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WEIGHTED ESTIMATES FOR TWO KINDS OF TOEPLITZ OPERATORS

ZHE CUI (1) AND YAN LIN (2)

ABSTRACT. In this paper, we establish the boundedness of a class of Toeplitz operators related to strongly singular Calderón-Zygmund operators and weighted BMO functions on weighted Morrey spaces. Moreover, the boundedness of another kind of Toeplitz operators related to strongly singular Calderón-Zygmund operators and weighted Lipschitz functions on weighted Morrey spaces is also obtained.

1. INTRODUCTION

As an important part of modern harmonic analysis, Calderón-Zygmund operator has been devoted to studying its boundedness in different function spaces and has achieved brilliant achievements. As the research goes on, Morrey spaces and weighted Morrey spaces have been proposed. In order to better study the local properties of the solutions of second order elliptic partial differential equations, Morrey [17] put forward the classical Morrey space. We can learn the properties and applications of Morrey spaces in [4, 20] and so on.

Adams [1] obtained the boundedness of Riesz potentials on Morrey spaces. Liu [16] proved the weighetd boundedness of Marcinkiewicz operators and Littlewood-Paley operators. Fu and Lu [6] established the boundedness of weighted Hardy operators

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and their commutators on Morrey spaces. Feng [5] proved the boundedness of Toeplitz operators on Morrey spaces.

Then in 2009, Komori and Shirai [10] defined the weighted Morrey spaces and established the boundedness of the fractional integral operator, the Hardy-Littlewood maximal operator on these weighted spaces. In 2012, Wang [22] studied the boundedness of commutators generated by classical Calderón-Zygmund operators and weighted BMO functions on weighted Morrey spaces. In 2013, the authors in [23] obtained the boundedness of the Toeplitz operator associated with the singular integral operator with the non-smooth kernel on the weighted Morrey spaces.

Let b be a locally integrable function on $\mathbb{R}^n$ and let T be a Calderón-Zygmund singular integral operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

Krantz and Li [11] discussed the Toeplitz operator in 2001. The commutator generated by the Calderón-Zygmund operator and a locally integrable function b can be regarded as a special case of the Toeplitz operator $T_b = \sum_{j=1}^{m} T_{j,1} M_b T_{j,2}$, where $T_{j,1}$ and $T_{j,2}$ are the Calderón-Zygmund operators or $\pm I$ (I is the identity operator), $M_b f(x) = b(x)f(x)$.

Lin and Lu [13] obtained the boundedness of the commutators of strongly singular Calderón-Zygmund operators on Hardy-type spaces. Morever, Lin and Lu [14] proved the boundedness of this kind of commutators on Morrey spaces. Lin and Sun [15] proved the boundedness of the commutators generated by strongly singular Calderón-Zygmund operators and weighted BMO functions on weighted Morrey spaces. Lin, Liu and Cong [12] established the boundedness of commutators generated by weighted Lipschitz functions and strongly singular Calderón-Zygmund operators on weighted Morrey spaces.

Inspired by the above results, in this paper we are interested in the boundedness of a class of Toeplitz operators related to strongly singular Calderón-Zygmund operators and weighted BMO functions or weighted Lipschitz functions on weighted Morrey spaces.

Before stating our main results, let us first recall some necessary definitions and notations.

Definition 1.1. ([2]) Let $\mathcal{S}$ be the space of all Schwartz functions on $\mathbb{R}^n$, $\mathcal{S}'$ its dual space, the class of all tempered distribution on $\mathbb{R}^n$. Let T :$\mathcal{S} \to \mathcal{S}'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following three conditions are satisfied.

(1) T can be extended into a continuous operator from $L^2(\mathbb{R}^n)$ into itself.

(2) There exists a function $K(x,y)$ continuous away from the diagonal $\{(x,y) : x \neq y\}$ such that

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leq C\frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\alpha}}}$$

if $2|y-z|^\alpha \leq |x-z|$ and $0 < \delta \leq 1, 0 < \alpha < 1$. And

$$\langle Tf, g\rangle = \int\int K(x,y)f(y)g(x)dydx,$$

for $f, g \in \mathcal{S}$ with disjoint supports.

(3) For some $\frac{n(1-\alpha)}{2} \leq \beta < \frac{n}{2}$, both T and its conjugate operator T^* can be extended into continuous operators from L^q to L^2, where $\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}$.

Definition 1.2. ([18]) A non-negative measurable function ω is said to be in the Muckenhoupt class A_p with $1 < p < \infty$ if for every cube Q in $\mathbb{R}^n$, there exists a positive constant C independent of Q such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\right)^{p-1} \leq C,$$

where Q denotes a cube in $\mathbb{R}^n$ with the side parallel to the coordinate axes and $1/p + 1/p' = 1$.

When $p = 1$, a non-negative measurable function ω is said to belong to A_1, if there exists a constant $C > 0$ such that for any cube Q,

$$\frac{1}{|Q|}\int_Q \omega(y)dy \leq C\omega(x), \quad a.e.\, x \in Q.$$

It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$, and $\omega \in A_q$ for some $1 < q < p$.

