Jordan Journal of Mathematics and Statistics (JJMS) 12(2), 2019, pp 175 - 190

ON THE DUAL OF WEAKLY PRIME AND SEMIPRIME MODULES

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Abstract. The weakly second modules (the dual of weakly prime modules) was introduced in [6]. In this paper we introduce and study the semisecond and strongly second modules. Let R be a ring and M be an R-module. We show that M is semisecond if and only if $MI = MI^2$ for any ideal I of R. It is shown that every sum of the second submodules of M is a semisecond submodule of M. Also if M is an Artinian module, then M has only a finite number of maximal semisecond submodules. We prove that every strongly second submodule of M is second and every minimal submodule of M is strongly second. If every nonzero submodule of M is (weakly) second, then M is called fully (weakly) second. It is shown that if R is a commutative ring, then M is fully second if and only if M is fully weakly second, if and only if M is a homogeneous semisimple module.

1. Introduction

Throughout the paper, all rings will have identity elements and all modules will be right unitary. The notation "$\subset$" is used to denote strict inclusion. Also, R denotes an arbitrary ring with identity element. Let M be an R-module. Then the annihilator of M (in R) is the ideal $\mathrm{ann}_R(M) = \{r \in R \mid Mr = 0\}$. For any submodule N of M and any ideal I of R, the submodule $\{x \in M \mid xI \subseteq N\}$ of M is denoted by $(N :_M I)$. A proper submodule N of a right R-module M is said to be a *prime submodule* of

2000 *Mathematics Subject Classification.* 16D10, 16D80, 16N60.

Key words and phrases. weakly second modules, semisecond modules, weakly prime modules.

Received: Feb. 15, 2018 Accepted: Sept. 3, 2018 .

M if for any submodule K of M and any ideal I of R, $KI \subseteq N$ implies that $K \subseteq N$ or $MI \subseteq N$, i.e., $\mathrm{ann}_R(M/N) = \mathrm{ann}_R(K/N)$, for any $N \subset K \leq M$ (see [8]). Also a proper submodule N of a right R-module M is said to be a *weakly prime submodule* of M if for any submodule K of M and any two ideals I, J of R, $KIJ \subseteq N$ implies that $KI \subseteq N$ or $KJ \subseteq N$, i.e., $\mathrm{ann}_R(K/N)$ is a prime ideal of R, for any $N \subset K \leq M$. Moreover, a module M is called a *prime module* (resp., *weakly prime module*) if (0) is a prime (resp., weakly prime) submodule of M. The notion of weakly prime modules is a generalization of prime modules and has been introduced by Behboodi and Koohi in [5]. Also if R is commutative, then it is easy to see that N is weakly prime if and only if for any $K \leq M$ and two elements a and b of R, $Kab \subseteq N$ implies that $Ka \subseteq N$ or $Kb \subseteq N$, see for example [4].

Let us mention another notion used for weakly prime submodule in the literature. Let M be an R-module over a commutative ring R and N be a proper submodule of M. For instance in [9, 12], N is called a *weakly prime submodule* of M, if for any $m \in M$ and $r \in R$, $0 \neq mr \in N$ implies that $m \in N$ or $Mr \subseteq N$. In the following, we show that the notion of weakly prime used in [9, 12] and what we use in this paper are independent. We note that the zero submodule in any module is always weakly prime regarding the sense used in [9, 12]. However, in $\mathbb{Z}_{30}$ as a $\mathbb{Z}$-module, (0) is not weakly prime submodule. Because $2\mathbb{Z}_{30}(3)(5) = 0$, but $2\mathbb{Z}_{30}(3) \neq 0$ and $2\mathbb{Z}_{30}(5) \neq 0$. On the other hand, it is easy to see that in $\mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module, $(0) \oplus 2\mathbb{Z}$ is a weakly prime submodule. However, $0 \neq ((0) \oplus \mathbb{Z})(2) \subseteq (0) \oplus 2\mathbb{Z}$, but $(0) \oplus \mathbb{Z} \nsubseteq (0) \oplus 2\mathbb{Z}$ and $(\mathbb{Z} \oplus \mathbb{Z})(2) \nsubseteq (0) \oplus 2\mathbb{Z}$ which means $(0) \oplus 2\mathbb{Z}$ is not weakly prime regarding the sense used in [9, 12].

