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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE QUASI-CONVEX

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Abstract. In this paper, we establish new Hermite-Hadamard's inequalities using a new identity for parameter functions via quasi-convexity. Several known results are derived. Applications to special means are also given.

1. Introduction

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on I, where I is a nonempty interval of real numbers, if

$$f(tx + (1-t)y) \ \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$ (see [12]).

If $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on $[a,b] \subset I$ such that $a < b$, then

$$f(\tfrac{a+b}{2}) \le \tfrac{1}{b-a}\int_a^b f(t)dt \le \tfrac{f(a)+f(b)}{2}. \tag{1.1}$$

The above inequality is known in the literature as Hermite-Hadamard's inequality.

Over the last century, inequality (1.1) has not ceased to draw the attention of researchers, various generalizations, refinements, extensions and variants have appeared

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in the literature, one can mention [3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16] and references therein.

We also recall that a function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex on I, if

$$f(tx + (1-t)y) \ \leq \max\{f(x), f(y)\}$$

holds for all $x, y \in I$ and all $t \in [0,1]$ (see [9]).

We note that the concept of quasi-convexity represents a generalization of the concept of classical convexity.

The famous Hölder's inequality can be stated as follows: for continuous functions $f, g : [a,b] \to \mathbb{R}$, we have

$$\int_a^b f(t)g(t)\,dt \leq \left(\int_a^b f^p(t)dt\right)^{\frac{1}{p}} \left(\int_a^b g^q(t)dt\right)^{\frac{1}{q}},$$

where $p, q \geq 1$ such that $\frac{1}{q} + \frac{1}{p} = 1$.

In what follows $L[a,b]$ represents the space of the integrable functions.

Recently, Alomari et al. [2] established the following Hadamard-type inequalities for quasi-convex functions

Theorem 1.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}$$
$$\times \left(\sup\left\{|f'(a)|, \left|f'(\tfrac{a+b}{2})\right|\right\} + \sup\left\{\left|f'(\tfrac{a+b}{2})\right|, |f'(b|\right\}\right).$$

Theorem 1.2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^p$is quasi-convex on $[a,b]$, such that $q > 1$*

with $\frac{1}{q}+\frac{1}{p}=1$, *then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}$$
$$\times\left(\left(\sup\left\{|f'(a)|^q,\left|f'(\tfrac{a+b}{2})\right|^q\right\}\right)^{\frac{1}{q}}+\left(\sup\left\{\left|f'(\tfrac{a+b}{2})\right|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Theorem 1.3. *Let* $f: I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I°*such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^p$*is quasi-convex on* $[a,b]$, *such that* $q\geq 1$, *then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}$$
$$\times\left(\left(\sup\left\{|f'(a)|^q,\left|f'(\tfrac{a+b}{2})\right|^q\right\}\right)^{\frac{1}{q}}+\left(\sup\left\{\left|f'(\tfrac{a+b}{2})\right|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Yildiz et al. [17] gave the following generalizations of the above results

Theorem 1.4. *Let* $f: I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I°*such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f'|$*is quasi-convex on* $[a,b]$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\left(\frac{(x-a)^2}{2(b-a)}\max\{|f'(a)|,|f'(x)|\}+\frac{(b-x)^2}{2(b-a)}\max\{|f'(x)|,|f'(b|\}\right).$$

Theorem 1.5. *Let* $f: I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* I°*such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $|f'|^p$*is quasi-convex on* $[a,b]$, *such that* $q>1$ *with* $\frac{1}{q}+\frac{1}{p}=1$, *then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq\left(\frac{(x-a)^2}{(b-a)(p+1)^{\frac{1}{p}}}\left(\max\left\{|f'(a)|^q,|f'(x)|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^2}{(b-a)(p+1)^{\frac{1}{p}}}\left(\max\left\{|f'(x)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Theorem 1.6. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^p$ is quasi-convex on $[a, b]$, such that $q \geq 1$, then the following inequality holds:*

$$\left|\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \left(\frac{(x-a)^2}{2(b-a)}\left(\max\left\{|f'(a)|^q, |f'(x)|^q\right\}\right)^{\frac{1}{q}} + \frac{(b-x)^2}{2(b-a)}\left(\max\left\{|f'(x)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Motivated by the results obtaind by [1, 2, 17], in this paper, we establish new Hermite-Hadamard inequalities using a new identity for parameter functions via quasi-convexity. Several known results are derived. Applications to special means are also given.

