Jordan Journal of Mathematics and Statistics (JJMS) 12(1), 2019, pp1 - 13

t-NORMS OVER Q-FUZZY SUBGROUPS OF A GROUP

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Abstract. In this paper, Q-fuzzy subgroups and normal Q-fuzzy subgroups of group G with respect to t-norm T are defined and investigated some of their properties and structured characteristics. Next the properties of them under homomorphisms and anti-homomorphisms are discussed.

1. Introduction

Since the concept of fuzzy group was introduced by Rosenfeldin 1971 [19], the theories and approaches on different fuzzy algebraic structures developed rapidly. Yuan and Lee [22] defined the fuzzy subgroup and fuzzy subring based on the theory of falling shadows. Liu [6] gave the definition of fuzzy invariant subgroups. By far, two books on fuzzy algebra have been published [8, 7]. The triangular norm, T-norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T-norm. Later, Hohle [5], Alsina et al. [2] introduced the T-norm into fuzzy set theory and suggested that the T-norm be used for the intersection of fuzzy sets. Since then, many other researchers have presented various types of T-norms for particular purposes [4, 21]. Anthony and Sherwood [3] gave the definition of fuzzy subgroup based on t-norm. A. Solairaju and R. Nagarajan [20] introduced the notion of Q- fuzzy groups. The author by using

2010 *Mathematics Subject Classification.* 08A72, 20XX, 47A30, 20N25, 20K27, 20K30.

Key words and phrases. Fuzzy algebraic structures, group theory, norms, Q-fuzzy subgroups, normal Q-fuzzy subgroups, homomorphisms.

Received: May 29, 2017 Accepted: Aug. 1, 2018 .

norms, investigated some properties of fuzzy submodules, fuzzy subrings, fuzzy ideals of subtraction semigroups, intuitionistic fuzzy subrings and ideals of a ring, fuzzy Lie algebra, fuzzy subgroups on direct product of groups, characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms, characterization of Q-fuzzy subrings (anti Q-fuzzy subrings) and fuzzy submodules of $R \times M$ ([9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). In this work, by using t-norm T, we introduce the notion of Q-fuzzy subgroups and normal Q-fuzzy subgroups of group G, and investigate some of their properties. Also we use a t-norm to construct Q-fuzzy subgroups in the finite direct product of Q-fuzzy subgroups. Finally we obtain some new results of Q-fuzzy subgroups and normal Q-fuzzy subgroups with respect to t-norm T under homomorphisms and anti-homomorphisms of groups.

2. Q-fuzzy subgroups with respect to T-norms and their properties

Definition 2.1. (See [7]) Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into $[0, 1]$. The set of all fuzzy subsets of G is called the $[0, 1]$-power set of G and is denoted $[0, 1]^G$.

Definition 2.2. (See [1]) A t-norm T is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(1) $T(x, 1) = x$ (neutral element),
(2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(3) $T(x, y) = T(y, x)$ (commutativity),
(4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$.

We say that T is idempotent if for all $x \in [0, 1]$,$T(x, x) = x$.

Example 2.1. *The basic t-norms are $T_m(x,y) = min\{x,y\}$,$T_b(x,y) = max\{0, x + y - 1\}$ and $T_p(x,y) = xy$, which are called standard intersection, bounded sum and algebraic product respectively for all $x, y \in [0,1]$.*

Definition 2.3. (See [20]) Let $(G, .)$ be a group and Q be a non empty set. $\mu \in [0,1]^{G\times Q}$ is said to be a Q-fuzzy subgroup of G if the following conditions are satisfied:
(1) $\mu(xy, q) \geq \mu(x, q) \wedge \mu(y, q)$,
(2) $\mu(x^{-1}, q) \geq \mu(x, q)$,
for all $x, y \in G$ and $q \in Q$. Throughout this paper the set of all Q-fuzzy subgroup of G will be denoted by $QFS(G)$.

Now we define Q-fuzzy subgroup of G with respect to t-norm T.

