$ fuzzy subpolygroup of P and w_A is an $(\in, \in \vee q)$ π-fuzzy subpolygroup of P.*

Proof. Let A be an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P. This is equivalent to having the conditions of Theorem 4.1 satisfied and we can rewrite them as follows:

(1) $r_A(x) \wedge r_A(y) \wedge 0.5 \leq r_A(z)$ and $w_A(x) \wedge w_A(y) \wedge \pi \leq w_A(z)$ for all $z \in x \circ y$,

(2) $r_A(x) \wedge 0.5 \leq r_A(x^{-1})$ and $w_A(x) \wedge \pi \leq w_A(x^{-1})$ for all $x \in P$.

The latter conditions are equivalent to having r_A an $(\in, \in \vee q)$ fuzzy subpolygroup of P and w_A an $(\in, \in \vee q)$ π-fuzzy subpolygroup of P as the conditions of Theorem 2.1 are satisfied for both: r_A and w_A. □

Theorem 4.4. *Let $(P, \circ)$ be a polygroup and A, B be (homogeneous) complex fuzzy subsets of P that are homogeneous with the complex number "$0.5e^{i\pi}$" and such that A is homogeneous with B. If A and B are $(\in, \in \vee q)$ complex fuzzy subpolygroups of P then $A \cap B$ is an $(\in, \in \vee q)$ complex fuzzy subpolygroup of P.*

Proof. In Proposition 3.4, we proved that $A \cap B$ is a homogeneous complex fuzzy subset of P. It is easy to see that $A \cap B$ is homogeneous with the complex number "$0.5e^{i\pi}$". We need to show that the conditions of Theorem 4.1 are satisfied for $A \cap B$, i.e.

(1) $\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y) \wedge 0.5e^{i\pi} \leq \mu_{A\cap B}(z)$ for all $z \in x \circ y$,

(2) $\mu_{A\cap B}(x) \wedge 0.5e^{i\pi} \leq \mu_{A\cap B}(x^{-1})$ for all $x \in P$.

To prove Condition 1, we have two cases: $\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y) \wedge 0.5e^{i\pi} = 0.5e^{i\pi}$ and $\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y) \wedge 0.5e^{i\pi} \neq 0.5e^{i\pi}$.

- Case $\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y) \wedge 0.5e^{i\pi} = 0.5e^{i\pi}$. We can rewrite this case as $r_{A\cap B}(x) \wedge r_{A\cap B}(y) \wedge 0.5 = 0.5$ and $w_{A\cap B}(x) \wedge w_{A\cap B}(y) \wedge \pi = \pi$. Having A and B $(\in, \in \vee q)$ complex fuzzy subpolygroups of P implies that for all $z \in x \circ y$, we have:
  (1) $0.5 = r_A(x) \wedge r_A(y) \wedge 0.5 \leq r_A(z)$;
  (2) $\pi = w_A(x) \wedge w_A(y) \wedge \pi \leq w_A(z)$
  (3) $0.5 = r_B(x) \wedge r_B(y) \wedge 0.5 \leq r_B(z)$;
  (4) $\pi = w_B(x) \wedge w_B(y) \wedge \pi \leq w_B(z)$.

  We get now that $0.5e^{i\pi} = \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y) \wedge 0.5e^{i\pi} \leq \mu_{A\cap B}(z)$.
- Case $\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y) \wedge 0.5e^{i\pi} \neq 0.5e^{i\pi}$. This case is equivalent to having $\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y) < 0.5e^{i\pi}$. We get that either $r_{A\cap B}(x) \wedge r_{A\cap B}(y) \wedge 0.5 \neq 0.5$ or $w_{A\cap B}(x) \wedge w_{A\cap B}(y) \wedge \pi \neq \pi$ (or both). We consider the case $r_{A\cap B}(x) \wedge r_{A\cap B}(y) \wedge 0.5 \neq 0.5$ and the other is done in a similar manner. Without loss of generality, let $r_A(x) = r_{A\cap B}(x) \wedge r_{A\cap B}(y) \wedge 0.5 \neq 0.5$. Having A and B $(\in, \in \vee q)$ complex fuzzy subpolygroups of P implies that for all $z \in x \circ y$, we have:
  (1) $r_A(x) = r_A(x) \wedge r_A(y) \wedge 0.5 \leq r_A(z)$;
  (2) $w_A(x) \wedge w_A(y) \wedge \pi \leq w_A(z)$
  (3) $r_A(x) \leq r_B(x) \wedge r_B(y) \wedge 0.5 \leq r_B(z)$;
  (4) $w_B(x) \wedge w_B(y) \wedge \pi \leq w_B(z)$.