Definition 1.3. ([10]) Let $1 \leq p < \infty, 0 < k < 1$, and ω be a weighted function. Then the weighted Morrey space $L^{p,k}(\omega)$ is defined by

$$L^{p,k}(\omega) = \{f \in L^p_{loc}(\omega) : \|f\|_{L^{p,k}(\omega)} < \infty\},$$

where $f \in L^p_{loc}(\omega)$ if and only if for any compact set K there is $\left(\int_K |f(x)|^p\omega(x)dx\right)^{\frac{1}{p}} < \infty$,

$$\|f\|_{L^{p,k}(\omega)} = \sup_Q \left(\frac{1}{\omega(Q)^k}\int_Q |f(x)|^p\omega(x)dx\right)^{\frac{1}{p}},$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Definition 1.4. ([10]) Let $1 \leq p < \infty, 0 < k < 1$. Then for two weighed functions u and v, the weighted Morrey space $L^{p,k}(u, v)$ is defined by

$$L^{p,k}(u, v) = \{f \in L^p_{loc}(u) : \|f\|_{L^{p,k}(u,v)} < \infty\},$$

where

$$\|f\|_{L^{p,k}(u,v)} = \sup_Q \left(\frac{1}{v(Q)^k}\int_Q |f(x)|^p u(x)dx\right)^{\frac{1}{p}},$$

and the supremum is taken over all cubes Q in $\mathbb{R}^n$.

Definition 1.5. ([7]) Let $1 \leq p < \infty$ and ω be a weighted function. A locally integrable function b is said to be in the weighted BMO space $BMO_p(\omega)$, if

$$\|b\|_{BMO_p(\omega)} = \sup_Q \left(\frac{1}{\omega(Q)} \int_Q |b(x) - b_Q|^p \omega(x)^{1-p} dx \right)^{\frac{1}{p}} < \infty,$$

where $b_Q = \frac{1}{|Q|} \int_Q b(y)dy$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

Moreover, we denote simply $BMO(\omega)$ when $p = 1$.

Definition 1.6. ([7]) Let $1 \leq p < \infty, 0 < \beta_0 < 1$ and ω be a weighted function. A locally integrable function b is said to be in the weighted Lipschitz space $Lip^p_{\beta_0}(\omega)$, if

$$\|b\|_{Lip^p_{\beta_0}(\omega)} = \sup_Q \frac{1}{\omega(Q)^{\frac{\beta_0}{n}}} \left(\frac{1}{\omega(Q)} \int_Q |b(x) - b_Q|^p \omega(x)^{1-p} du \right)^{\frac{1}{p}} < \infty,$$

where $b_Q = \frac{1}{|Q|} \int_Q b(y)dy$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Moreover, we denote simply by $Lip_{\beta_0}(\omega)$ when $p = 1$.

Definition 1.7. ([9, 21]) The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

We set $M_s(f) = M(|f|^s)^{\frac{1}{s}}$, where $0 < s < \infty$.

The sharp maximal operator $M^\sharp$ is defined by

$$\begin{aligned} M^\sharp(f)(x) &= \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy \\ &\sim \sup_{Q \ni x} \inf_{a \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - a|dy, \end{aligned}$$

where $f_Q = \frac{1}{|Q|} \int_Q f(x)dx$. For $0 < t < 1$, we define the t-sharp maximal operator $M^\sharp_t(f) = M^\sharp(|f|^t)^{\frac{1}{t}}$.

Let ω be a weight. The weighted maximal operator M_ω is defined by

$$M_\omega(f)(x) = \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f(y)|\omega(y)dy.$$

We also set $M_{s,\omega}(f) = M_\omega(|f|^s)^{\frac{1}{s}}$, where $0 < s < \infty$.

Definition 1.8. ([8]) For $0 < \beta_0 < n, 1 \leq r < \infty$, the fractional maximal operator $M_{\beta_0,r}$ is defined by

$$M_{\beta_0,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-\frac{\beta_0 r}{n}}} \int_Q |f(y)|^r dy \right)^{\frac{1}{r}},$$

where the above supremum is taken over all cubes Q containing x.

Definition 1.9. ([8]) For $0 < \beta_0 < n, 1 \leq r < \infty$, and a weight ω, the weighted fractional maximal operator $M_{\beta_0,r,\omega}$ is defined by

$$M_{\beta_0,r,\omega}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{\omega(Q)^{1-\frac{\beta_0 r}{n}}} \int_Q |f(y)|^r \omega(y) dy \right)^{\frac{1}{r}},$$

where the above supremum is taken over all cubes Q containing x.

Definition 1.10. ([8]) A weighted function ω belongs to the reverse Hölder class RH_r, if there exists two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^r dx \right)^{\frac{1}{r}} \leq C \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right)$$

holds for every cube Q in $\mathbb{R}^n$. Denote r_ω the critical index of ω for the reverse Hölder condition. That is $r_\omega = \sup\{r > 1 : \omega \in RH_r\}$.

It is well known that if $\omega \in A_p$ with $1 \leq p < \infty$, then there exists a $r > 1$ such that $\omega \in RH_r$. It follows directly from Hölder inequality that $\omega \in RH_r$ implies $\omega \in RH_s$ for all $1 < s < r$.

2. Main Results

Now we state our main results as follows.

Theorem 2.1. *Let α, β, δ be the same as those in Definition 1.1, $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2} (n \geq 2)$, $T_{j,1}$ be the strongly singular Calderón-Zygmund operators or*

$\pm I$, $T_{j,2}$ be the bounded linear operators on $L^{p,k}(\omega)$. When $f \in C_c^\infty(\mathbb{R}^n)$, $T_1(f) = 0$. Suppose $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty, 0 < k < 1$, and $\omega \in A_1 \cap RH_r$ with $r > \frac{(n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$. If $b \in BMO(\omega)$, then T_b is bounded from $L^{p,k}(\omega)$ to $L^{p,k}(\omega^{1-p}, \omega)$.