Definition 2.4. Let $(G, .)$ be a group and Q be a non empty set. $\mu \in [0,1]^{G\times Q}$ is said to be a Q-fuzzy subgroup of G with respect to t-norm T if the following conditions are satisfied:
(1) $\mu(xy, q) \geq T(\mu(x, q), \mu(y, q))$,
(2) $\mu(x^{-1}, q) \geq \mu(x, q)$,
for all $x, y \in G$ and $q \in Q$. Throughout this paper the set of all Q-fuzzy subgroup of G with respect to t-norm T will be denoted by $QFST(G)$.

Remark 1. The condition (2) of Definition 2.3 implies that $\mu(x, q) = \mu((x^{-1})^{-1}, q) \geq \mu(x^{-1}, q) \geq \mu(x, q)$ and then $\mu(x, q) = \mu(x^{-1}, q)$.

Lemma 2.1. *Let $\mu \in QFST(G)$ and T be idempotent. Then $\mu(e_G, q) \geq \mu(x, q)$ for all $x \in G$ and $q \in Q$.*

Proof. As $\mu \in QFST(G)$ so

$$\mu(e_G, q) = \mu(xx^{-1}, q) \geq T(\mu(x, q), \mu(x^{-1}, q)) = T(\mu(x, q), \mu(x, q)) = \mu(x, q)$$

for all $x \in G, q \in Q$. □

Proposition 2.1. *Let $\mu \in QFST(G)$ and T be idempotent. If $\mu(xy^{-1}, q) = \mu(e_G, q)$, then $\mu(x, q) = \mu(y, q)$ for all $x, y \in G$ and $q \in Q$.*

Proof. Let $x, y \in G$ and $q \in Q$. Then

$$\mu(x,q) = \mu(xy^{-1}y, q) \geq T(\mu(xy^{-1}, q), \mu(y,q)) = T(\mu(e_G, q), \mu(y,q)) \geq T(\mu(y,q), \mu(y,q))$$
$$= \mu(y,q) = \mu(yx^{-1}x, q) \geq T(\mu(yx^{-1}, q), \mu(x,q)) = T(\mu((xy^{-1})^{-1}, q), \mu(x,q))$$
$$= T(\mu(xy^{-1}, q), \mu(x,q)) = T(\mu(e_G, q), \mu(x,q)) \geq T(\mu(x,q), \mu(x,q)) = \mu(x,q).$$

Thus $\mu(x, q) = \mu(y, q)$. □

Proposition 2.2. *Let T be idempotent. Then $\mu \in QFST(G)$ if and only if $\mu(xy^{-1}, q) \geq T(\mu(x, q), \mu(y, q))$ for all $x, y \in G$ and $q \in Q$.*

Proof. Let $\mu \in QFST(G)$ and $x, y \in G, q \in Q$. Then $\mu(xy^{-1}, q) \geq T(\mu(x, q), \mu(y^{-1}, q)) \geq T(\mu(x, q), \mu(y, q))$.
Conversely, let $\mu(xy^{-1}, q) \geq T(\mu(x, q), \mu(y, q))$. Then $\mu(x^{-1}, q) = \mu(e_G x^{-1}, q) \geq T(\mu(e_G, q), \mu(x, q)) \geq T(\mu(x, q), \mu(x, q)) = \mu(x, q)$ and then $\mu(x^{-1}, q) \geq \mu(x, q)$.
Also $\mu(xy, q) = \mu(x(y^{-1})^{-1}, q) \geq T(\mu(x, q), \mu(y^{-1}, q)) \geq T(\mu(x, q), \mu(y, q))$. Hence $\mu \in QFST(G)$. □

Proposition 2.3. *Let $\mu \in [0,1]^{G \times Q}$ such that $\mu(e_G, q) = 1$ and $\mu(xy^{-1}, q) \geq T(\mu(x, q), \mu(y, q))$ for all $x, y \in G$ and $q \in Q$. Then $\mu \in QFST(G)$*