2. Main results

We first prove the following identity

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $[a, b]$. Then for any $\lambda \in [\frac{1}{2}, 1]$, one has the equality*

$$\frac{x-a}{b-a} f(\lambda a + (1-\lambda) b) + \frac{b-x}{b-a} f((1-\lambda) a + \lambda b) - \frac{1}{b-a}\int_a^b f(t)dt$$

$$= \frac{1}{b-a}\int_a^b k(x, t) f'(t)dt, \tag{2.1}$$

where

$$k(x, t) = \begin{cases} t - a \ \text{if}\ t \in [a, \lambda a + (1-\lambda) b) \\ t - x \ \text{if}\ t \in [\lambda a + (1-\lambda) b, (1-\lambda) a + \lambda b) \\ t - b \ \text{if}\ t \in [(1-\lambda) a + \lambda b, b], \end{cases} \tag{2.2}$$

and $x \in [\lambda a + (1-\lambda) b, (1-\lambda) a + \lambda b]$.

Proof. by integration by parts, we have

$$I_1 = \frac{1}{b-a} \int_a^{\lambda a+(1-\lambda)b} (t-a) f'(t)dt = (1-\lambda) f(\lambda a + (1-\lambda) b)$$

$$- \frac{1}{b-a} \int_a^{\lambda a+(1-\lambda)b} f(t)dt \tag{2.3}$$

and

$$I_2 = \frac{1}{b-a} \int_{\lambda a+(1-\lambda)b}^{(1-\lambda)a+\lambda b} (t-x) f'(t)dt = \left(\lambda - \frac{x-a}{b-a}\right) f((1-\lambda) a + \lambda b)$$

$$+ \left(\lambda - \frac{b-x}{b-a}\right) f(\lambda a + (1-\lambda) b) - \frac{1}{b-a} \int_{\lambda a+(1-\lambda)b}^{(1-\lambda)a+\lambda b} f(t)dt, \tag{2.4}$$

and

$$I_3 = \frac{1}{b-a} \int_{(1-\lambda)a+\lambda b}^{b} (t-b) f'(t)dt = (1-\lambda) f((1-\lambda) a + \lambda b)$$

$$- \frac{1}{b-a} \int_{(1-\lambda)a+\lambda b}^{b} f(t)dt. \tag{2.5}$$

Adding (2.3)-(2.5), we get the desired inequality. □

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $[a,b]$ such that $f' \in L([a,b])$. If $|f'|$ is quasi-convex, then the following inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \left(\frac{x-a}{b-a} f(\lambda a + (1-\lambda) b) + \frac{b-x}{b-a} f((1-\lambda) a + \lambda b) \right) \right|$$

$$\leq \frac{b-a}{2} \left((1-\lambda)^2 \max\{|f'(a)|, |f'(\lambda a + (1-\lambda) b)|\} \right.$$

$$+ \left(\lambda - \frac{b-x}{b-a}\right)^2 \max\{|f'(\lambda a + (1-\lambda) b)|, |f'(x)|\}$$

$$+ \left(\lambda - \frac{x-a}{b-a}\right)^2 \max\{|f'(x)|, |f'((1-\lambda) a + \lambda b)|\}$$

$$+ (1-\lambda)^2 \max\{|f'((1-\lambda)a+\lambda b)|, |f'(b)|\}\big) \tag{2.6}$$

holds for all $x \in [\lambda a + (1-\lambda) b, (1-\lambda) a + \lambda b]$ *with* $\lambda \in \left[\frac{1}{2}, 1\right]$.

Proof. From Lemma 2.1, and property of modulus, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \left(\tfrac{x-a}{b-a} f(\lambda a + (1-\lambda) b) + \tfrac{b-x}{b-a} f((1-\lambda) a + \lambda b)\right)\right|$$

$$= \frac{1}{b-a}\left|\int_a^b k(x,t) f'(t)dt\right|$$

$$\leq \frac{1}{b-a}\left(\int_a^{\lambda a+(1-\lambda)b} (t-a)|f'(t)|\,dt + \int_{\lambda a+(1-\lambda)b}^{x} (x-t)|f'(t)|\,dt\right.$$

$$\left. + \int_x^{(1-\lambda)a+\lambda b} (t-x)|f'(t)|\,dt + \int_{(1-\lambda)a+\lambda b}^{b} (b-t)|f'(t)|\,dt\right)$$