Theorem 2.2. *Let α, β, δ be the same as those in Definition 1.1, $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}(n \geq 2)$, $T_{j,1}$ be the strongly singular Calderón-Zygmund operators or $\pm I$, $T_{j,2}$ be the bounded linear operators on $L^{p,k}(\omega)$. When $f \in C_c^\infty(\mathbb{R}^n)$, $T_1(f) = 0$. Suppose $0 < \beta_0 < 1, \frac{n(1-\alpha)+2\beta}{2\beta} < p < \frac{n}{\beta_0}, \frac{1}{q} = \frac{1}{p} - \frac{\beta_0}{n}, 0 < k < \frac{p}{q}$, and $\omega \in A_1$ with $r_\omega > \frac{(n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$. If $b \in Lip_{\beta_0}(\omega)$, then T_b is bounded from $L^{p,k}(\omega)$ to $L^{q,\frac{kq}{p}}(\omega^{1-p}, \omega)$.*

If we consider the special cases, then the Toeplitz operator comes back to the commutator, where the boundedness of the strongly singular Calderón-Zygmund operator on $L^{p,k}(\omega)$ can be seen in Lemma 3.2 in the next section. Thus we can obtain the following results as corollaries.

Corollary 2.1. *Let T be a strongly singular Calderón-Zygmund operator, α, β, δ be given as in Definition 1.1 and $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}(n \geq 2)$. Suppose $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty, 0 < k < 1$, and $\omega \in A_1 \cap RH_r$ with $r > \frac{(n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$. If $b \in BMO(\omega)$, then $[b, T]$ is bounded from $L^{p,k}(\omega)$ to $L^{p,k}(\omega^{1-p}, \omega)$.*

Corollary 2.2. *Let T be a strongly singular Calderón-Zygmund operator, α, β, δ be given as in Definition 1.1 and $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}(n \geq 2)$. Suppose $0 < \beta_0 < 1, \frac{n(1-\alpha)+2\beta}{2\beta} < p < \frac{n}{\beta_0}, \frac{1}{q} = \frac{1}{p} - \frac{\beta_0}{n}, 0 < k < \frac{p}{q}$, and $\omega \in A_1$ with $r_\omega > \frac{(n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$. If $b \in Lip_{\beta_0}(\omega)$, then $[b, T]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,\frac{kq}{p}}(\omega^{1-p}, \omega)$.*

Remark 2.1. *As matter of fact, the results of Corollary 2.1 and 2.2 have been obtained in* [15, 12]. *Thus Theorem 2.1 and Theorem 2.2 can be regarded as generalizations of the corresponding results in* [15, 12]. *And from this point of view, the ranges of the index in Theorem 2.1 and Theorem 2.2 are reasonable.*

3. Preliminaries

In this section, we will introduce some requisite lemmas used in the proof of our main results in Section 4.

Firstly, we need the boundedness of the strongly singular Calderón-Zygmund operators on Lebesgue spaces and weighted Morrey spaces, respectively.

Lemma 3.1. ([2, 3]) *If T is a strongly singular Calderón-Zygmund operator, then T is the type of weak $(1,1)$ and can be defined to be a continuous operator from L^∞ to BMO.*

Remark 3.1. *By using Lemma 3.1, Definition 1.1 and the interpolation theory, we can deduce two kinds of boundedness properties for the strongly singular Calderón-Zygmund operator T on Lebesgue spaces. T is bounded on L^t for $1 < t < \infty$, and T is bounded from L^μ to L^ν, where $\frac{n(1-\alpha)+2\beta}{2\beta} \le \mu < \infty$, and $0 < \frac{\mu}{\nu} \le \alpha$. In particular, if we restrict $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}$ in (3) of Definition 1.1, then T is bounded from L^μ to L^ν, where $\frac{n(1-\alpha)+2\beta}{2\beta} < \mu < \infty$, and $0 < \frac{\mu}{\nu} < \alpha$.*

Lemma 3.2. ([15]) *Let T be a strongly singular Calderón-Zygmund operator, and α, β, δ be given as in Definition 1.1. If $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty, 0 < k < 1$ and $\omega \in A_{\frac{2\beta p}{n(1-\alpha)+2\beta}}$, then T is bounded on $L^{p,k}(\omega)$.*

Then we need the norm properties for functions on weighted BMO spaces.

Lemma 3.3. ([7, 22]) *Let $\omega \in A_1$. Then for any $1 \le p < \infty$, there exists an absolute constant $C > 0$, such that*

$$\|b\|_{BMO_p(\omega)} \le C\|b\|_{BMO(\omega)}.$$

Lemma 3.4. ([15]) *Let $\omega \in A_1$ and $f \in BMO(\omega)$. Suppose $1 \le p < \infty, x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_{B(x,r_2)}|^p \omega(y)^{1-p} dy\right)^{\frac{1}{p}}$$

$$\leq C\|f\|_{BMO(\omega)}\omega(x)\left(1+\ln|\frac{r_2}{r_1}|\right)\left(\frac{\omega(B(x,r_1))}{|B(x,r_1)|}\right)^{\frac{1}{-p'}}.$$

And the properties of maximal operators on weighted Morrey spaces are also necessary.

Lemma 3.5. ([20]) *Let* $1 < p < \infty, 0 < k < 1$, *and* $\omega \in A_\infty$, *then for any* $1 < s < p$, *we have*

$$\|M_{s,\omega}(f)\|_{L^{p,k}(\omega)} \leq C\|f\|_{L^{p,k}(\omega)}.$$

Lemma 3.6. ([22]) *Let* $0 < t < 1, 1 < p \leq \infty$, *and* $0 < k < 1$. *If* $\mu, \nu \in A_\infty$, *then we have*

$$\|M_t(f)\|_{L^{p,k}(\mu,\nu)} \leq C\|M_t^\sharp(f)\|_{L^{p,k}(\mu,\nu)}$$

for all functions f *such that the left hand side is finite. In particular, when* $\mu = \nu = \omega$ *and* $\omega \in A_\infty$, *we have*

$$\|M_s(f)\|_{L^{p,k}(\omega)} \leq C\|M_t^\sharp(f)\|_{L^{p,k}(\omega)}$$

for all functions f *such that the left hand side is finite.*

Lemma 3.7. *If* $\epsilon > 0$, *then* $\ln x \leq \frac{1}{\epsilon}x^\epsilon$, *for all* $x \geq 1$.

