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COMPACT COMPOSITION OPERATORS ON MODEL SPACES WITH UNIVALENT SYMBOLS

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ABSTRACT. We give a sufficient condition and a necessary condition for the compactness of the composition operators on model spaces $C_\varphi : K_\theta \to H^2$, where φ is univalent. This is a generalization to a result of Shapiro for the composition operator $C_\varphi : H^2 \to H^2$, [11].

1. INTRODUCTION

Suppose φ is an analytic self–map of the unit disc $\mathbb{D} = \{z : |z| < 1\}$. Then the composition operator is defined as follows:

$$C_\varphi : f \mapsto f \circ \varphi.$$

The composition operator was studied on many different function spaces, and the monographs [2, 11] give the basic foundations and basic results of the composition operator.

We will concentrate on the Hardy spaces, [3, 7]. The Hardy space H^2 is the space of all analytic functions on the $\mathbb{D}$ such that

$$\|f\| = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

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If $f \in H^2$ then Fatou's Theorem guarantees that the radial limits $f(e^{i\theta}) := \lim_{r \to ^{-}1} f(re^{i\theta})$ exist almost everywhere for $\theta \in [0, 2\pi]$, and the resulting function belongs to $L^2(\mathbb{T})$, where $\mathbb{T} = \{z : |z| = 1\}$.

It is well-known that C_φ maps H^2 into itself, moreover $C_\varphi : H^2 \to H^2$ is bounded, that is a direct consequence of the Littlewoods Subordination Principle,[2, 11]. The story for the compactness is different. C_φ is not necessarily compact, for example $\varphi(z) = (1+z)/2$ induces a non–compact operator on H^2, [11].

If φ is univalent then the following theorem gives a characterization for the compactness of the composition operator on H^2.

Theorem 1.1. [11] *Suppose φ is a univalent self–map of $\mathbb{D}$. Then $C_\varphi : H^2 \to H^2$ is compactness if and only if*

$$\lim_{|z| \to 1^-} \frac{1 - |\varphi(z)|}{1 - |z|} = \infty. \tag{1.1}$$

This discussion can be shifted to subspaces in H^2. Let θ be an inner function on $\mathbb{D}$, that is bounded, analytic and $|\theta| = 1$ $a.e$ on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. Then the model space is defined as follows

$$K_\theta := H^2 \ominus \theta H^2 = (\theta H^2)^\perp,$$

or

$$K_\theta = \{f \in H^2 : < f, \theta g >= 0, \quad \text{for all } g \in H^2\}.$$

These are the non-trivial closed shift invariant subspaces of H^2, [10, 5]. It is well-known that the model space K_θ is a reproducing kernel Hilbert space. It is not hard to show that its kernel, for $\lambda \in \mathbb{D}$, is given by

$$k_\lambda^\theta(z) = \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda} z},$$

and for all $f \in K_\theta$ we have $f(\lambda) =< f, k_\lambda^\theta >$.

Mashreghi and Shabankhah, in [8, 9], studied the composition operators on these model spaces. A similar question is discussed in [4]. Lyubarskii and Malinnikova, in [6], characterized the compactness of the composition operator $C_\varphi : K_\theta \to H^2$.

In this article we want to obtain a similar result to Theorem 1.1 for model spaces K_θ. More precisely, we have the following theorems:

Theorem 1.2. *Let φ be a univalent self map of $\mathbb{D}$. If, for some $0 \le p < 1$,*

$$\lim_{|z|\to 1^-} \frac{1 - |\varphi(z)|}{(1 - |z|)(1 - |\theta(\varphi(z))|)^p} = \infty. \tag{1.2}$$

Then the composition operator $C_\varphi : K_\theta \to H^2$ is compact.

Theorem 1.3. *Let φ be a univalent self map of $\mathbb{D}$. If, the composition operator $C_\varphi : K_\theta \to H^2$ is compact, then*

$$\lim_{|z|\to 1^-} \frac{1 - |\varphi(z)|}{(1 - |z|)(1 - |\theta(\varphi(z))|)} = \infty. \tag{1.3}$$

Remark 1. Lyubarskii and Malinnikova's condition for the compactness works for any inner function, and it is written in terms of the Nivanlinna counting function $N_\varphi(w) = -\sum_{w=\varphi(z)} \log|z|$. So if φ is univalent then (1.2) is nothing but a special case of Lyubarskii and Malinnikova's condition.

2. The proof of 1.2

The goal of this section is to prove Theorem 1.2. To do so, we need the following results.

Theorem 2.1. [11] *If f is analytic on the unit disc $\mathbb{D}$ then*

$$\|f\|^2 \approx |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2) dA(z). \tag{2.1}$$

Theorem 2.2. [1, Page 187] *If f is K_θ, $0 \le p < 1$, then*

$$\|f\|^2 \gtrsim |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)}{(1-|\theta(\varphi(z))|)^p} dA(z). \tag{2.2}$$

Proof of Theorem 1.2 . Suppose φ is a univalent and satisfies (1.2). That means, for any $\epsilon > 0$ there is $0 \le r < 1$ such that

$$\frac{1-|\varphi(z)|}{(1-|z|)(1-|\theta(\varphi(z))|)^p} > \frac{1}{\epsilon}, \qquad r < |z| < 1,$$

or equivalently

$$1 - |z| \le \epsilon \frac{1-|\varphi(z)|}{(1-|\theta(\varphi(z))|)^p}, \qquad r < |z| < 1. \tag{2.3}$$

Suppose f_n is a sequence in the closed unit ball of K_θ and converges uniformly to 0 on compact subsets of $\mathbb{D}$.