$$= (b-a)\left((1-\lambda)^2 \int_0^1 \alpha\,|f'((1-\alpha)a + \alpha(\lambda a + (1-\lambda) b))|\,d\alpha\right.$$

$$+ \left(\lambda - \tfrac{b-x}{b-a}\right)^2 \int_0^1 (1-\alpha)\,|f'(((1-\alpha)(\lambda a + (1-\lambda) b) + \alpha x))|\,d\alpha$$

$$+ \left(\lambda - \tfrac{x-a}{b-a}\right)^2 \int_0^1 \alpha\,|f'((1-\alpha)x + \alpha((1-\lambda) a + \lambda b))|\,d\alpha$$

$$\left. + (1-\lambda)^2 \int_0^1 (1-\alpha)\,|f'((1-\alpha)((1-\lambda) a + \lambda b) + \alpha b)|\,d\alpha\right). \tag{2.7}$$

Using the quasi-convexity of $|f'|$, we get

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \left(\tfrac{x-a}{b-a} f(\lambda a + (1-\lambda) b) + \tfrac{b-x}{b-a} f((1-\lambda) a + \lambda b)\right)\right|$$

$$\leq \frac{b-a}{2}\left((1-\lambda)^2 \max\{|f'(a)|, |f'(\lambda a+(1-\lambda)b)|\}\right.$$
$$+\left(\lambda-\tfrac{b-x}{b-a}\right)^2 \max\{|f'(\lambda a+(1-\lambda)b)|, |f'(x)|\}$$
$$+\left(\lambda-\tfrac{x-a}{b-a}\right)^2 \max\{|f'(x)|, |f'((1-\lambda)a+\lambda b)|\}$$
$$\left.+(1-\lambda)^2 \max\{|f'((1-\lambda)a+\lambda b)|, |f'(b)|\}\right),$$

which is the desired result. □

Corollary 2.1. *In Theorem 2.1, if we choose* $x=\frac{a+b}{2}$ *and* $\lambda=\frac{2}{3}$*, we obtain the following two-point open Newton-Cotes inequality*

$$\left|\tfrac{1}{b-a}\int_a^b f(t)dt-\frac{f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)}{2}\right| \leq \tfrac{b-a}{72}$$
$$\times\left(4\max\left\{|f'(a)|, \left|f'(\tfrac{2a+b}{3})\right|\right\}+\max\left\{\left|f'(\tfrac{2a+b}{3})\right|, \left|f'(\tfrac{a+b}{2})\right|\right\}\right.$$
$$\left.+\max\left\{\left|f'(\tfrac{a+b}{2})\right|, \left|f'(\tfrac{a+2b}{3})\right|\right\}+4\max\left\{\left|f'(\tfrac{a+2b}{3})\right|, |f'(b)|\right\}\right).$$

Remark 1. Theorem 2.1 will be reduced to Theorem 6 from [17], if we take $\lambda=1$. Moreover if we choose $x=\frac{a+b}{2}$ we obtain Theorem 2.2 from [2]. And if we take $\lambda=\frac{1}{2}$ we obtain Theorem 7 from [1].

Theorem 2.2. *Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* $[a,b]$ *such that* $f'\in L([a,b])$*. If* $|f'|^q$ $(q>1)$ *is quasi-convex, then the following inequality*

$$\left|\tfrac{1}{b-a}\int_a^b f(t)dt-\left(\tfrac{x-a}{b-a}f(\lambda a+(1-\lambda)b)+\tfrac{b-x}{b-a}f((1-\lambda)a+\lambda b)\right)\right|$$
$$\leq \tfrac{b-a}{2}\left((1-\lambda)^2\left(\left(\max\left\{|f'(a)|^q, |f'(\lambda a+(1-\lambda)b)|^q\right\}\right)\right)^{\frac{1}{q}}\right.$$
$$+\left(\lambda-\tfrac{b-x}{b-a}\right)^2\left(\left(\max\left\{|f'(\lambda a+(1-\lambda)b)|^q, |f'(x)|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+\left(\lambda-\tfrac{x-a}{b-a}\right)^2\left(\left(\max\left\{|f'(x)|^q, |f'((1-\lambda)a+\lambda b)|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$\left.+(1-\lambda)^2\left(\left(\max\left\{|f'((1-\lambda)a+\lambda b)|^q, |f'(b)|^q\right\}\right)\right)^{\frac{1}{q}}\right) \tag{2.8}$$

holds for all $x \in [\lambda a + (1-\lambda) b, (1-\lambda) a + \lambda b]$ *with* $\lambda \in \left[\frac{1}{2}, 1\right]$.

