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IRREDUCIBLE & STRONGLY IRREDUCIBLE BI-IDEALS OF Γ-SO-RINGS

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Abstract. The set of all partial functions over a set under a natural addition (disjoint-domain sum), functional composition and functional relation on them, forms a Γ-so-ring. In this paper we introduce the notions of irreducible bi-ideal, strongly irreducible bi-ideal and strongly prime bi-ideals of Γ-so-rings and we prove that a bi-ideal is strongly irreducible if and only if it is strongly prime in a class of Γ-so-rings.

1. Introduction

In 2014, by extending the binary operation addition in Γ-semirings to partially defined infinitary operation Σ, M. Siva Mala[10], introduced the notion of a partial Γ-semiring as a common generalization of partial semiring by Arbib, manes and Benson[3],[4] and Γ-semiring. Also the author developed the ideal theory for the Γ-so-rings[11] to [16]. In [17] and [18], we introduced the notions of bi-ideal, prime & semiprime bi-ideals in Γ-so-rings and obtained various characteristics of them. In this paper, we introduce the notions of irreducible, strongly irreducible and strongly prime bi-ideals of Γ-so-rings and obtained characterizations of prime, semiprime, irreducible and strongly irreducible bi-ideals in regular Γ-so-rings.

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2. Preliminaries

In this section we collect important definitions from the literature.

Let M be a nonempty set, and let I be a set. An $I - indexed\ family$ in M is a function $x : I \rightarrow M$. Such a family is denoted by $(x_i : i \in I)$, where $x_i = ix$ for each i in I. The *cardinality* of the family $(x_i : i \in I)$ is the cardinality of its index set I. Two families $(x_i : i \in I)$ and $(y_k : k \in K)$ are *isomorphic* if there is a bijection $\sigma : I \rightarrow K$ with $y_{i\sigma} = x_i$ for each i in I. A $sub\ family$ of $(x_i : i \in I)$ is a family $(x_j : j \in J)$ such that $J \subseteq I$. The $empty\ family$ is the unique such family indexed by $\emptyset$.

Now let us consider an infinitary operation Σ which takes families in M to elements of M, but which may not be defined for all families in M. By "infinitary", we mean that Σ may be applied to a family $(x_i : i \in I)$ in M, for which the cardinality of the index set I is infinite. Since $\Sigma(x_i : i \in I)$ need not be defined for an arbitrary family $(x_i : i \in I)$ in M, Σ is said to be *partially-defined.* A family $(x_i : i \in I)$ in M is said to be *summable* if $\Sigma(x_i : i \in I)$ is defined and is in M. We use the notations $\Sigma(x_i : i \in I)$, and $\Sigma_i x_i$ interchangebly.

Definition 2.1. [4] *A* partial monoid *is a pair* (M, Σ) *where* M *is a nonempty set and* Σ *is a partial addition defined on some, but not necessarily all, families* $(x_i : i \in I)$ *in* M *subject to the following axioms:*
(1) Unary Sum Axiom. If $(x_i : i \in I)$ *is a one element family in* M *and* $I = \{j\}$, *then* $\Sigma(x_i : i \in I)$ *is defined and equals* x_j.
(2) Partition-Associativity Axiom. If $(x_i : i \in I)$ *is a family in* M *and* $(I_j : j \in J)$ *is a partition of* I, *then* $(x_i : i \in I)$ *is summable if and only if* $(x_i : i \in I_j)$ *is summable for every* j *in* J, $(\Sigma(x_i : i \in I_j) : j \in J)$ *is summable, and* $\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j) : j \in J)$.

Example 2.2. [4] *Let D and E be two sets and let the set of all partial functions from D to E be denoted by $Pfn(D,E)$. A family $(x_i : i \in I)$ is summable if and only if for i,j in I, and $i \neq j$, $dom(x_i) \cap dom(x_j) = \emptyset$. If $(x_i : i \in I)$ is summable, then for any d in D*

$$d(\Sigma_i x_i) = \begin{cases} dx_i, \ if \ d \in dom(x_i) \ for \ some \ (necessarily \ unique) \ i \in I; \\ undefined, \ otherwise. \end{cases}$$

Then $(Pfn(D,E),\Sigma)$ is a partial monoid.

Definition 2.3. [4] *Let (M,Σ) and (M',Σ') be partial monoids. Then (M',Σ') is said to be a* partial submonoid *of (M,Σ) if it satisfies the following:*
(1) M' is a subset of M, and
(2) $(x_i : i \in I)$ is a summable family in M' implies that $(x_i : i \in I)$ is a summable family in M and $\Sigma'_i x_i = \Sigma_i x_i$.