A nonzero R-module M is called a *second module* (the dual of a prime module) if $\mathrm{ann}_R(M) = \mathrm{ann}_R(M/N)$ for every proper submodule N of M. This notion was introduced and studied by Yassemi in [14], for modules over commutative rings. Moreover, in [7], the authors generalized second modules from commutative rings to noncommutative setting. The dual notion of a weakly prime module over noncommutative rings was introduced by the author in [6] and some properties of this class of modules have been considered. A nonzero R-module M is a *weakly second* (resp., *semisecond*) *module* if $\mathrm{ann}_R(M/N)$ is a prime (resp., semiprime) ideal of R for every proper submodule N of M. By a *second* (resp., *weakly second, semisecond*) *submodule* of a module we mean a submodule which is also a second (resp., weakly second, semisecond) module.

Prime and weakly prime modules are interesting topics which have been studied by many researchers, see [2, 5, 7, 8, 14]. It is natural to ask the following question: to what extent dose the dual of these results hold for weakly second modules. The purpose of this paper is to obtain more information about this class of modules.

Let M be an R-module and N be a submodule of M. In Section 2, we show that the proper submodule N of M is weakly prime if and only if for any two ideals I and J of R, $(N :_M IJ) = (N :_M I)$ or $(N :_M IJ) = (N :_M J)$ (Proposition 2.1). Some characterizations of semisecond modules are given (Proposition 2.2). $0 \neq N$ is called a *secondary submodule* of M if for each ideal I of R, $NI = N$ or $NI^n = 0$, for some integer number n. It is shown that if N is a secondary and semisecond submodule of M, then N is weakly second (Proposition 2.3). Also if M is an Artinian module, then M has only a finite number of maximal semisecond submodules (Theorem 2.1). As an interesting result, we prove that every nonzero submodule of M is semisecond if and only if every proper submodule of M is semiprime (Theorem 2.2). A nonzero submodule S of an R-module M is called *strongly second* if for every two submodules L_1 and L_2 of M and nonzero ideal I of R, $S \subseteq (L_1 :_M I\mathrm{ann}_R(L_2 \cap S))$ implies that $SI \subseteq L_1$ or $S \subseteq L_2$. We show that if V is a vector space over a division ring and W is a subspace of V, then W is a minimal subspace of V if and only if W is a strongly second subspace of V (Proposition 2.5).

In Section 3, we study the fully weakly second modules. A nonzero R-module M is called *fully* (*weakly*) *prime* if each proper submodule of M is a (weakly) prime submodule. Also we say that M is *fully* (*weakly*) *second* if each nonzero submodule of M is a (weakly) second submodule. In Theorem 3.1, fully weakly second modules are characterized. It is shown that M is fully weakly prime if and only if M is fully weakly second (Lemma 3.1). Finally, in Theorem 3.2, fully weakly second modules over a commutative ring are characterized.

2. Weakly second and semisecond modules

Let M be a nonzero R-module. As in [7], M is called a *second module* if for every proper submodule N of M, $\mathrm{ann}_R(M/N) = \mathrm{ann}_R(M)$. We say that M is a *weakly second* (resp., *semisecond*) *module* if for every proper submodule N of M, $\mathrm{ann}_R(M/N)$ is a prime (resp., semiprime) ideal of R. It is easy to see that

$$M \text{ is second} \Rightarrow M \text{ is weakly second} \Rightarrow M \text{ is semisecond.}$$

In general, non of implications is reversible (see Example 2.1).

Example 2.1. (*a*) *It is clear to see that every homogenous semisimple module is weakly second and every semisimple module is semisecond. Also, the $\mathbb{Z}$-module $\mathbb{Z}_n$ is semisecond if and only if n is a square-free number. Moreover, $\mathbb{Z}_n$ is a weakly second $\mathbb{Z}$-module if and only if n is a prime number. In particular, for any two distinct prime numbers p and q, the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is not weakly second because $(\mathbb{Z}_p \oplus \mathbb{Z}_q)p\mathbb{Z} \neq 0$ and $(\mathbb{Z}_p \oplus \mathbb{Z}_q)q\mathbb{Z} \neq 0$ but $(\mathbb{Z}_p \oplus \mathbb{Z}_q)p\mathbb{Z}q\mathbb{Z} = 0$. On the other hand, $(\mathbb{Z}_p \oplus \mathbb{Z}_q)n\mathbb{Z} = (\mathbb{Z}_p \oplus \mathbb{Z}_q)n^2\mathbb{Z}$ for any $n \in \mathbb{N}$. Thus $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is semisecond.*

(*b*) *Let $V = \oplus_{i=1}^{\infty} e_i D$ be a vector space over a division ring D, and set $R = End(V_D)$ and $T = \{f \in R \mid \ rank f < \infty\}$. It is known that R has only three ideals (0), R and*

T. So T is a maximal ideal and (0) is a prime ideal of R. Now it is easy to check that R as a left R-module is weakly second but is not a second R-module.