The above result comes from the monotone property of the function
$\varphi(x) = \ln x - \frac{1}{\epsilon}x^\epsilon, x \geq 1$.

Lemma 3.8. ([22]) *Let* $0 < \beta_0 < n, 1 < p < \frac{n}{\beta_0}, \frac{1}{s} = \frac{1}{p} - \frac{\beta_0}{n}, 0 < k < \frac{p}{s}$ *and* $\omega \in A_\infty$. *Then for every* $1 < r < p$, *we have*

$$\|M_{\beta_0,r,\omega}(f)\|_{L^{s,\frac{ks}{p}}(\omega)} \leq C\|f\|_{L^{p,k}(\omega)}.$$

Then we also need the norm properties for functions on weighted Lipschitz spaces as follows.

Lemma 3.9. ([7, 19]) *Let $0 < \beta_0 < 1$, and $\omega \in A_1$. Then for any $1 \le p < \infty$, there exists an absolute constant $C > 0$, such that*

$$\|b\|_{Lip^p_{\beta_0}(\omega)} \le C\|b\|_{Lip_{\beta_0}(\omega)}.$$

Lemma 3.10. ([12]) *Let $0 < \beta_0 < 1$, $\omega \in A_1$, and $f \in Lip_{\beta_0(\omega)}$. Suppose $1 \le p < \infty, x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)} |f(y) - f_B(x,r_2)|^p \omega(y)^{1-p}dy\right)^{\frac{1}{p}}$$
$$\le C\|f\|_{Lip_{\beta_0}(\omega)}\omega(x)(1+\ln|\frac{r_2}{r_1}|)\left(\frac{\omega(B(x,r_1))}{|B(x,r_1)|}\right)^{-\frac{1}{p'}} \max_{i=1,2}\omega(B(x,r_i))^{\frac{\beta_0}{n}}.$$

The main method to obtain our results is to dominate the maximal functions of Toeplitz operators related to strongly singular Calderón-Zygmund operators.

Lemma 3.11. *Let α, β, δ be the same as those in Definition 1.1, $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}(n \ge 2)$, $T_{j,1}$ be the strongly singular Calderón-Zygmund operator or $\pm I$. When $f \in C_c^\infty(\mathbb{R}^n)$, $T_1(f) = 0$. Suppose $0 < t < 1, \frac{n(1-\alpha)+2\beta}{2\beta} < s < \infty$, $\omega \in A_1 \cap RH_r$, where $r > \frac{(n(1-\alpha)+2\beta)(s-1)}{2\beta s - n(1-\alpha)-2\beta}$, and $b \in BMO(\omega)$. Then*

$$M_t^\sharp(T_bf)(x) \le C\|b\|_{BMO(\omega)}\omega(x)\sum_{j=1}^m \left((\|T_{j,1}\|+1)M_{s,\omega}(T_{j,2}f)(x)\right), a.e.\, x \in \mathbb{R}^n.$$

Proof. For any ball $B = B(x, r_B)$ with the center x and radius r_B, there are two cases.

Case 1: $r_B > 1$.

Since $T_b(f) = T_{(b-b_{2B})\chi_{2B}}(f) + T_{(b-b_{2B})\chi_{(2B)^c}}(f), 0 < t < 1$, then we have

$$\left(\frac{1}{|B|}\int_B \Big||T_b(f)(y)|^t - |T_{(b-b_{2B})\chi_{(2B)^c}}(f)(x)|^t\Big|dy\right)^{\frac{1}{t}}$$
$$\le C\left(\frac{1}{|B|}\int_B |T_{(b-b_{2B})\chi_{2B}}(f)(y)|^t dy\right)^{\frac{1}{t}}$$
$$+C\left(\frac{1}{|B|}\int_B |T_{(b-b_{2B})\chi_{(2B)^c}}(f)(y) - T_{(b-b_{2B})\chi_{(2B)^c}}(f)(x)|^t dy\right)^{\frac{1}{t}}$$

$$:= I_1 + I_2.$$

To estimate I_1,

$$I_1 \leq C \sum_{j=1}^{m} \left(\frac{1}{|B|} \int_B |T_{j,1} M_{(b-b_{2B})\chi_{2B}} T_{j,2}(f)(y)|^t dy \right)^{\frac{1}{t}} := C \sum_{j=1}^{m} I_{1j}.$$

If $T_{j,1}$ is the strongly singular Calderón-Zygmund operator, then by Hölder's inequality, Kolmogorov's inequality, and Lemma 3.3, we have

$$\begin{aligned} I_{1j} &\leq C\|T_{j,1}\| |B|^{-\frac{1}{t}} \left(|B|^{1-t} \|(b-b_{2B})\chi_{2B}(T_{j,2}f)\|_1^t \right)^{\frac{1}{t}} \\ &\leq C\|T_{j,1}\| \frac{1}{|B|} \left(\int_{2B} |b(y)-b_{2B}|^{s'} \omega(y)^{1-s'} dy \right)^{\frac{1}{s'}} \left(\int_{2B} |T_{j,2}(f)(y)|^s \omega(y) dy \right)^{\frac{1}{s}} \\ &\leq C\|T_{j,1}\| \|b\|_{BMO_{(\omega)}} \frac{\omega(2B)}{|2B|} \left(\frac{1}{\omega(2B)} \int_{2B} |T_{j,2}(f)(y)|^s \omega(y) dy \right)^{\frac{1}{s}} \\ &\leq C\|T_{j,1}\| \|b\|_{BMO_{(\omega)}} \omega(x) M_{s,\omega}(T_{j,2}f)(x). \end{aligned}$$