Definition 2.4. [10] *Let (R,Σ) and (Γ,Σ') be two partial monoids. Then R is said to be a* partial Γ-semiring *if there exists a mapping $R \times \Gamma \times R \longrightarrow R$ (images to be denoted by $x\gamma y$ for $x,y \in R$ and $\gamma \in \Gamma$) satisfying the following axioms for all $x,y,z,(x_i : i \in I)$ in R and $\mu,\gamma,$ $(\gamma_i : i \in I)$ in Γ*
(1) $x\gamma(y\mu z) = (x\gamma y)\mu z$,
(2) a family $(x_i : i \in I)$ is summable in R implies $(x\gamma x_i : i \in I)$ is summable in R and $x\gamma[\Sigma(x_i : i \in I)] = \Sigma(x\gamma x_i : i \in I)$, $[\Sigma(x_i : i \in I)]\gamma x = \Sigma(x_i\gamma x : i \in I)$,
(3) a family $(\gamma_i : i \in I)$ is summable in Γ implies $(x\gamma_i y : i \in I)$ is summable in R and $x[\Sigma'(\gamma_i : i \in I)]y = \Sigma(x\gamma_i y : i \in I)$.

Definition 2.5. [10] *Let R be a partial Γ-semiring. Let A be a nonempty subset of R and Γ' be a nonempty subset of Γ. Then the pair (A,Γ') is said to be a* partial sub Γ-semiring *if*
(i) A is a partial submonoid of R,

(ii) Γ' *is a partial submonoid of* Γ, *and*

(iii) $A\Gamma' A \subseteq A$.

Definition 2.6. [20] *The* sum ordering $\leq$ *on a partial monoid* (M, Σ) *is the binary relation such that* $x \leq y$ *if and only if there exists an element* h *in* M *such that* $y = x + h$ *for* $x, y \in M$.

It is trivial to observe that the sum ordering is a quasi order (i.e., reflexive and transitive).

Definition 2.7. [20] *A* sum-ordered partial monoid *or* so-monoid, *in short, is a partial monoid in which the sum ordering is a partial ordering.*

Definition 2.8. [10] *A partial* Γ*-semiring* R *is said be a* sum-ordered partial Γ-semiring *(in short* Γ*-so-ring) if the partial monoids* R *and* Γ *are so-monoids.*

The *support* of a family $(x_i : i \in I)$ in M is defined to be the subfamily $(x_i : i \in J)$ where $J = \{i \in I \mid x_i \neq 0\}$.

Definition 2.9. [20] *A partial semiring is a quadruple* $(R, \Sigma, \cdot, 1)$, *where* (R, Σ) *is a partial monoid with partial addition* Σ, $(R, \cdot, 1)$ *is an monoid with multiplicative operation* $\cdot$ *and unit* 1, *and the additive and multiplicative structures obey the following distributive laws.*

If $\Sigma(x_i : i \in I)$ *is defined in* R, *then for all* y *in* R, $\Sigma(y{\cdot}x_i : i \in I)$ *and* $\Sigma(x_i{\cdot}y : i \in I)$ *are defined and* $y \cdot [\Sigma_i x_i] = \Sigma_i(y \cdot x_i)$; $[\Sigma_i x_i] \cdot y = \Sigma_i(x_i \cdot y)$.

Definition 2.10. [4] *A sum ordered partial semiring or so-ring for short, is a partial semiring in which the sum ordering is a partial ordering.*

Example 2.11. [10] *Let* $R = \Gamma := Z^- \bigcup \{0\}$, *the set of all nonpositive integers. Then* R *and* Γ *are partial monoids with finite support addition. Now* R *is a partial*

Γ-semiring with usual multiplication of integers. Also R is a Γ-so-ring with the partial order "usual less than or equal to". However R is not a so-ring. Since $-2, -3 \in R$ and $(-2)(-3) = 6 \notin R$.

Example 2.12. [10] *Let D, E be any two sets. Then $Pfn(D, E)$ and $Pfn(E, D)$ are partial monoids with the summations defined as in the Example 2.2. Consider the mapping $(f, \gamma, g) \mapsto f\gamma g$ of $Pfn(D, E) \times Pfn(E, D) \times Pfn(D, E)$ into $Pfn(D, E)$ where $d(f\gamma g) = (((df)\gamma)g)$, for any $d \in D$. Then $Pfn(D, E)$ is a partial $Pfn(E, D)$-semiring.*

In general $Pfn(D, E)$ is not a $Pfn(E, D)$-semiring, since a family in the partial $Pfn(E, D)$-semiring $Pfn(D, E)$ need not be summable.