Proof. From Lemma 2.1, property of modulus, and power mean inequality, we have

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \left(\frac{x-a}{b-a} f(\lambda a + (1-\lambda) b) + \frac{b-x}{b-a} f((1-\lambda) a + \lambda b) \right) \right|$$

$$= (b-a) \left((1-\lambda)^2 \int_0^1 \alpha \left| f'((1-\alpha) a + \alpha (\lambda a + (1-\lambda) b)) \right| d\alpha \right.$$

$$+ \left(\lambda - \tfrac{b-x}{b-a}\right)^2 \int_0^1 (1-\alpha) \left| f'(((1-\alpha)(\lambda a + (1-\lambda) b) + \alpha x)) \right| d\alpha$$

$$+ \left(\lambda - \tfrac{x-a}{b-a}\right)^2 \int_0^1 \alpha \left| f'((1-\alpha) x + \alpha ((1-\lambda) a + \lambda b)) \right| d\alpha$$

$$\left. + (1-\lambda)^2 \int_0^1 (1-\alpha) \left| f'((1-\alpha)((1-\lambda) a + \lambda b) + \alpha b) \right| d\alpha \right)$$

$$\leq \frac{b-a}{2^{1-\frac{1}{q}}} \left((1-\lambda)^2 \left(\int_0^1 \alpha \left| f'((1-\alpha) a + \alpha (\lambda a + (1-\lambda) b)) \right|^q d\alpha \right)^{\frac{1}{q}} \right.$$

$$+ \left(\lambda - \tfrac{b-x}{b-a}\right)^2 \left(\int_0^1 (1-\alpha) \left| f'(((1-\alpha)(\lambda a + (1-\lambda) b) + \alpha x)) \right|^q d\alpha \right)^{\frac{1}{q}}$$

$$+ \left(\lambda - \tfrac{x-a}{b-a}\right)^2 \left(\int_0^1 \alpha \left| f'((1-\alpha) x + \alpha ((1-\lambda) a + \lambda b)) \right|^q d\alpha \right)^{\frac{1}{q}}$$

$$\left. + (1-\lambda)^2 \left(\int_0^1 (1-\alpha) \left| f'((1-\alpha)((1-\lambda) a + \lambda b) + \alpha b) \right|^q d\alpha \right)^{\frac{1}{q}} \right). \tag{2.9}$$

Using the quasi-convexity of $|f'|^q$, we obtain

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \left(\frac{x-a}{b-a} f(\lambda a + (1-\lambda) b) + \frac{b-x}{b-a} f((1-\lambda) a + \lambda b) \right) \right|$$

$$\leq \tfrac{b-a}{2}\Big((1-\lambda)^2\left(\left(\max\left\{|f'(a)|^q, |f'(\lambda a+(1-\lambda)b)|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+\left(\lambda-\tfrac{b-x}{b-a}\right)^2\left(\left(\max\left\{|f'(\lambda a+(1-\lambda)b)|^q, |f'(x)|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+\left(\lambda-\tfrac{x-a}{b-a}\right)^2\left(\left(\max\left\{|f'(x)|^q, |f'((1-\lambda)a+\lambda b)|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+(1-\lambda)^2\left(\left(\max\left\{|f'((1-\lambda)a+\lambda b)|^q, |f'(b)|^q\right\}\right)\right)^{\frac{1}{q}}\Big),$$

which is the desired result. □

Corollary 2.2. *In Theorem 2.2, if we choose $x=\frac{a+b}{2}$ and $\lambda=\frac{2}{3}$, we obtain the following two-point open Newton-Cotes inequality*

$$\left|\tfrac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left(f(\tfrac{2a+b}{3})+f(\tfrac{a+2b}{3})\right)\right|\leq\frac{b-a}{72}$$
$$\times\Big(4\left(\left(\max\left\{|f'(a)|^q, \left|f'(\tfrac{2a+b}{3})\right|^q\right\}\right)\right)^{\frac{1}{q}}+\left(\left(\max\left\{\left|f'(\tfrac{2a+b}{3})\right|^q, \left|f'(\tfrac{a+b}{2})\right|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+\left(\left(\max\left\{\left|f'(\tfrac{a+b}{2})\right|^q, \left|f'(\tfrac{a+2b}{3})\right|^q\right\}\right)\right)^{\frac{1}{q}}+4\left(\left(\max\left\{\left|f'(\tfrac{a+2b}{3})\right|^q, |f'(b)|^q\right\}\right)\right)^{\frac{1}{q}}\Big).$$

Remark 2. Theorem 2.2 will be reduced to Theorem 8 from [17], if we take $\lambda=1$. Moreover if we choose $x=\frac{a+b}{2}$ we obtain Theorem 2.4 from [2]. And if we take $\lambda=\frac{1}{2}$ we obtain Theorem 9 from [1].