Example 2.2. *Let M be a right R-module. Then for each maximal ideal P of R, $MP = M$ or M/MP is a second R-module. To see this, suppose that $M \neq MP$ and $0 \neq K/MP$ is a submodule of M/MP. Then $P \subseteq ann_R(M/MP) \subseteq ann_R(K/MP)$ and since P is maximal, $P = ann_R(M/MP) = ann_R(K/MP)$.*

It is easy to see that a nonzero submodule N of a right R-module M is weakly second if and only if for any two ideals I and J of R, $NIJ = NI$ or $NIJ = NJ$. We give a similar result for a weakly prime submodule of a module.

Proposition 2.1. *Let M be a right R-module and N be a proper submodule of M. Then N is weakly prime if and only if for any two ideals I and J of R, $(N :_M IJ) = (N :_M I)$ or $(N :_M IJ) = (N :_M J)$.*

Proof. Suppose that N is weakly prime. It is easy to check that $(N :_M IJ) = (N :_M I) \cup (N :_M J)$. Since $(N :_M IJ)$ is a submodule of M, $(N :_M I) \subseteq (N :_M J)$ or $(N :_M J) \subseteq (N :_M I)$. Thus $(N :_M IJ) = (N :_M I)$ or $(N :_M IJ) = (N :_M J)$. Conversely, assume that for any two ideals I and J of R, $(N :_M IJ) = (N :_M I)$ or $(N :_M IJ) = (N :_M J)$. Also suppose that $KIJ \subseteq N$, where K is a submodule of M and I, J are two ideals of R. Then $K \subseteq (N :_M IJ)$ and by the hypothesis, $K \subseteq (N :_M I)$ or $K \subseteq (N :_M J)$ and so $KI \subseteq N$ or $KJ \subseteq N$. □

A proper submodule N of an R-module M is said to be *completely irreducible* if $N = \cap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. Every submodule of M is an intersection of completely irreducible submodules of M. Thus, the intersection of all completely irreducible submodules of M is zero (see [11]).

Proposition 2.2. *For any right R-module M, the following are equivalent:*

(1) *M is a semisecond module;*

(2) *Every nonzero quotient of M is a semisecond module;*

(3) *For each proper completely irreducible submodule L of M, $ann_R(M/L)$ is a semiprime ideal of R;*

(4) *For any ideal I of R, $MI = MI^2$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1). Let N be a proper submodule of M. Since every submodule of M is an intersection of completely irreducible submodules of M, we set $N = \cap_{L\in T} L$, where T is a set of completely irreducible submodules of M. Suppose $a \in R$ and $aRa \subseteq \text{ann}_R(M/N) = \text{ann}_R(M/\cap_{L\in T} L)$. Then $MaRa \subseteq L$ for each $L \in T$. By (3), $Ma \subseteq L$ for each $L \in T$. Thus $Ma \subseteq \cap_{L\in T} L$ and so $a \in \text{ann}_R(M/N)$.

(1) $\Rightarrow$ (4). Let I be an ideal of R. If $MI^2 = M$, then $MI^2 = MI = M$. Thus we assume that MI^2 is a proper submodule of M. Then $\text{ann}_R(M/MI^2)$ is a semiprime ideal of R and since $I^2 \subseteq \text{ann}_R(M/MI^2)$, we have $I \subseteq \text{ann}_R(M/MI^2)$, i.e., $MI = MI^2$.