Definition 2.13. [10] *A nonempty subset A of a Γ-so-ring R is said to be* Γ-subso-ring *if*
(i) A is a subso-monoid of R, i.e., A is closed under the partial addition defined on R, and
(ii) $A\Gamma A \subseteq A$.

Definition 2.14. [10] *A partial Γ-semiring R is said to have a* left (right) unity *if there exists a family $(e_i : i \in I)$ of elements of R and a family $(\gamma_i : i \in I)$ of elements of Γ such that $\Sigma_i e_i\gamma_i r = r$ ($\Sigma_i r\gamma_i e_i = r$) for any r in R.*

Definition 2.15. [12] *A Γ-so-ring R is said to be a* complete Γ-so-ring *if every family of elements in R is summable and every family of elements in Γ is summable.*

Definition 2.16. [12] *Let R be a partial Γ-semiring, A be a nonempty subset of R and Ω be a nonempty subset of Γ. Then the pair (A, Ω) of (R, Γ) is said to be a* left (right) partial Γ-ideal *of R if it satisfies the following:*
(i) $(x_i : i \in I)$ is a summable family in R and $x_i \in A$ $\forall i \in I$ implies $\Sigma_i x_i \in A$,

(ii) $(\alpha_i : i \in I)$ *is a summable family in* Γ *and* $\alpha_i \in \Omega$ $\forall i \in I$ *implies* $\Sigma_i \alpha_i \in \Omega$*, and*
(iii) for all $x \in R,\ y \in A$ *and* $\alpha \in \Omega$*,* $x\alpha y \in A$ $(y\alpha x \in A)$*.*

If (A, Ω) is both left and right partial Γ-ideal of a partial Γ-semiring R, then (A, Ω) is called a *partial* Γ*-ideal* of R. If $\Omega = \Gamma$, then A is called a *partial ideal* of R.

Definition 2.17. [12] *Let* R *be a* Γ*-so-ring,* A *be a nonempty subset of* R *and* Ω *be a nonempty subset of* Γ*. Then the pair* (A, Ω) *is said to be a* left (right) Γ-ideal *of* R *if it satisfies the following:*
(i) (A, Ω) *is a left (right) partial* Γ*-ideal of* R*,*
(ii) for any $x \in R$ *and* $y \in A$ *such that* $x \leq y$ *implies* $x \in A$*, and*
(iii) for all $\alpha \in \Gamma$ *and* $\beta \in \Omega$ *such that* $\alpha \leq \beta$ *implies* $\alpha \in \Omega$*.*

If (A, Ω) is both left and right Γ-ideal of a Γ-so-ring R, then (A, Ω) is called a Γ*-ideal* of R. If $\Omega = \Gamma$, then A is called an *ideal* of (R, Γ).

Definition 2.18. [12] *Let* R *be a* Γ*-so-ring. If* A, B *are subsets of* R *and* Γ_1 *is a subset of* Γ*, then we define* $A\Gamma_1 B$ *as the set* $\{x \in R \mid\ \exists\ a_i \in A, \gamma_i \in \Gamma_1, b_i \in B, \Sigma_i a_i \gamma_i b_i$ *exists and* $x \leq \Sigma_i a_i \gamma_i b_i\}$*.*

If $A = \{a\}$ then we also denote $A\Gamma_1 B$ by $a\Gamma_1 B$. If $B = \{b\}$ then we also denote $A\Gamma_1 B$ by $A\Gamma_1 b$. Similarly if $A = \{a\}$ and $B = \{b\}$, we denote $A\Gamma_1 B$ by $a\Gamma_1 b$ and thus $a\Gamma_1 b = \{x \in R \mid x \leq a\gamma b$ for some $\gamma \in \Gamma_1\}$. Also, if A is a left ideal and B is a right ideal of R, then $A\Gamma R\Gamma B = \{x \in R \mid x \leq \Sigma_i a_i \alpha_i r_i \beta_i b_i$ for some $a_i \in A,\ b_i \in B,\ r_i \in R,\ \alpha_i,\ \beta_i \in \Gamma\}$.