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $[a,b]$ such that $f'\in L([a,b])$. If $|f'|^q$ $(q>1)$ is quasi-convex, then the following inequality*

$$\left|\tfrac{1}{b-a}\int_a^b f(t)dt-\left(\tfrac{x-a}{b-a}f(\lambda a+(1-\lambda)b)+\tfrac{b-x}{b-a}f((1-\lambda)a+\lambda b)\right)\right|$$
$$\leq\tfrac{b-a}{(1+p)^{\frac{1}{p}}}\Big((1-\lambda)^2\left(\left(\max\left\{|f'(a)|^q, |f'(\lambda a+(1-\lambda)b)|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+\left(\lambda-\tfrac{b-x}{b-a}\right)^2\left(\left(\max\left\{|f'(\lambda a+(1-\lambda)b)|^q, |f'(x)|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+\left(\lambda-\tfrac{x-a}{b-a}\right)^2\left(\left(\max\left\{|f'(x)|^q, |f'((1-\lambda)a+\lambda b)|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+(1-\lambda)^2\left(\left(\max\left\{|f'((1-\lambda)a+\lambda b)|^q, |f'(b)|^q\right\}\right)\right)^{\frac{1}{q}}\Big) \tag{2.10}$$

holds for all $x \in [\lambda a + (1-\lambda) b, (1-\lambda) a + \lambda b]$ *with* $\lambda \in \left[\frac{1}{2}, 1\right]$, *and* $\frac{1}{q} + \frac{1}{p} = 1$.

Proof. From Lemma 2.1, property of modulus, Hölder inequality, and quasi-convexity of $|f'|^q$, we get

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t)dt - \left(\frac{x-a}{b-a} f(\lambda a + (1-\lambda) b) + \frac{b-x}{b-a} f((1-\lambda) a + \lambda b) \right) \right| \\
&= (b-a) \left((1-\lambda)^2 \int_0^1 \alpha \left| f'((1-\alpha) a + \alpha (\lambda a + (1-\lambda) b)) \right| d\alpha \right. \\
&\quad + \left(\lambda - \frac{b-x}{b-a} \right)^2 \int_0^1 (1-\alpha) \left| f'(((1-\alpha) (\lambda a + (1-\lambda) b) + \alpha x)) \right| d\alpha \\
&\quad + \left(\lambda - \frac{x-a}{b-a} \right)^2 \int_0^1 \alpha \left| f' ((1-\alpha) x + \alpha ((1-\lambda) a + \lambda b)) \right| d\alpha \\
&\quad \left. + (1-\lambda)^2 \int_0^1 (1-\alpha) \left| f' ((1-\alpha) ((1-\lambda) a + \lambda b) + \alpha b) \right| d\alpha \right) \\
&\le \frac{b-a}{(1+p)^{\frac{1}{p}}} \left((1-\lambda)^2 \left(\int_0^1 \left| f'((1-\alpha) a + \alpha (\lambda a + (1-\lambda) b)) \right|^q d\alpha \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\lambda - \frac{b-x}{b-a} \right)^2 \left(\int_0^1 \left| f'(((1-\alpha) (\lambda a + (1-\lambda) b) + \alpha x)) \right|^q d\alpha \right)^{\frac{1}{q}} \\
&\quad + \left(\lambda - \frac{x-a}{b-a} \right)^2 \left(\int_0^1 \left| f' ((1-\alpha) x + \alpha ((1-\lambda) a + \lambda b)) \right|^q d\alpha \right)^{\frac{1}{q}} \\
&\quad \left. + (1-\lambda)^2 \left(\int_0^1 \left| f' ((1-\alpha) ((1-\lambda) a + \lambda b) + \alpha b) \right|^q d\alpha \right)^{\frac{1}{q}} \right) \\
&\le \frac{b-a}{(1+p)^{\frac{1}{p}}} \left((1-\lambda)^2 \left(\left(\max \left\{ |f'(a)|^q, |f'(\lambda a + (1-\lambda) b)|^q \right\} \right) \right)^{\frac{1}{q}} \right.
\end{aligned}$$