Proof. Let $x, y \in G$ and $q \in Q$. Then $\mu(x^{-1}, q) = \mu(e_G x^{-1}, q) \geq T(\mu(e_G, q), \mu(x, q)) = T(1, \mu(x, q)) = \mu(x, q)$ and so $\mu(x^{-1}, q) \geq \mu(x, q)$. Now $\mu(xy, q) = \mu(x((y)^{-1})^{-1}, q) \geq T(\mu(x, q), \mu(y^{-1}, q)) \geq T(\mu(x, q), \mu(y, q))$ and so $\mu(xy, q) \geq T(\mu(x, q), \mu(y, q))$. Thus $\mu \in QFST(G)$. □

Proposition 2.4. *If $\mu \in QFST(G)$, then $H = \{x \in G \mid f_{G \times Q}(x, q) = 1\}$ is a subgroup of G.*

Proof. Let $x, y \in H$ and $q \in Q$. As $\mu \in QFST(G)$, then $\mu(xy^{-1}, q) \geq T(\mu(x, q), \mu(y, q)) = T(1, 1) = 1$. This implies that $\mu(xy^{-1}, q) = 1$ and $xy^{-1} \in H$ and then H will be a subgroup of G. □

Proposition 2.5. *Let $\mu \in QFST(G)$ and T be idempotent. Then $H = \{x \in G \mid \mu(x, q) = \mu(e_G, q)\}$ is a subgroup of G.*

Proof. Let $x, y \in H, q \in Q$ and $\mu \in QFST(G)$. Then $\mu(xy^{-1}, q) \geq T(\mu(x, q), \mu(y, q)) = T(\mu(e_G, q), \mu(e_G, q)) = \mu(e_G, q) \geq \mu(xy^{-1}, q)$ and so $\mu(xy^{-1}, q) = \mu(e_G, q)$. Then $xy^{-1} \in H$ and hence H will be a subgroup of G. □

Proposition 2.6. *Let $\mu \in QFST(G)$ and $\mu(xy^{-1}, q) = 1$, then $\mu(x, q) = \mu(y, q)$ for all $x, y \in G$ and $q \in Q$.*

Proof. Assume that $\mu \in QFST(G)$ and $x, y \in G, q \in Q$. Then

$$\mu(x, q) = \mu(xy^{-1}y, q) \geq T(\mu(xy^{-1}, q), \mu(y, q)) = T(1, \mu(y, q)) = \mu(y, q)$$

$$= \mu(y^{-1}, q) = \mu(x^{-1}xy^{-1}, q) \geq T(\mu(x^{-1}, q), \mu(xy^{-1}, q))$$

$$= T(\mu(x^{-1}, q), 1) = \mu(x^{-1}, q) = \mu(x, q).$$

Hence $\mu(x, q) = \mu(y, q)$. □

Proposition 2.7. *Let $\mu \in QFST(G)$. Then $\mu(xy, q) = \mu(yx, q)$ if and only if $\mu(x, q) = \mu(y^{-1}xy, q)$ for all $x, y \in G$ and $q \in Q$.*

Proof. Let $x, y \in G, q \in Q$ and $\mu(xy, q) = \mu(yx, q)$. Then $\mu(y^{-1}xy, q) = \mu(y^{-1}(xy), q) = \mu(xyy^{-1}, q) = \mu(xe_G, q) = \mu(x, q)$. Conversely, let $\mu(x, q) = \mu(y^{-1}xy, q)$ then we obtain $\mu(xy, q) = \mu(x(yx)x^{-1}, q) = \mu(yx, q)$. □

Proposition 2.8. *Let $\mu \in QFST(G)$. If $\mu(xy^{-1}, q) = 0$, then either $\mu(x, q) = 0$ or $\mu(y, q) = 0$ for all $x, y \in G$ and $q \in Q$.*

Proof. As $\mu \in QFST(G)$ so for all $x, y \in G$ and $q \in Q$ we have that $0 = \mu(xy^{-1}, q) \geq T(\mu(x, q), \mu(y, q))$ and then either $\mu(x, q) = 0$ or $\mu(y, q) = 0$. □

Now we define the intersection of two Q-fuzzy subgroups of G with respect to t-norm T.