Definition 2.19. [16] *Let* R *be a* Γ*-so-ring. An element 'a' of* R *is said to be* regular *if* $a \in a\Gamma R\Gamma a$*. If every element of* R *is regular then* R *is called a regular* Γ*-so-ring.*

Lemma 2.20. [16] *Let* R *be a complete* Γ*-so-ring with left unity. Then* R *is regular if and only if* $B\Gamma A = A \bigcap B$ *for any left ideal* A *and right ideal* B *of* R*.*

Definition 2.21. [17] *Let R be a Γ-so-ring. A Γ-subso-ring B of R is said to be a* bi-ideal *of R if and only if $B\Gamma R\Gamma B \subseteq B$.*

Theorem 2.22. [17] *Let R be a complete Γ-so-ring and A be a nonempty subset of R. Then the bi-ideal of R generated by A is $< A >_b = \{x \in R \mid x \leq \Sigma_i x_i + \Sigma_j x_j \alpha_j x'_j + \Sigma_k x''_k \alpha'_k r_k \alpha''_k x'''_k$, where $x_i, x_j, x'_j, x''_k, x'''_k \in A, \alpha_j, \alpha'_k, \alpha''_k \in \Gamma, r_k \in R\}$.*

Definition 2.23. [18] *Let R be a Γ-so-ring and P be a proper bi-ideal of R. Then P is called a* prime bi-ideal *of R if and only if for any bi-ideals A, B of R, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

Theorem 2.24. [18] *Let R be a complete Γ-so-ring and P be a proper bi-ideal of R. Then the following conditions are equivalent:*
(i) P is a prime bi-ideal of R.
(ii) If A is a right ideal and B is a left ideal of R such that $A\Gamma B \subseteq P$ then $A \subseteq P$ or $B \subseteq P$.

Definition 2.25. [18] *A proper bi-ideal P of a Γ-so-ring R is said to be a* semiprime bi-ideal *of R if and only if for any bi-ideal A of R, $A\Gamma A \subseteq P$ implies $A \subseteq P$.*

Theorem 2.26. [18] *Let R be a complete Γ-so-ring and P be a proper bi-ideal of R. Then the following conditions are equivalent:*
(i) P is semiprime bi-ideal of R.
(ii) If A is a left (right) ideal of R such that $A\Gamma A \subseteq P$ then $A \subseteq P$.

3. Irreducible & Strongly Irreducible Bi-Ideals

We introduce the notions of irreducible and strongly irreducible bi-ideals in Γ-so-rings as follows:

Definition 3.1. *A bi-ideal P of a Γ-so-ring R is said to be* irreducible bi-ideal *if and only if for any bi-ideals H and K of R, $H \bigcap K = P$ implies $H = P$ or $K = P$.*

Definition 3.2. *A bi-ideal P of a Γ-so-ring R is said to be* strongly irreducible bi-ideal *if and only if for any bi-ideals H and K of R, $H\bigcap K \subseteq P$ implies $H \subseteq P$ or $K \subseteq P$.*

Clearly every strongly irreducible bi-ideal of R is irreducible bi-ideal. The following is an example of an irreducible bi-ideal which is not a strongly irreducible bi-ideal of a Γ-so-ring.

Example 3.3. *Consider the Γ-so-ring R as in the Example 3.3 of* [16]. *In that example $R = \{0, a, b, c, d, e\}$, Σ is defined on R as*

$$\Sigma_i x_i = \begin{cases} x_j,\ if\ x_i = 0\ \forall i \neq j,\ for\ some\ j, \\ d,\ if\ (x_j = a,\ x_k = b\ or\ x_j = b,\ x_k = c\ for\ some\ j,\ k)\ and\ x_i = 0\ \forall i \neq j, k, \\ undefined,\ otherwise. \end{cases}$$

Then R is a so-monoid.

And $\Gamma = \{0', 1'\}$, Σ' is defined on Γ as

$$\Sigma'_i \alpha_i = \begin{cases} 1',\ if\ \alpha_i = 0'\ \forall i \neq j\ for\ some\ j \\ undefined,\ otherwise. \end{cases}$$

Then Γ is a so-monoid.

The mapping $R \times \Gamma \times R \to R$ is defined as follows:

$0'$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	0	0
e	0	0	0	0	0	0

$1'$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
e	0	a	b	c	d	e

Then R is a Γ-so-ring. For the bi-ideals $A = \{0, a\}$, $B = \{0, b\}$ and $C = \{0, c\}$ of

R, $B \bigcap C = \{0, b\} \bigcap \{0, c\} = \{0\} \subset A$ and $B \nsubseteq A$, $C \nsubseteq A$. Hence $A = \{0, a\}$ is not a strongly irreducible bi-ideal of R. However the bi-ideal $A = \{0, a\}$ is an irreducible bi-ideal of R, since there are no bi-ideals H, K other than $A = \{0, a\}$ such that $H \bigcap K = A$.