(4) $\Rightarrow$ (1). Let N be a proper submodule of M and $I^2 \subseteq \text{ann}_R(M/N)$. Then $MI^2 \subseteq N$ and by (4), $MI \subseteq N$. Thus $I \subseteq \text{ann}_R(M/N)$ and so $\text{ann}_R(M/N)$ is a semiprime ideal of R, as desired. □

Example 2.3. *Every sum of the second submodules of a right R-module M is a semisecond submodule of M. To see this, let $\{N_i\}_{i\in I}$ be a family of second submodules of M and $K \subsetneqq \sum_{i\in I} N_i$. We claim that $ann_R(\frac{\sum_{i\in I} N_i}{K})$ is a semiprime ideal of R. Suppose that $a \in R$ and $aRa \subseteq ann_R(\frac{\sum_{i\in I} N_i}{K})$. Then $(\sum_{i\in I} N_i)aRa \subseteq K$ and for any $i \in I$, $N_i a = (N_i RaR)a = N_i aRa \subseteq K$ (since N_i is second). Thus $(\sum_{i\in I} N_i)a = \sum_{i\in I} N_i a \subseteq K$ and so $a \in ann_R(\frac{\sum_{i\in I} N_i}{K})$.*

In [13], I.G. Macdonald introduced the notion of secondary modules. Let M be a module over commutative ring R. A nonzero submodule N of M is said to be *secondary* if for each r in R, $Nr = N$ or $Nr^n = 0$, for some integer number n. This notion has been studied by several authors, for example see [3, 10]. In the following we define the secondary submodules when R is an arbitrary ring.

Definition 2.1. Let M be a nonzero right R-module.
(1) A nonzero submodule N of M is called a *secondary submodule* of M if for each ideal I of R, $NI = N$ or $NI^n = 0$, for some integer number n.
(2) A proper submodule N of M is called a *primary submodule* of M if for any submodule K of M and any ideal I of R, $KI \subseteq N$ implies that $K \subseteq N$ or $MI^n \subseteq N$ for some integer number n.

It is clear that every second submodule is a secondary submodule. But the converse is not true in general. Because for $n \geq 3$, in $\mathbb{Z}_{2^n}$ as a $\mathbb{Z}$-module, the submodule $2\mathbb{Z}_{2^n}$ is secondary while is not second.

Proposition 2.3. *Let N be a submodule of a right R-module M. Then we have the following.*
(1) *If N is a secondary and semisecond submodule of M, then N is a weakly second submodule of M;*
(2) *If N is a primary and semiprime submodule of M, then N is a weakly prime submodule of M;*
(3) *If there exist maximal ideals $m_1, \ldots, m_n$ of R such that $m_1 \cap \ldots \cap m_n \subseteq ann_R(N)$ and $N \neq 0$, then N is a semisecond submodule of M.*

Proof. (1). Let I and J be two ideals of R. If $NI = N$, then $NIJ = NJ$. If $NI \neq N$, then $NI^n = 0$, for some integer number n. Now since N is semisecond, $NI = 0$ and so $NI = NIJ = 0$, as desired.

(2). Suppose that $KIJ \subseteq N$, where K is a submodule of M and I,J are two ideals of R. Then $(KI)J \subseteq N$ and since N is primary, $KI \subseteq N$ or $MJ^n \subseteq N$ for some $n \in \mathbb{N}$. This implies that $KI \subseteq N$ or $MJ \subseteq N$ because N is semiprime. Thus $KI \subseteq N$ or $KJ \subseteq N$, as desired.

(3). Let I be an ideal of R and r be a nonzero element of I. We show that $Nr \subseteq NI^2$. After a suitable rearrangement on $m_1, \ldots, m_n$, there can be found i $(0 \leq i \leq n-1)$ such that $r \in m_1 \cap \ldots \cap m_i$ and $r \notin m_{i+1} \cup \ldots \cup m_n$. Thus $R = m_j + RrR$ for $i+1 \leq j \leq n$ and so $1 = x_j + \sum_l r_{jl} r s_{jl}$ for some $x_j \in m_j$, $r_{jl}, s_{jl} \in R$ and $i+1 \leq j \leq n$. Therefore there exists $a \in I$ such that $1 = x_{i+1}x_{i+2}\ldots x_n + a$ and hence $r = x_{i+1}x_{i+2}\ldots x_n r + ar$. Since $x_{i+1}x_{i+2}\ldots x_n r \in \mathrm{ann}_R(N)$, we have $Nr \subseteq Nx_{i+1}x_{i+2}\ldots x_n r + Nar \subseteq NI^2$. □

Let M be a right R-module. By a maximal semisecond submodule of M, we mean a semisecond submodule L of M such that L is not properly contained in another semisecond submodule of M. By applying Zorn's Lemma, it is easy to see that each semisecond submodule of M is contained in a maximal semisecond submodule of M.