Definition 2.5. The intersection of $\mu, \nu \in [0, 1]^{G\times Q}$ is defined by $(\mu \cap \nu)(x, q) = T(\mu(x, q), \nu(x, q))$ for all $x \in G$ and $q \in Q$.

Lemma 2.2. *(See [1]) Let T be a t-norm. Then*

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

Proposition 2.9. *Let $\mu, \nu \in QFST(G)$. Then $\mu \cap \nu \in QFST(G)$.*

Proof. Let $x, y \in G, q \in Q$. Then

$$(\mu \cap \nu)(xy, q) = T(\mu(xy, q), \nu(xy, q)) \geq T(T(\mu(x, q), \mu(y, q)), T(\nu(x, q), \nu(y, q)))$$

$$= T(T(\mu(x, q), \nu(x, q)), T(\mu(y, q), \nu(y, q))) = T((\mu \cap \nu)(x, q), (\mu \cap \nu)(y, q)),$$

also $(\mu \cap \nu)(x^{-1}, q) = T(\mu(x^{-1}, q), \nu(x^{-1}, q)) \geq T(\mu(x, q), \nu(x, q)) = (\mu \cap \nu)(x, q)$. Hence $\mu \cap \nu \in QFST(G)$. □

Proposition 2.10. *Let $\mu \in QFST(G)$ and $x, y \in G, q \in Q$. If T be idempotent and $\mu(x, q) \neq \mu(y, q)$, then $\mu(xy, q) = T(\mu(x, q), \mu(y, q))$.*

Proof. Let $x, y \in G$ and $q \in Q$. From $\mu \in QFST(G)$ we have that $\mu(xy, q) \geq T(\mu(x, q), \mu(y, q))$. Now let $\mu(x, q) > \mu(y, q)$ then

$$\mu(xy, q) = T(\mu(xy, q), \mu(xy, q)) \leq T(\mu(x, q), \mu(xy, q))$$

$$\leq T(\mu(x^{-1}, q), \mu(xy, q)) \leq \mu(x^{-1}xy, q) = \mu(y, q)$$

$$= T(\mu(y, q), \mu(y, q)) \leq T(\mu(x, q), \mu(y, q))$$

and then $\mu(xy, q) \leq T(\mu(x, q), \mu(y, q))$. Thus $\mu(xy, q) = T(\mu(x, q), \mu(y, q))$. □

Definition 2.6. (See [20]) We say that $\mu \in QFS(G)$ is a normal if $\mu(xyx^{-1}, q) = \mu(y, q)$ for all $x, y \in G$ and $q \in Q$.

Now we define the normal Q-fuzzy subgroups of G with respect to t-norm T.

Definition 2.7. We say that $\mu \in QFST(G)$ is a normal if $\mu(xyx^{-1}, q) = \mu(y, q)$ for all $x, y \in G$ and $q \in Q$. We denote by $NQFST(G)$ the set of all normal Q-fuzzy subgroups of G with respect to t-norm T.

Proposition 2.11. *Let $\mu_1, \mu_2 \in NQFST(G)$. Then $\mu_1 \cap \mu_2 \in NQFST(G)$.*

Proof. If $x, y, \in G$ and $q \in Q$, then $(\mu_1 \cap \mu_2)(xyx^{-1}, q) = T(\mu_1(xyx^{-1}, q), \mu_2(xyx^{-1}, q)) = T(\mu_1(y, q), \mu_2(y, q)) = (\mu_1 \cap \mu_2)(y, q)$. □

Corollary 2.1. *Let $I_n = \{1, 2, ..., n\}$. If $\{\mu_i \mid i \in I_n\} \subseteq NQFST(G)$, Then $\mu = \cap_{i \in I_n} \mu_i \in NQFST(G)$.*

In the following we define the product of two Q-fuzzy subgroups of G with respect to t-norm T.