If $T_{j,1} = \pm I$, then

$$\begin{aligned} I_{1j} &\leq \frac{1}{|B|} \int_B |b(y)-b_{2B}||T_{j,2}(f)(y)| dy \\ &\leq C \frac{1}{|B|} \left(\int_B |b(y)-b_{2B}|^{s'} \omega(y)^{1-s'} dy \right)^{\frac{1}{s'}} \left(\int_B |T_{j,2}(f)(y)|^s \omega(y) dy \right)^{\frac{1}{s}} \\ &\leq C\|b\|_{BMO_{(\omega)}} \frac{\omega(2B)}{|2B|} \left(\frac{1}{\omega(2B)} \int_{2B} |T_{j,2}(f)(y)|^s \omega(y) dy \right)^{\frac{1}{s}} \\ &\leq C\|b\|_{BMO_{(\omega)}} \omega(x) M_{s,\omega}(T_{j,2}f)(x). \end{aligned}$$

So we get

$$I_1 \leq C\|b\|_{BMO_{(\omega)}} \omega(x) \sum_{j=1}^{m} M_{s,\omega}(T_{j,2}f)(x)(\|T_{j,1}\| + 1).$$

Then we estimate I_2. Since $r_B > 1$ and $2|y-x|^\alpha \leq 2r^\alpha \leq 2r \leq |z-x|$ for any $y \in B$ and $z \in (2B)^c$, by (2) of Definition 1.1, we have

$$I_2 \leq C \frac{1}{|B|} \int_B |T_{(b-b_{2B})\chi_{(2B)^c}}(f)(y) - T_{(b-b_{2B})\chi_{(2B)^c}}(f)(x)| dy$$

$$\begin{aligned}
&\le C\sum_{j=1}^{m}\frac{1}{|B|}\int_B |T_{j,1}M_{(b-b_{2B})\chi_{(2B)^c}}T_{j,2}(f)(y)-T_{j,1}M_{(b-b_{2B})\chi_{(2B)^c}}T_{j,2}(f)(x)|dy\\
&:= C\sum_{j=1}^{m} I_{2j}.
\end{aligned}$$

If $T_{j,1}$ is the strongly singular Calderón-Zygmund operator, we have

$$\begin{aligned}
I_{2j} &\le C\frac{1}{|B|}\int_B\int_{(2B)^c}|K(y,z)-K(x,z)||b(z)-b_{2B}||T_{j,2}(f)(z)|dzdy\\
&\le C\frac{1}{|B|}\int_B\sum_{i=1}^{\infty}\int_{2^{i+1}B\setminus 2^iB}\frac{|y-x|^{\delta}}{|z-x|^{n+\frac{\delta}{\alpha}}}|b(z)-b_{2B}||T_{j,2}(f)(z)|dzdy\\
&\le C\sum_{i=1}^{\infty} r_B^{\delta-\frac{\delta}{\alpha}}\frac{1}{(2^i)^{\frac{\delta}{\alpha}}}\frac{1}{|2^{i+1}B|}\int_{2^{i+1}B}|b(z)-b_{2B}||T_{j,2}(f)(z)|dz.
\end{aligned}$$

Applying Hölder's inequality and Lemma 3.4, we get

$$\begin{aligned}
I_{2j} &\le C\sum_{i=1}^{\infty}(2^i)^{-\frac{\delta}{\alpha}}\frac{1}{|2^{i+1}B|}\left(\int_{2^{i+1}B}|b(z)-b_{2B}|^{s'}\omega(z)^{1-s'}dz\right)^{\frac{1}{s'}}\\
&\quad\times\left(\int_{2^{i+1}B}|T_{j,2}(f)(z)|^s\omega(z)dz\right)^{\frac{1}{s}}\\
&\le C\sum_{i=1}^{\infty}(2^i)^{-\frac{\delta}{\alpha}}\frac{1}{|2^{i+1}B|}|2^{i+1}B|^{\frac{1}{s'}}\left(\int_{2^{i+1}B}|T_{j,2}(f)(z)|^s\omega(z)dz\right)^{\frac{1}{s}}\\
&\quad\times\|b\|_{BMO(\omega)}\omega(x)\left(1+\ln|\frac{2r}{2^{i+1}r}|\right)\left(\frac{\omega(2^{i+1}B)}{|2^{i+1}B|}\right)^{-\frac{1}{s}}\\
&\le C\sum_{i=1}^{\infty}(2^i)^{-\frac{\delta}{\alpha}}i\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x)\\
&\le C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x).
\end{aligned}$$

If $T_{j,1}=\pm I$, then $I_{2j}=0$. We have

$$I_2\le C\|b\|_{BMO(\omega)}\omega(x)\sum_{j=1}^{m}M_{s,\omega}(T_{j,2}f)(x).$$

Case 2: $0<r_B\le 1$.

Since $r > \frac{(n(1-\alpha)+2\beta)(s-1)}{2\beta s-n(1-\alpha)-2\beta}$, that is $\frac{n(1-\alpha)+2\beta}{2\beta} < \frac{rs}{s+r-1}$, there exists an s_0 such that $\frac{n(1-\alpha)+2\beta}{2\beta} < s_0 < \frac{rs}{s+r-1}$. For the index s_0 which we chose, by Remark 3.1 there exists an l_0 such that T is bounded from L^{s_0} to L^{l_0} and $0 < \frac{s_0}{l_0} < \alpha$. Then we can take a θ satisfying $\frac{s_0}{l_0} < \theta < \alpha$.