Theorem 2.1. *Let M be an Artinian right R-module. Then M has only a finite number of maximal semisecond submodules.*

Proof. Suppose that the result is false. Let Σ denote the collection of nonzero submodules N of M such that N has an infinite number of maximal semisecond submodules. The collection Σ is nonempty because $M \in \Sigma$ and hence has a minimal element, K say. Clearly, K is not a semisecond submodule of M. Thus there exists an ideal I of R such that $KI \neq KI^2$. Let V be a maximal semisecond submodule of M contained in K. Then $V \subseteq (KI^2 :_K I) \subset K$. By the choice of K, the module $(KI^2 :_K I)$ has only finitely many maximal semisecond submodules. Therefore there is only a finite number of possibilities for the module K, which is a desired contradiction. □

Theorem 2.2. *For any R-module M, the following statements are equivalent:*

(1) *Every nonzero submodule of M is semisecond;*

(2) *For each ideal I of R and each submodule N of M, $(N :_M I) = (N :_M I^2)$;*

(2′) *For each ideal I of R and each completely irreducible submodule L of M, $(L :_M I) = (L :_M I^2)$;*

(3) *Every proper submodule of M is semiprime.*

Proof. (1) ⇒ (2). Let I be an ideal of R and N be a submodule of M. Clearly $(N :_M I) \subseteq (N :_M I^2)$. Now suppose that $0 \neq x \in (N :_M I^2)$. Then $xI = xRI = xRI^2 = xI^2 \subseteq N$ because xR is semisecond. Thus $(N :_M I) = (N :_M I^2)$.

(2) ⇒ (2′) is trivial.

(2′) ⇒ (2). Let I be an ideal of R and N be a submodule of M. Then $N = \cap_{\alpha \in A} L_\alpha$, for some completely irreducible submodules L_α of M. Thus $(N :_M I) = (\cap_{\alpha \in A} L_\alpha :_M I) = \cap_{\alpha \in A}(L_\alpha :_M I) = \cap_{\alpha \in A}(L_\alpha :_M I^2) = (\cap_{\alpha \in A} L_\alpha :_M I^2) = (N :_M I^2)$.

(2) ⇒ (3). Let N be a proper submodule of M. Suppose $I^2 \subseteq \mathrm{ann}_R(K/N)$, where I is an ideal of R and $N \subset K \leq M$ is a submodule of M. Then $KI^2 \subseteq N$ and so $K \subseteq (N :_M I^2) = (N :_M I)$. Thus $KI \subseteq N$ and hence $I \subseteq \mathrm{ann}_R(K/N)$.

(3) ⇒ (1). Let I be an ideal of R and N be a nonzero submodule of M. Clearly $NI^2 \subseteq NI$. Since $I^2 \subseteq \mathrm{ann}_R(N/NI^2)$ and $\mathrm{ann}_R(N/NI^2)$ is semiprime, $I \subseteq \mathrm{ann}_R(N/NI^2)$ and so $NI \subseteq NI^2$, as desired. □

Definition 2.2. A nonzero submodule S of an R-module M is called *strongly second* if for every two submodules L_1 and L_2 of M and nonzero ideal I of R, $S \subseteq (L_1 :_M I\mathrm{ann}_R(L_2 \cap S))$ implies that $SI \subseteq L_1$ or $S \subseteq L_2$. Also we say that S is *strongly semisecond* if for every submodule L of M and nonzero ideal I of R, $S \subseteq (L :_M I\mathrm{ann}_R(L \cap S))$ implies that $SI \subseteq L$.

We note that every submodule of an R-module M is an intersection of completely irreducible submodules of M. Thus it is easy to see that a nonzero submodule S

is strongly second if for every two completely irreducible submodules L_1 and L_2 of M and nonzero ideal I of R, $S \subseteq (L_1 :_M I\text{ann}_R(L_2 \cap S))$ implies that $SI \subseteq L_1$ or $S \subseteq L_2$.