Definition 2.8. Let $(G, .), (H, .)$ be any two groups such that $\mu \in [0, 1]^{G \times Q}$ and $\nu \in [0, 1]^{H \times Q}$. The product of μ and ν, denoted by $\mu \times \nu \in [0, 1]^{(G \times H) \times Q}$, is defined as $(\mu \times \nu)((x, y), q) = T(\mu(x, q), \nu(y, q))$ for all $x \in G, y \in H, q \in Q$. Throughout this paper, H denotes an arbitrary group with identity element e_H.

Proposition 2.12. *If $\mu \in QFST(G)$ and $\nu \in QFST(H)$, then $\mu \times \nu \in QFST(G \times H)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in G \times H$ and $q \in Q$. Then

$$(\mu \times \nu)((x_1, y_1)(x_2, y_2), q) = (\mu \times \nu)((x_1x_2, y_1y_2), q)$$

$$= T(\mu(x_1x_2, q), \nu(y_1y_2, q)) \geq T(T(\mu(x_1, q), \mu(x_2, q)), T(\nu(y_1, q), \nu(y_2, q)))$$

$$= T(T(\mu(x_1, q), \nu(y_1, q)), T(\mu(x_2, q), \nu(y_2, q))) = T((\mu\times\nu)((x_1, y_1), q), (\mu\times\nu)((x_2, y_2), q)).$$

Also

$$(\mu \times \nu)((x_1, y_1)^{-1}, q) = (\mu \times \nu)((x_1^{-1}, y_1^{-1}), q) = T(\mu(x_1^{-1}, q), \nu(y_1^{-1}, q))$$

$$\geq T(\mu(x_1, q), \nu(y_1, q)) = (\mu \times \nu)((x_1, y_1), q).$$

Hence $\mu \times \nu \in QFST(G \times H)$. □

Proposition 2.13. *Let $\mu \in [0, 1]^{G\times Q}$ and $\nu \in [0, 1]^{H\times Q}$. If T be idempotent and $\mu \times \nu \in QFST(G \times H)$, then at least one of the following two statements must hold.*
(1) $\nu(e_H, q) \geq \mu(x, q)$, for all $x \in G$ and $q \in Q$,
(2) $\mu(e_G, q) \geq \nu(y, q)$, for all $y \in H$ and $q \in Q$.

Proof. Let none of the statements (1) and (2) holds, then we can find $g \in G$ and $h \in H$ such that $\mu(g, q) > \nu(e_H, q)$ and $\nu(h, q) > \mu(e_G, q)$. Now $(\mu \times \nu)((g, h), q) = T(\mu(g, q), \nu(h, q)) > T(\mu(e_G, q), \nu(e_H, q)) = (\mu\times\nu)((e_G, e_H), q)$ and it is contradiction with $\mu \times \nu \in QFST(G \times H)$ (Lemma 2.1). This complets the proof. □

Proposition 2.14. *Let $\mu \in [0, 1]^{G\times Q}$ and $\nu \in [0, 1]^{H\times Q}$. Moreover let $T = T_m(a, b) = min\{a, b\}$ for all $a, b \in [0, 1]$ and $\mu\times\nu \in QFST(G\times H)$. Then we obtain the following statements.*
(1) If $\mu(x, q) \leq \nu(e_H, q)$, then $\mu \in QFST(G)$ for all $x, y \in G$ and $q \in Q$.
(2) If $\nu(x, q) \leq \mu(e_G, q)$, then $\nu \in QFST(H)$ for all $x \in H$ and $q \in Q$.
(3) Either $\mu \in QFST(G)$ or $\nu \in QFST(H)$.

Proof. (1) Let $x, y \in G, q \in Q$ and $\mu(x, q) \leq \nu(e_H, q)$. Then

$$\mu(xy^{-1}, q) = T(\mu(xy^{-1}, q), \mu(xy^{-1}, q)) \geq T(\mu(xy^{-1}, q), \nu(e_H, q))$$

$$= T(\mu(xy^{-1}, q), \nu(e_He_H, q)) = (\mu \times \nu)((xy^{-1}, e_He_H), q) = (\mu \times \nu)((x, e_H)(y^{-1}, e_H), q)$$

$$\geq T((\mu\times\nu)((x, e_H), q), (\mu\times\nu)((y^{-1}, e_H), q)) \geq T((\mu\times\nu)((x, e_H), q), (\mu\times\nu)((y, e_H), q))$$

$$= T(T(\mu(x, q), \nu(e_H, q)), T(\mu(y, q), \nu(e_H, q))) \geq T(T(\mu(x, q), \mu(x, q)), T(\mu(y, q), \mu(y, q)))$$

$$= T(\mu(x, q), \mu(y, q)).$$

Now from Proposition 2.2 we obtain that $\mu \in QFST(G)$.