Hence, for $0 \le r < 1$, Theorem 2.1 implies,

$$\begin{aligned} \|C_\varphi f_n\|^2 &\lesssim |f_n(\varphi(0))|^2 + \int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 (1-|z|^2) dA(z) \\ &= \underbrace{|f_n(\varphi(0))|^2}_{I_1} + \underbrace{\int_{r\mathbb{D}} |(f_n \circ \varphi)'(z)|^2 (1-|z|^2) dA(z)}_{I_2} \\ &+ \underbrace{\int_{\mathbb{D}\setminus r\mathbb{D}} |(f)n \circ \varphi)'(z)|^2 (1-|z|^2) dA(z)}_{I_3} \end{aligned}$$

It is clear that $I_1 \to 0$ as $n \to \infty$ (f_n converges uniformly to 0). For I_2, since $f_n \circ \varphi$ converges uniformly to 0 on compact subsets, then $(f \circ \varphi)'$ converges uniformly to 0 on compact subsets. That means, $I_2 \to 0$ as $n \to \infty$. For I_3, for any $\epsilon > 0$ we have

$$\begin{aligned} I_3 &\leq \int_{\mathbb{D}} |f_n'(\varphi(z))\varphi'(z)|^2(1-|z|^2)dA(z) \\ &\leq \int_{\mathbb{D}} |f_n'(\varphi(z))\varphi'(z)|^2 \epsilon \frac{1-|\varphi(z)|}{(1-|\theta(\varphi(z))|)^p} dA(z), \quad \text{by (2.3)} \\ &\leq \epsilon \int_{\mathbb{D}} |f_n'(w)|^2 \frac{1-|w|}{(1-|\theta(w)|)^p} dA(w), \quad \text{(change of variables)} \\ &\leq \epsilon(\|f_n\|^2 - |f(0)|^2), \quad \text{(by 2.2)} \\ &\leq \epsilon(1-|f(0)|^2). \end{aligned}$$

Since ϵ is arbitrary, I_3 approaches 0 as $n \to \infty$. Hence $\|C_\varphi f_n\| \to 0$. Hence C_φ is compact.

□

3. The proof of Theorem 1.3

In this section we will prove the necessity of (1.3) for the compactness of $C_\varphi : K_\theta \to H^2$. We begin with the following lemma.

Lemma 3.1. *If $C_\varphi : K_\theta \to H^2$ then $C_\varphi^* k_\lambda = k_{\varphi(\lambda)}^\theta$*

Proof. Let f be in K_θ. Then,

$$\begin{aligned} < f, C_\varphi^* k_\lambda > &= < C_\varphi f, k_\lambda > \\ &= f(\varphi(\lambda)) \\ &= < f, k_{\varphi(\lambda)}^\theta > . \end{aligned}$$

□

Proof of Theorem 1.3. Suppose that $C_\varphi : K_\theta \to H^2$ is compact. This implies that $C_\varphi^* : H^2 \to K_\theta$ is compact, [11, Section 3.4].

The normalized reproducing kernel of H^2 is

$$\widetilde{k_\lambda}(z) = \frac{k_\lambda(z)}{\|k_\lambda\|} = \frac{\sqrt{1-|\lambda|^2}}{1-\bar{\lambda}z},$$

where $\lambda \in \mathbb{D}$. Let $\mathcal{A} = \{C_\varphi^* \widetilde{k_\lambda},\ \lambda \in \mathbb{D}\}$. Since C_φ^* is compact, $\mathcal{A}$ is relatively compact subset in K_θ. Hence every sequence in $\mathcal{A}$ has a convergent subsequence. Let $\lambda_n \in \mathbb{D}$ be such that $|\lambda_n| \to 1$, and $C_\varphi^* \widetilde{k_{\lambda_n}}$ is convergent to $g \in K_\theta$.

Therefore, for h in $K_\theta^\infty := K_\theta \cap H^\infty$ we have,

$$\begin{aligned}
< h, g > &= \lim_{n\to\infty} < h, C_\varphi^* \widetilde{k_{\lambda_n}} > \\
&= \lim_{n\to\infty} \sqrt{1-|\lambda_n|^2} < h, C_\varphi^* k_{\lambda_n} > \\
&= \lim_{n\to\infty} \sqrt{1-|\lambda_n|^2} < h, k^\theta_{\varphi(\lambda_n)} > \qquad \text{(Lemma 3.1)} \\
&= \lim_{n\to\infty} h(\varphi(\lambda_n))\sqrt{1-|\lambda_n|^2} \\
&= 0 \qquad (h \text{ is bounded}).
\end{aligned}$$

Since K_θ^∞ is dense in K_θ, g must be zero. Hence,

$$C_\varphi^* \widetilde{k_{\lambda_n}} \to 0,$$

as $n \to \infty$. That means

$$\|C_\varphi^* \widetilde{k_{\lambda_n}}\| \to 0.$$

But

$$\begin{aligned}
\|C_\varphi^* \widetilde{k_{\lambda_n}}\|^2 &= (1-|\lambda_n|^2)\, k^\theta_{\varphi(\lambda_n)} \\
&= (1-|\lambda_n|^2)\frac{1-|\theta(\varphi(\lambda_n)|^2}{1-|\varphi(\lambda_n)|^2}.
\end{aligned}$$

Which implies (1.3) and completes the proof. □

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