Let $\widetilde{B} = B(x, r_B^\theta)$. Since $T_b(f) = T_{(b-b_{2B})\chi_{2\widetilde{B}}}(f) + T_{(b-b_{2B})\chi_{(2\widetilde{B})^c}}(f)$, we get

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \left| |T_b(f)(y)|^t - |T_{(b-b_{2B})\chi_{(2\widetilde{B})^c}}(f)(x)|^t \right| dy\right)^{\frac{1}{t}} \\
&\le C\left(\frac{1}{|B|}\int_B |T_{(b-b_{2B})\chi_{2\widetilde{B}}}(f)(y)|^t dy\right)^{\frac{1}{t}} \\
&\quad + C\left(\frac{1}{|B|}\int_B |T_{(b-b_{2B})\chi_{(2\widetilde{B})^c}}(f)(y) - T_{(b-b_{2B})\chi_{(2\widetilde{B})^c}}(f)(x)|^t dy\right)^{\frac{1}{t}} \\
&:= I_3 + I_4.
\end{aligned}$$

To estimate I_3,

$$I_3 \le C\sum_{j=1}^m \frac{1}{|B|}\int_B |T_{j,1}M_{(b-b_{2B})\chi_{2\widetilde{B}}}T_{j,2}(f)(y)|dy := C\sum_{j=1}^m I_{3j}.$$

If $T_{j,1}$ is the strongly singular Calderón-Zygmund operator, we have

$$\begin{aligned}
I_{3j} &\le C|B|^{-\frac{1}{l_0}}\|T_{j,1}\|\left(\int_{2\widetilde{B}} |b(y)-b_{2B}|^{s_0}|T_{j,2}(f)(y)|^{s_0}dy\right)^{\frac{1}{s_0}} \\
&\le C|B|^{-\frac{1}{l_0}}\|T_{j,1}\|\left(\int_{2\widetilde{B}} |b(y)-b_{2B}|^{l}\omega(y)^{-\frac{l}{s}}dy\right)^{\frac{1}{l}} \\
&\quad \times\left(\int_{2\widetilde{B}} |T_{j,2}(f)(y)|^{s}\omega(y)dy\right)^{\frac{1}{s}} \\
&\le C\|T_{j,1}\||B|^{-\frac{1}{l_0}}M_{s,\omega}(T_{j,2}f)(x)\omega(2\widetilde{B})^{\frac{1}{s}}\left(\int_{2\widetilde{B}} |b(y)-b_{2B}|^{l}\omega(y)^{-\frac{l}{s}}dy\right)^{\frac{1}{l}}.
\end{aligned}$$

Let $p_0 = \frac{(r-1)(s-s_0)}{s(s_0-1)}$. Since $s_0 < \frac{rs}{s+r-1}$, we get $1 < p_0 < \infty$. So $p_0' = \frac{(r-1)(s-s_0)}{rs-(s+r-1)s_0}$ and $1 < p_0' < \infty$. Applying Hölder's inequality, Lemma 3.4 and noticing that $r = \frac{lp_0}{s'} - \frac{p_0}{p_0'}$, we get

$$\begin{aligned}
I_{3j} \quad &\leq \quad C\|T_{j,1}\||B|^{-\frac{1}{l_0}} M_{s,\omega}(T_{j,2}f)(x)\omega(2\widetilde{B})^{\frac{1}{s}} \left(\int_{2\widetilde{B}} |b(y)-b_{2B}|^{lp_0'}\omega(y)^{1-lp_0'}dy\right)^{\frac{1}{lp_0'}} \\
&\quad \times \left(\int_{2\widetilde{B}} \omega(y)^{\frac{lp_0}{s'}-\frac{p}{p_0'}}dy\right)^{\frac{1}{lp_0}} \\
&\leq \quad C\|T_{j,1}\||B|^{-\frac{1}{l_0}}\omega(2\widetilde{B})^{\frac{1}{s}} M_{s,\omega}(T_{j,2}f)(x)|2\widetilde{B}|^{\frac{1}{lp_0'}}\|b\|_{BMO(\omega)}\omega(x) \\
&\quad \times \left(1+\left|\ln\frac{r_B^\theta}{r_B}\right|\right)\left(\frac{\omega(2\widetilde{B})}{|2\widetilde{B}|}\right)^{-\frac{1}{(lp_0')'}}\left[\left(\frac{1}{|2\widetilde{B}|}\int_{2\widetilde{B}}\omega(y)^r dy\right)^{\frac{1}{r}}\right]^{\frac{r}{lp_0}} |2\widetilde{B}|^{\frac{1}{lp_0}} \\
&\leq \quad C\|T_{j,1}\|M_{s,\omega}(T_{j,2}f)(x)\omega(2\widetilde{B})^{\frac{1}{s}}|B|^{-\frac{1}{l_0}}|2\widetilde{B}|^{\frac{1}{l}}\|b\|_{BMO(\omega)}\omega(x)\left(1+\left|\ln\frac{r_B^\theta}{r_B}\right|\right) \\
&\quad \times \left(\frac{\omega(2\widetilde{B})}{|2\widetilde{B}|}\right)^{-\frac{1}{(lp_0')'}}\left(\frac{\omega(2\widetilde{B})}{|2\widetilde{B}|}\right)^{\frac{r}{lp_0}} \\
&\leq \quad C\|T_{j,1}\|M_{s,\omega}(T_{j,2}f)(x)\|b\|_{BMO(\omega)}\omega(x)\omega(2\widetilde{B})^{\frac{1}{s}}|B|^{-\frac{1}{l_0}}|2\widetilde{B}|^{\frac{1}{l}} \\
&\quad \times \left(1+(1-\theta)\ln\frac{1}{r_B}\right)\left(\frac{\omega(2\widetilde{B})}{|2\widetilde{B}|}\right)^{-\frac{1}{s}}.
\end{aligned}$$