Proposition 2.4. *Let M be a right R-module. Then*
(1) *Every strongly second submodule of M is second;*
(2) *Every minimal submodule of M is strongly second.*

Proof. (1). Suppose that S is a strongly second submodule of M which is not second. Then there is a proper submodule L_1 of S such that $\text{ann}_R(S) \subset \text{ann}_R(S/L_1)$. Let $I = \text{ann}_R(S/L_1)$ and so $SI \subseteq L_1$ and $SI \neq 0$. Then there exists a completely irreducible submodule L_2 of M such that $SI \nsubseteq L_2$. Now $S \subseteq (L_2 :_M I\text{ann}_R(L_1 \cap S))$. But $SI \nsubseteq L_2$ and $S \nsubseteq L_1$, a contradiction.
(2). Suppose that S is a minimal submodule of M and L_1, L_2 are two submodules of M with $S \subseteq (L_1 :_M I\text{ann}_R(L_2 \cap S))$, where I is a nonzero ideal of R. If $S \nsubseteq L_2$, then $S \cap L_2 = 0$. Thus $S \subseteq (L_1 :_M I\text{ann}_R(L_2 \cap S)) = (L_1 :_M I)$ and so $SI \subseteq L_1$, as desired. □

The following example shows that a second submodule need not be a strongly second submodule.

Example 2.4. *Set $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ and $N =< \frac{1}{p} + \mathbb{Z} >$ where p is a prime number. Then $N \oplus N$ is a second submodule of the $\mathbb{Z}$-module M, but it is not a strongly second submodule of M. We note that $N \oplus N \nsubseteq N \oplus (0)$, but $N \oplus N \subseteq (N \oplus (0) :_M \text{ann}_{\mathbb{Z}}((N \oplus N) \cap (N \oplus (0))))$.*

Proposition 2.5. *Let V be a vector space over a division ring F and W be a subspace of V. Then W is a minimal subspace of V if and only if W is a strongly second subspace of V.*

Proof. By Proposition 2.4, every minimal subspace is strongly second. Conversely, suppose that W is a strongly second subspace of V which is not a minimal subspace of V. Then there exists a completely irreducible submodule L of V such that $L \cap W \neq 0$ and $W \nsubseteq L$. Therefore for every completely irreducible submodule L_1 of V, we have

$$W \subseteq V = (L_1 :_V 0) = (L_1 :_V \text{ann}_F(W \cap L)).$$

It follows that $W \subseteq L_1$, and hence $W = 0$, a contradiction. □

Let N be a nonzero submodule of an R-module M. We define the strongly second socle of N as the sum of all strongly second submodules of M contained in N and denoted by $S.soc(N)$. If there is no strongly second submodule contained in N, then we put $S.soc(N) = 0$. A family $\{N_i\}_{i \in I}$ of submodules of an R-module M is said to be an inverse family of submodules of M if the intersection of two of its submodules again contains a module in $\{N_i\}_{i \in I}$. Also M satisfies the property $AB5^*$ if for every submodule K of M and every inverse family $\{N_i\}_{i \in I}$ of submodules of M, $K + \cap_{i \in I} N_i = \cap_{i \in I}(K + N_i)$.

Theorem 2.3. *Let M be an R-module which satisfies the property $AB5^*$, and N be a nonzero submodule of M. If N is strongly semisecond, then $N = S.soc(N)$.*

Proof. It is enough to show that $N \subseteq S.soc(N)$. Let L be a completely irreducible submodule of M with $N \nsubseteq L$. We define the set $T = \{L_0, L_1, \dots\}$ of completely irreducible submodules of M inductively as follows:

$$L_0 = L,\ N \nsubseteq L_i,\ (L_i :_M \text{ann}_R(L_i \cap N)) \subseteq L_{i+1},\ i \in \mathbb{N}.$$

Set

$$\Omega = \{K : K \text{ is a submodule of } N \text{ and } K \nsubseteq L_i, \text{ for each } L_i \in T\}.$$