  We get now that $r_A(x) = r_{A\cap B}(x) \wedge r_{A\cap B}(y) \wedge 0.5 \leq r_A(z)$. Since A and B are homogeneous, it follows that $w_A(x) = w_A(x) \wedge w_B(x) \wedge w_A(y) \wedge w_B(y) = w_{A\cap B}(x) \wedge w_{A\cap B}(y)$. Having A homogeneous with $0.5e^{i\pi}$ and $r_A(x) \leq 0.5$ implies that $w_A(x) \leq \pi$. We get now that $w_A(z) \geq w_A(x) = w_A(x) \wedge w_B(x) \wedge w_A(y) \wedge w_B(y) \wedge \pi = w_{A\cap B}(x) \wedge w_{A\cap B}(y) \wedge \pi$. Consequently, we get that $\mu_A(z) \geq \mu_A(x) = \mu_A(x) \wedge \mu_B(x) \wedge \mu_A(y) \wedge \mu_B(y) \wedge 0.5e^{i\pi} = \mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y) \wedge 0.5e^{i\pi}$.

To prove Condition 2 ($\mu_{A\cap B}(x) \wedge 0.5e^{i\pi} \leq \mu_{A\cap B}(x^{-1})$), we use the fact that A, B are $(\in, \in \vee q)$ complex fuzzy subpolygroup of P. We get that $\mu_A(x) \wedge 0.5e^{i\pi} \leq \mu_A(x^{-1})$

and $\mu_B(x) \wedge 0.5e^{i\pi} \leq \mu_B(x^{-1})$. The latter implies that $\mu_{A\cap B}(x) \wedge 0.5e^{i\pi} = (\mu_A(x) \wedge 0.5e^{i\pi}) \wedge (\mu_B(x) \wedge 0.5e^{i\pi}) \leq \mu_A(x^{-1}) \wedge \mu_B(x^{-1}) = \mu_{A\cap B}(x^{-1})$. □

Corollary 4.1. *Let* $(P, \circ)$ *be a polygroup and* A_i *be a (homogeneous) complex fuzzy subset of* P *for all* $i = 1, 2, \ldots, n$. *If* A_i *is an* $(\in, \in \vee q)$ *complex fuzzy subpolygroup of* P *then* $\bigcap_{i=1}^{n} A_i$ *is an* $(\in, \in \vee q)$ *complex fuzzy subpolygroup of* P.

Proof. The proof results from using mathematical induction and Theorem 4.4. □

In Section 3 (Proposition 3.3), we proved that μ and its complement μ^c are both complex fuzzy subpolygroups if and only if μ is a constant complex fuzzy subset. This is not applicable for $(\in, \in \vee q)$ complex fuzzy subpolygroups. We present an example where a non constant complex fuzzy subset μ and its complement μ^c are both complex $(\in, \in \vee q)$ fuzzy subpolygroups.

Example 4.3. *Let* $(P, \circ)$ *be the polygroup given by the following table:*

$\circ$	0	a	b
0	0	a	b
a	a	$\{0, b\}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{0, a\}$

Define μ *on* P *as follows:* $\mu(0) = 0.5e^{i\frac{\pi}{2}}$, $\mu(a) = 0.7e^{i\frac{3\pi}{2}}$, $\mu(b) = 0.7e^{i\frac{3\pi}{2}}$ *and* μ^c *is defined on* P *as follows:* $\mu^c(0) = 0.5e^{i\frac{3\pi}{2}}$, $\mu(a) = 0.3e^{i\frac{\pi}{2}}$, $\mu(b) = 0.3e^{i\frac{\pi}{2}}$. *It is easy to see that both:* μ *and* μ^c *are* $(\in, \in \vee q)$ *complex fuzzy subpolygroups of* P.

5. Conclusion

This paper contributed to the study of fuzzy subpolygroups by extending the traditional concept to the concepts of complex fuzzy and generalized complex fuzzy subpolygroups. Different examples using the extended concepts were provided and several properties were proved.

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