(2) Use similar method as in the proof of (1).

(3) Straight forward. □

3. PROPERTIES OF $QFST(G)$ AND $NQFST(G)$ UNDER HOMOMORPHISMS AND ANTI-HOMOMORPHISMS

In this section we prove many results in homomorphism and anti-homomorphism in Q-fuzzy subgroups.

Definition 3.1. (See [7]) Let $(G, .), (H, .)$ be any two groups. The function $f : G \to H$ is called a homomorphism (anti-homomorphism) if $f(xy) = f(x)f(y)(f(xy) = f(y)f(x))$, for all $x, y \in G$.

Example 3.1. *(1) Consider the cyclic group* $\dfrac{\mathbf{Z}}{3\mathbf{Z}} = \{\bar{0}, \bar{1}, \bar{2}\}$ *and the group of integers* $\mathbf{Z}$ *with addition. The map* $h : \mathbf{Z} \to \dfrac{\mathbf{Z}}{3\mathbf{Z}}$ *with* $h(u) = \bar{u}$ *is a group homomorphism.*

(2) In ring theory, an antihomomorphism is a map between two rings that preserves addition, but reverses the order of multiplication. So $\varphi : X \to Y$ *is a ring antihomomorphism if and only if:*

(1) $\varphi(1) = 1$,

(2) $\varphi(x + y) = \varphi(x) + \varphi(y)$,

(3) $\varphi(xy) = \varphi(y)\varphi(x)$,
for all $x, y \in X$.

Definition 3.2. (See [7]) Let $\varphi : A \to B$ be a function such that $\mu \in [0,1]^{A\times Q}$ and $\nu \in [0,1]^{B\times Q}$. Then fuzzy image $\varphi(\mu)$ of μ under φ is defined by

$$\varphi(\mu)(y,q) = \begin{cases} \sup\{\mu(x,q) \mid (x,q) \in A \times Q, \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \emptyset \\ 0 & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

and fuzzy pre-image (or fuzzy inverse image) of ν under φ is $\varphi^{-1}(\nu)(x,q) = \nu(\varphi(x), q)$ for all $(x,q) \in A \times Q$.

Proposition 3.1. *Let φ be an epimorphism from group G into group H. If $\mu \in QFST(G)$, then $\varphi(\mu) \in QFST(H)$.*

Proof. Let $h_1, h_2 \in H$ and $q \in Q$. Then

$$\varphi(\mu)(h_1h_2, q) = \sup\{\mu(g_1g_2, q) \mid g_1, g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\}$$

$$\geq \sup\{T(\mu(g_1, q), \mu(g_2, q)) \mid g_1, g_2 \in G, \varphi(g_1) = h_1, \varphi(g_2) = h_2\}$$

$$= T((\sup\{\mu(g_1, q) \mid g_1 \in G, \varphi(g_1) = h_1\}), (\sup\{\mu(g_2, q) \mid g_2 \in G, \varphi(g_2) = h_2\}))$$

$$= T(\varphi(\mu)(h_1, q), \varphi(\mu)(h_2, q)).$$

Also

$$\varphi(\mu)(h_1^{-1}, q) = \sup\{\mu(g_1^{-1}, q) \mid g_1 \in G, \varphi(g_1^{-1}) = h_1^{-1}\}$$

$$\geq \sup\{\mu(g_1, q) \mid g_1 \in G, \varphi(g_1, q) = h_1\} = \varphi(\mu)(h_1, q).$$