$\Omega \neq \emptyset$ because $N \in \Omega$. By the property $AB5^*$ and Zorn's Lemma, Ω has a minimal element, S say. We claim that S is a strongly second submodule of M. To see this, suppose that H_1 and H_2 are two completely irreducible submodules of M and I is a nonzero ideal of R with $SI \nsubseteq H_2$ and $S \nsubseteq H_1$, but $S \subseteq (H_2 :_M I \operatorname{ann}_R(H_1 \cap S))$. By the minimality of S, $S \cap H_1 \subseteq L_i$ and $S \cap H_2 \subseteq L_j$ for some L_i and L_j belong to T. Now $S \cap H_1 \subseteq L_i \cap N$ implies that $S \subseteq (H_2 :_M I \operatorname{ann}_R(L_i \cap N))$. If $i \leq j$, then $S \subseteq (L_j :_M I \operatorname{ann}_R(L_j \cap N)) \subseteq L_{j+1}$, a contradiction. If $j \leq i$, then $S \subseteq (L_i :_M I \operatorname{ann}_R(L_i \cap N)) \subseteq L_{i+1}$, which is again a contradiction. Therefore S is strongly second. Now since $S \nsubseteq L$, this implies that $S.soc(N) \nsubseteq L$, as desired. □

3. Fully weakly second modules

A ring R is called a *fully prime ring* if each proper ideal of R is a prime ideal. Also, an R-module M is called a *fully (weakly) prime module* if $M \neq 0$ and each proper submodule of M is a (weakly) prime submodule. On the other hand, an R-module M is called a *fully (weakly) second module* if $M \neq 0$ and each nonzero submodule of M is a (weakly) second submodule.

In the following theorem, fully weakly second modules are characterized.

Theorem 3.1. *Let M be a nonzero R-module. Then M is fully weakly second if and only if $(K :_M I)$ and $(K :_M J)$ are compatible and $(K :_M I) = (K :_M I^2)$ for every submodule K of M and every two ideals I, J of R.*

Proof. Suppose that M is fully weakly second. Let K be a submodule of M and I, J be two ideals of R. Clearly $(K :_M I) \subseteq (K :_M I^2)$. If $(K :_M I^2) = 0$, there is no thing to prove. Thus we assume $(K :_M I^2) \neq 0$. Then $(K :_M I^2)I^2 \subseteq K$ implies that $(K :_M I^2)I \subseteq K$ because $(K :_M I^2)$ is a weakly second submodule of M. Therefore $(K :_M I) = (K :_M I^2)$. Now we suppose that $(K :_M I) \neq 0$ and $(K :_M J) \neq 0$.

We note that $(K :_M I)IJ \subseteq K$ and $(K :_M J)IJ \subseteq K$. Thus $((K :_M I) + (K :_M J))IJ \subseteq (K :_M I)IJ+(K :_M J)IJ \subseteq K$. Since $(K :_M I)+(K :_M J)$ is weakly second, $((K :_M I)+(K :_M J))I \subseteq K$ or $((K :_M I)+(K :_M J))J \subseteq K$. Hence $(K :_M J)I \subseteq K$ or $(K :_M I)J \subseteq K$ and so $(K :_M J) \subseteq (K :_M I)$ or $(K :_M I) \subseteq (K :_M J)$.
Conversely, suppose that N is a nonzero submodule of M and $NIJ \subseteq K$, where I and J are two ideals of R and K is a submodule of M. By hypothesis, $(K :_M I) \subseteq (K :_M J)$ or $(K :_M J) \subseteq (K :_M I)$. If $(K :_M J) \subseteq (K :_M I)$, then $NIJ \subseteq K$ implies that $NI \subseteq (K :_M I)$. Therefore $NI^2 \subseteq K$ and so $N \subseteq (K :_M I^2) = (K :_M I)$. Thus $NI \subseteq K$. Now assume that $(K :_M I) \subseteq (K :_M J)$. Since $NIJ \subseteq K$, we have $N \subseteq (K :_M (JI)^2) = (K :_M JI)$ and so $NJI \subseteq K$. Thus $NJ \subseteq (K :_M J)$ because $(K :_M I) \subseteq (K :_M J)$. This implies that $NJ^2 \subseteq K$. Then $N \subseteq (K :_M J^2) = (K :_M J)$ and hence $NJ \subseteq K$. Thus N is weakly second. □

Lemma 3.1. *Let R be a ring. An R-module M is fully weakly prime if and only if M is fully weakly second.*

Proof. First suppose that M is fully weakly prime and N is a nonzero submodule of M. Let L be a proper submodule of N. Then L is a weakly prime submodule of M, i.e., M/L is a weakly prime module. Thus $\text{ann}_R(N/L)$ is a prime ideal and so N is a weakly second submodule. Conversely, suppose that M is fully weakly second and N is a proper submodule of M. Let $N \subset K$ be a submodule of M. Then K is a weakly second submodule and hence $\text{ann}_R(K/N)$ is a prime ideal. Thus M/N is a weakly prime module, i.e., N is a weakly prime submodule of M. □