$$+\left(\lambda-\tfrac{b-x}{b-a}\right)^2\left(\left(\max\left\{\left|f'(\lambda a+(1-\lambda)b)\right|^q,\left|f'(x)\right|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+\left(\lambda-\tfrac{x-a}{b-a}\right)^2\left(\left(\max\left\{\left|f'(x)\right|^q,\left|f'((1-\lambda)a+\lambda b)\right|^q\right\}\right)\right)^{\frac{1}{q}}$$
$$+(1-\lambda)^2\left(\left(\max\left\{\left|f'((1-\lambda)a+\lambda b)\right|^q,\left|f'(b)\right|^q\right\}\right)\right)^{\frac{1}{q}}\Big),$$

which is the desired result. □

Corollary 2.3. *In Theorem 2.3, if we choose* $x=\frac{a+b}{2}$ *and* $\lambda=\frac{2}{3}$*, we obtain the following two-point open Newton-Cotes inequality*

$$\left|\tfrac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left(f\left(\tfrac{2a+b}{3}\right)+f\left(\tfrac{a+2b}{3}\right)\right)\right|\leq\tfrac{b-a}{36\times(p+1)^{\frac{1}{p}}}$$
$$\times\left(4\left(\left(\max\left\{\left|f'(a)\right|^q,\left|f'\tfrac{2a+b}{3})\right|^q\right\}\right)^{\frac{1}{q}}+\left(\max\left\{\left|f'(\tfrac{a+2b}{3})\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\left(\max\left\{\left|f'\tfrac{2a+b}{3})\right|^q,\left|f'(\tfrac{a+b}{2})\right|^q\right\}\right)^{\frac{1}{q}}+\left(\max\left\{\left|f'(\tfrac{a+b}{2})\right|^q,\left|f'(\tfrac{a+2b}{3})\right|^q\right\}\right)^{\frac{1}{q}}\right).$$

Remark 3. Theorem 2.3 will be reduced to Theorem 7 from [17], if we take $\lambda=1$. Moreover if we choose $x=\frac{a+b}{2}$ we obtain Theorem 2.3 from [2]. And if we take $\lambda=\frac{1}{2}$ we obtain Theorem 8 from [1] .

3. Applications to special means

We recall the following means for positive real numbers α,β for $\alpha\neq\beta$, which are well known in the literature

The arithmetic mean: $A(\alpha,\beta)=\frac{\alpha+\beta}{2}$.

The generalized log-mean: $L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}$ with $p\neq-1,0$.

Proposition 3.1. *Let* a,b *two real numbers with* $0<a<b$*, then the following inequality holds:*

$$\left|L_4^4(a,b)-\frac{\left(\frac{2a+b}{3}\right)^4+\left(\frac{a+2b}{3}\right)^4}{2}\right|$$
$$leq\frac{b-a}{9}\left(4A\left(\left(\tfrac{2a+b}{3}\right)^3,b^3\right)+A\left(\left(\tfrac{a+b}{2}\right)^3,\left(\tfrac{a+2b}{3}\right)^3\right)\right).$$

Proof. The proof is immediate from Theorem 2.1 with $x = \frac{a+b}{2}$ and $\lambda = \frac{2}{3}$, applied for $f(x) = \frac{1}{4}x^4$ with $x > 0$. □

Proposition 3.2. *Let a, b two real numbers with $1 < a < b$, then for any $q > 2$, the following inequality holds:*

$$\left| L_{\frac{2}{q}+1}^{\frac{2}{q}+1}(a,b) - A\left(a^{\frac{2}{q}+1}, b^{\frac{2}{q}+1}\right) \right| \leq \frac{(q+2)(q-1)^{\frac{q-1}{q}}(b-a)}{2q(2q-1)^{\frac{q-1}{q}}} A\left(\left(\frac{a+b}{2}\right)^{\frac{2}{q}}, a^{\frac{2}{p}}\right).$$

Proof. The proof is immediate from Theorem 1.3 with $x = \frac{a+b}{2}$ and $\lambda = 1$ applied for $f(x) = \frac{q}{q+2}x^{\frac{2}{q}+1}$, and $x > 0$. □

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