Therefore $\varphi(\mu) \in QFST(H)$. □

Proposition 3.2. *Let φ be a homorphism from group G into group H. If $\nu \in QFST(H)$, then $\varphi^{-1}(\nu) \in QFST(G)$.*

Proof. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$\varphi^{-1}(\nu)(g_1g_2, q) = \nu(\varphi(g_1g_2), q) = \nu(\varphi(g_1)\varphi(g_2), q)$$

$$\geq T(\nu(\varphi(g_1), q), \nu(\varphi(g_2), q)) = T(\varphi^{-1}(\nu)(g_1, q), \varphi^{-1}(\nu)(g_2, q)).$$

Moreover $\varphi^{-1}(\nu)(g_1^{-1}, q) = \nu(\varphi(g_1^{-1}), q) = \nu(\varphi^{-1}(g_1), q) \geq \nu(\varphi(g_1), q) = \varphi^{-1}(\nu)(g_1, q)$. Then $\varphi^{-1}(\nu) \in QFST(G)$. □

Proposition 3.3. *Let φ be an anti-homorphism from group G into group H. If $\nu \in QFST(H)$, then $\varphi^{-1}(\nu) \in QFST(G)$.*

Proof. Let $g_1, g_2 \in G$ and $q \in Q$. Then

$$\varphi^{-1}(\nu)(g_1g_2, q) = \nu(\varphi(g_1g_2), q) = \nu(\varphi(g_2)\varphi(g_1), q) \geq T(\nu(\varphi(g_2), q), \nu(\varphi(g_1), q))$$

$$= T(\varphi^{-1}(\nu)(g_2, q), \varphi^{-1}(\nu)(g_1, q)) = T(\varphi^{-1}(\nu)(g_1, q), \varphi^{-1}(\nu)(g_2, q)).$$

Also $\varphi^{-1}(\nu)(g_1^{-1}, q) = \nu(\varphi(g_1^{-1}), q) = \nu(\varphi^{-1}(g_1), q) \geq \nu(\varphi(g_1), q) = \varphi^{-1}(\nu)(g_1, q)$. Thus $\varphi^{-1}(\nu) \in QFST(G)$. □

Proposition 3.4. *Let $\mu \in NQFST(G)$ and H be a group. Suppose that φ is an epimorphism of G onto H. Then $f(\mu) \in NQFST(H)$.*

Proof. By Proposition 3.1 we have $\varphi(\mu) \in QFST(H)$. Let $x, y \in H$ and $q \in Q$. Since φ is a surjection, $\varphi(u) = x$ for some $u \in G$. Then

$$\varphi(\mu)(xyx^{-1}, q) = \sup\{\mu(w, q) \mid w \in G, \varphi(w) = xyx^{-1}\}$$

$$= \sup\{\mu(u^{-1}wu, q) \mid w \in G, \varphi(u^{-1}wu) = y\} = \sup\{\mu(w, q) \mid w \in G, \varphi(w) = y\}$$

$$= \varphi(\mu)(y, q).$$

□

Proposition 3.5. *Let H be a group and $\nu \in NQFST(H)$. Suppose that φ is a homomorphism of G into H. Then $\varphi^{-1}(\nu) \in NQFST(G)$.*

Proof. By Proposition 3.2 we obtain that $\varphi^{-1}(\nu) \in QFST(G)$. Now for any $x, y \in G$ and $q \in Q$ we obtain $\varphi^{-1}(\nu)(xyx^{-1}, q) = \nu(\varphi(xyx^{-1}), q) = \nu(\varphi(x)\varphi(y)\varphi(x^{-1}), q) = \nu(\varphi(x)\varphi(y)\varphi^{-1}(x), q) = \nu(\varphi(y), q) = \varphi^{-1}(\nu)(y, q)$. Therefore $\varphi^{-1}(\nu) \in NQFST(G)$. □

Conclusion

In this study, we define Q-fuzzy subgroups and normal Q-fuzzy subgroups of groups under t-norm T and investigated some of their properties and structured characteristics. One can investigate this concept in rings theory and obtains some new resuls as discussing First, Second, Third Isomorphism Theorems.

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