The inequality $0<\frac{s_0}{l_0}<\theta$ implies that $\varepsilon_1 := n(\frac{\theta}{s_0}-\frac{1}{l_0})>0$. By Lemma 3.7, we have

$$\begin{aligned}
I_{3j} \quad &\leq \quad C\|T_{j,1}\|\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x)|B|^{-\frac{1}{l_0}}|2\widetilde{B}|^{\frac{1}{s}+\frac{1}{l}}(1+\frac{1}{\varepsilon_1}r_B^{-\varepsilon_1}) \\
&\leq \quad C\|T_{j,1}\|\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x)r_B^{n(\frac{\theta}{s_0}-\frac{1}{l_0})-\varepsilon_1} \\
&= \quad C\|T_{j,1}\|\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x).
\end{aligned}$$

If $T_{j,1}=\pm I$, similarly to estimate I_{1j}, then

$$I_{3j} \leq C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x).$$

We have

$$I_3 \leq C\|b\|_{BMO(\omega)}\omega(x)\sum_{j=1}^m(\|T_{j,1}\|+1)M_{s,\omega}(T_{j,2}f)(x).$$

To estimate I_4,

$$\begin{aligned} I_4 &\leq C\sum_{j=1}^{m}\frac{1}{|B|}\int_B |T_{j,1}M_{(b-b_{2B})\chi_{(2B)^c}}T_{j,2}(f)(y)-T_{j,1}M_{(b-b_{2B})\chi_{(2B)^c}}T_{j,2}(f)(x)|dy \\ &:= C\sum_{j=1}^{m} I_{4j}. \end{aligned}$$

If $T_{j,1}$ is the strongly singular Calderón-Zygmund operator, the fact $\theta<\alpha$ implies that $\varepsilon_2 := \frac{\delta}{\alpha}(\alpha-\theta)>0$. For any $y\in B$ and $z\in(2\widetilde{B})^c$, we have $2|y-x|^\alpha\leq 2r_B^\alpha\leq 2r_B^\theta\leq|z-x|$ since $0<r_B\leq 1$. It follows from (2) of Definition 1.1, Hölder's inequality, Lemma 3.4 and Lemma 3.7, we have

$$\begin{aligned} I_{4j} &\leq C\frac{1}{|B|}\int_B\int_{(2\widetilde{B})^c}|K(y,z)-K(x,z)||b(z)-b_{2B}||T_{j,2}(f)(z)|dzdy \\ &\leq C\left(\frac{1}{|B|}\int_B\sum_{i=1}^{\infty}\int_{2^{i+1}\widetilde{B}\setminus 2^i\widetilde{B}}\frac{|y-x|^\delta}{|z-x|^{n+\frac{\delta}{\alpha}}}|b(z)-b_{2B}||T_{j,2}(f)(z)|\right)dzdy \\ &\leq C\sum_{i=1}^{\infty}r_B^{\delta-\frac{\theta\delta}{\alpha}}(2^i)^{-\frac{\delta}{\alpha}}\frac{1}{|2^{i+1}\widetilde{B}|}\left(\int_{2^{i+1}\widetilde{B}}|b(z)-b_{2B}|^{s'}\omega(y)^{1-s'}dy\right)^{\frac{1}{s'}} \\ &\quad\times\left(\int_{2^{i+1}\widetilde{B}}|T_{j,2}(f)(z)|^s\omega(y)dy\right)^{\frac{1}{s}} \\ &\leq C\sum_{i=1}^{\infty}r_B^{\delta-\frac{\theta\delta}{\alpha}}(2^i)^{-\frac{\delta}{\alpha}}\frac{1}{|2^{i+1}\widetilde{B}|}|2^{i+1}B|^{\frac{1}{s'}}\|b\|_{BMO(\omega)}\omega(x)(i+(1-\theta)\ln\frac{1}{r_B}) \\ &\quad\times\left(\frac{\omega(2^{i+1}\widetilde{B})}{|2^{i+1}\widetilde{B}|}\right)^{-\frac{1}{s}}M_{s,\omega}(T_{j,2}f)(x)\omega(2^{i+1}\widetilde{B})^{\frac{1}{s}} \\ &\leq Cr_B^{\delta-\frac{\theta\delta}{\alpha}}\sum_{i=1}^{\infty}(2^i)^{-\frac{\delta}{\alpha}}\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x)(i+(1-\theta)\ln\frac{1}{r_B}) \\ &\leq C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x)r_B^{\delta-\frac{\theta\delta}{\alpha}}\sum_{i=1}^{\infty}(2^i)^{-\frac{\delta}{\alpha}}(i+\frac{1}{\varepsilon_2 r_B^{-\varepsilon_2}}) \\ &\leq C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x)r_B^{\frac{\delta}{\alpha}(\alpha-\theta)-\varepsilon_2}\sum_{i=1}^{\infty}i(2^i)^{-\frac{\delta}{\alpha}} \\ &\leq C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(T_{j,2}f)(x). \end{aligned}$$

If $T_{j,1}=\pm I$, then $I_{4j}=0$. We have

$$I_4 \le C\|b\|_{BMO(\omega)}\omega(x)\sum_{j=1}^{m} M_{s,\omega}(T_{j,2}f)(x).$$

Combining the estimates in both cases, we have

$$\begin{aligned} M_t^{\sharp}(T_bf)(x) &\sim \sup_{r_B>0}\inf_{a\in\mathbb{C}}\left(\frac{1}{|B(x,r_B)|}\int_{B(x,r_B)}\Big||T_b(f)(y)|^t-a\Big|dy\right)^{\frac{1}{t}} \\ &\le C\|b\|_{BMO(\omega)}\omega(x)\sum_{j=1}^{m}(\|T_{j,1}\|+1)M_{s,\omega}(T_{j,2}f)(x). \end{aligned}$$