Corollary 3.1. *Let R be a ring and M be an R-module. Then M is a fully weakly second module if and only if for each submodule $K \subseteq M$ and each ideal I of R, $KI = KI^2$, and also for any two ideals A and B of R, KA and KB are comparable.*

Proof. By Lemma 3.1, and [5, Proposition 4.4]. □

Recall that a module M is *semisimple* if M is a direct sum of a family of simple submodules. Also M is called *homogeneous semisimple* if M is a direct sum of a family of pairwise isomorphic simple submodules. If the ring R is considered as right R-module, we use the notation R_R. Clearly, if R is a fully prime ring, then each nonzero R-module is weakly second module. Thus we have the following result which is a characterization of rings whose all nonzero modules are weakly second.

Proposition 3.1. *The following statements are equivalent:*

(1) *All nonzero right R-modules are weakly second;*

(2) *The R-module R_R is weakly second;*

(3) *R is a fully prime ring.*

Proof. Clear. □

Proposition 3.2. *The following statements are equivalent:*

(1) *R_R is a second R-module;*

(2) *All nonzero right R-modules are second;*

(3) *All nonzero right ideals of R are second;*

(4) *R is a simple ring.*

Proof. Clear. □

Corollary 3.2. *Let R be a ring. Then all nonzero right R-modules are prime if and only if all nonzero right R-modules are second.*

Proof. This is immediate from the above proposition. □

We conclude the paper with the following interesting result.

Theorem 3.2. *Let R be a commutative ring and M be a nonzero R-module. Then the following statements are equivalent:*

(1) *M is a fully second module;*

(2) *M is a fully weakly second module;*

(3) *Each nonzero cyclic submodule of M is a weakly second module;*

(4) *M is a homogeneous semisimple module.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4). Suppose that x is a nonzero element of M. Then $xR \cong R/\text{ann}_R(x)$ and since xR is a weakly second R-module, so is $R/\text{ann}_R(x)$. Let $\overline{A} = A/\text{ann}_R(x)$ be a proper ideal of $\overline{R} = R/\text{ann}_R(x)$. Then $\overline{R}/\overline{A} \cong R/A$ and so $\text{ann}_R(\overline{R}/\overline{A}) = \text{ann}_R(R/A) = A$. Since $\overline{R}$ is weakly second, A is a prime ideal of R and hence $\overline{A}$ is a prime ideal of $\overline{R}$. Thus every proper ideal of $\overline{R}$ is prime and so $\overline{R}$ is a field. It follows that $\text{ann}_R(x)$ is a maximal ideal of R and so xR is a simple R-module. Now suppose that $0 \neq y \in M$ such that $x \neq y$. Then $\text{ann}_R(x) \cap \text{ann}_R(y) \subseteq \text{ann}_R(x-y)$ implies that $\text{ann}_R(x) \subseteq \text{ann}_R(x-y)$ or $\text{ann}_R(y) \subseteq \text{ann}_R(x-y)$ and so $\text{ann}_R(x) = \text{ann}_R(x-y)$ or $\text{ann}_R(y) = \text{ann}_R(x-y)$. Thus $(x-y)\,\text{ann}_R(x) = 0$ or $(x-y)\,\text{ann}_R(y) = 0$ and hence $y\,\text{ann}_R(x) = 0$ or $x\,\text{ann}_R(y) = 0$. This shows that $\text{ann}_R(x) \subseteq \text{ann}_R(y)$ or $\text{ann}_R(y) \subseteq \text{ann}_R(x)$. Thus $\text{ann}_R(x) = \text{ann}_R(y)$. Since $M = \sum_{x \in M} xR$, $\text{ann}_R(M) = \text{ann}_R(x)$ for each nonzero element x of M. Therefore M is a homogeneous semisimple R-module.

(4) $\Rightarrow$ (1). Clearly, every homogeneous semisimple module is a second module. Also by [1, Proposition 9.4], all submodules and all factor modules of a homogeneous semisimple module are homogeneous semisimple. Thus (1) is obtained. □

Acknowledgement

The author would like to thank the referees for helpful comments that improved this paper.

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