□

Lemma 3.12. *Let α,β,δ be the same as those in Definition 1.1, $\frac{n(1-\alpha)}{2}<\beta<\frac{n}{2}(n\ge 2)$, $T_{j,1}$ be the strongly singular Calderón-Zygmund operator or $\pm I$. When $f\in C_c^{\infty}(\mathbb{R}^n)$, $T_1(f)=0$. Suppose $0<\beta_0<1, 0<t<1, \frac{n(1-\alpha)+2\beta}{2\beta}<s<\infty, \omega\in A_1\cap RH_r$, where $r>\frac{(n(1-\alpha)+2\beta)(s-1)}{2\beta s-n(1-\alpha)-2\beta}$, and $b\in Lip_{\beta_0}(\omega)$. Then we have*

$$M_t^{\sharp}(T_bf)(x)\le C\|b\|_{Lip_{\beta_0}(\omega)}\omega(x)\sum_{j=1}^{m}\left((\|T_{j,1}\|+1)M_{\beta_0,s,\omega}(T_{j,2}f)(x)\right), a.e.\, x\in\mathbb{R}^n.$$

The proof process of Lemma 3.12 is similar to that of Lemma 3.11, so we omit it.

4. Proof of the Main Results

Now we are able to prove our main results. Firstly, we give the proof of Theorem 2.1 as follows.

Proof. Since $r>\frac{n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$, that is $p>\frac{n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta}$, there exists an s,such that $p>s>\frac{n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta}>\frac{n(1-\alpha)+2\beta}{2\beta}$. Since $s>\frac{n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta}$, we have $r>\frac{n(1-\alpha)+2\beta)(s-1)}{2\beta s-n(1-\alpha)-2\beta}$. Applying Lemma 3.6 and Lemma 3.11 we thus have

$$\begin{aligned} &\|T_bf\|_{L^{p,k}(\omega^{1-p},\omega)}\le\|M_t(T_bf)\|_{L^{p,k}(\omega^{1-p},\omega)} \\ &\le C\|M_t^{\sharp}(T_bf)\|_{L^{p,k}(\omega^{1-p},\omega)} \end{aligned}$$

$$\leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{m}(\|T_{j,1}\|+1)\|\omega(\cdot)M_{s,\omega}(T_{j,2}f)\|_{L^{p,k}(\omega^{1-p},\omega)}$$
$$= C\|b\|_{BMO(\omega)}\sum_{j=1}^{m}(\|T_{j,1}\|+1)\|M_{s,\omega}(T_{j,2}f)\|_{L^{p,k}(\omega)}.$$

Then, by Lemma 3.5 and the boundedness of $T_{j,2}$ on $L^{p,k}(\omega)$, we have

$$\|T_bf\|_{L^{p,k}(\omega^{1-p},\omega)} \leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{m}(\|T_{j,1}\|+1)\|T_{j,2}f\|_{L^{p,k}(\omega)}$$
$$\leq C\|b\|_{BMO(\omega)}\sum_{j=1}^{m}(\|T_{j,1}\|+1)\|f\|_{L^{p,k}(\omega)}.$$

This completes the proof of Theorem 2.1. □

Then we give the proof of Theorem 2.2 as follows.

Proof. Since $r_\omega > \frac{n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$, then $r > \frac{n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$ with $\omega \in RH_r$. Since $p > \frac{n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta}$, there exists an s, such that $p > s > \frac{n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta} > \frac{n(1-\alpha)+2\beta}{2\beta}$. Since $s > \frac{n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta}$, we have $r > \frac{n(1-\alpha)+2\beta)(s-1)}{2\beta s-n(1-\alpha)-2\beta}$. Applying Lemma 3.6 and Lemma 3.12 we thus have

$$\|T_bf\|_{L^{q,\frac{kq}{p}}(\omega^{1-q},\omega)} \leq \|M_t(T_bf)\|_{L^{q,\frac{kq}{p}}(\omega^{1-q},\omega)}$$
$$\leq C\|M_t^\sharp(T_bf)\|_{L^{q,\frac{kq}{p}}(\omega^{1-q},\omega)}$$
$$\leq C\|b\|_{Lip_{\beta_0}(\omega)}\sum_{j=1}^{m}(\|T_{j,1}\|+1)\|\omega(\cdot)M_{\beta_0,s,\omega}(T_{j,2}f)\|_{L^{q,\frac{kq}{p}}(\omega^{1-q},\omega)}$$
$$= C\|b\|_{Lip_{\beta_0}(\omega)}\sum_{j=1}^{m}(\|T_{j,1}\|+1)\|M_{\beta_0,s,\omega}(T_{j,2}f)\|_{L^{q,\frac{kq}{p}}(\omega)}.$$

Then, by Lemma 3.8 and the boundedness of $T_{j,2}$ on $L^{p,k}(\omega)$, we have

$$\|T_bf\|_{L^{q,\frac{kq}{p}}(\omega^{1-q},\omega)} \leq C\|b\|_{Lip_{\beta_0}(\omega)}\sum_{j=1}^{m}(\|T_{j,1}\|+1)\|T_{j,2}f\|_{L^{p,k}(\omega)}$$
$$\leq C\|b\|_{Lip_{\beta_0}(\omega)}\sum_{j=1}^{m}(\|T_{j,1}\|+1)\|f\|_{L^{p,k}(\omega)}.$$

This completes the proof of Theorem 2.2. □

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(1) School of Science, China University of Mining and Technology, Beijing, China.

E-mail address: `cuizhe@cumtb.edu.cn`

(2) School of Science, China University of Mining and Technology, Beijing, China.

E-mail address: `linyan@cumtb.edu.cn (Corresponding author)`