Theorem 3.4. *If P is a bi-ideal of a complete Γ-so-ring R then the following conditions are equivalent:*
(i) P is a strongly irreducible bi-ideal of R.
(ii) If x, y be elements of R such that $< x >_b \bigcap < y >_b \subseteq P$ then $x \in P$ or $y \in P$.

Proof. (i)⇒(ii): Suppose P is a strongly irreducible bi-ideal of R. Let x, y be elements of R such that $< x >_b \bigcap < y >_b \subseteq P$. Since P is strongly irreducible bi-ideal, $< x >_b \subseteq P$ or $< y >_b \subseteq P$. Hence $x \in P$ or $y \in P$.

(ii)⇒(i): Suppose x, y be elements of R such that $< x >_b \bigcap < y >_b \subseteq P$ implies $x \in P$ or $y \in P$. Let H, K be ideals of R such that $H \bigcap K \subseteq P$. Suppose that $H \nsubseteq P$. Then there exists an element $x \in H$ such that $x \notin P$. Let $y \in K$. Then $< x >_b \subseteq H$ and $< y >_b \subseteq K$, and thus $< x >_b \bigcap < y >_b \subseteq H \bigcap K \subseteq P$. Then by assumption $x \in P$ or $y \in P$. Since $x \notin P$, $y \in P$ and so $K \subseteq P$. Hence P is a strongly irreducible bi-ideal of R. □

Definition 3.5. *Let R be a Γ-so-ring and P be a proper bi-ideal of R. Then P is said to be a* strongly prime bi-ideal *of R if and only if for any bi-ideals A, B of R, $(A\Gamma B) \bigcap (B\Gamma A) \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.*

Example 3.6. *Let $R := [0, 1]$ be the unit interval of real numbers. For any family $(x_i : i \in I)$ in R, define $\Sigma(x_i : i \in I) = sup\{x_i \mid i \in I\}$. Then (R, Σ) is a so-monoid. Let $\Gamma := \mathbb{N} \bigcup \{0\}$, the set of all nonnegative integers. Then Γ is a so-monoid with finite support addition. Consider the mapping $(x, \alpha, y) \mapsto Inf\{x, \alpha, y\}$ of $R \times \Gamma \times R$ into R. Then R is a Γ-so-ring and every bi-ideal of R is in the form of $[0, a]$ for* *some* $a \in [0,1]$. *Let* $x \in R$. *Take* $P := [0,x]$. *Let* $A,\ B$ *be bi-ideals of* R *such that* $(A\Gamma B)\bigcap(B\Gamma A) \subseteq P$. *Then there exists* $y,\ z \in R$ *such that* $A = [0,y]$ *and* $B = [0,z]$. *Now* $A\Gamma B = [0,y]\Gamma[0,z] = [0, inf\{y,\alpha,z\}]$ *and* $B\Gamma A = [0,z]\Gamma[0,y] = [0, inf\{z,\alpha,y\}]$ *for every* $\alpha \in \Gamma$ *and so,* $[0, inf\{y,\alpha,z\}] \subseteq P = [0,x]$. *Then either* $y \leq x$ *or* $z \leq x$. *Thus* $A = [0,y] \subseteq [0,x] = P$ *or* $B = [0,z] \subseteq [0,x] = P$. *Hence* $P = [0,x]$ *is a strongly prime bi-ideal of* R.

Every strongly prime bi-ideal of R is a prime bi-ideal of R.

Theorem 3.7. *Every strongly irreducible and semiprime bi-ideal of a* Γ*-so-ring* R *is a strongly prime bi-ideal of* R.

Proof. Let P be a strongly irreducible and semiprime bi-ideal of R. Let $A,\ B$ be any bi-ideals of R such that $(A\Gamma B)\bigcap(B\Gamma A) \subseteq P$. Clearly $A\bigcap B$ is a bi-ideal of R, $(A\bigcap B)\Gamma(A\bigcap B) \subseteq A\Gamma B$ and $(A\bigcap B)\Gamma(A\bigcap B) \subseteq B\Gamma A$. Thus $(A\bigcap B)\Gamma(A\bigcap B) \subseteq (A\Gamma B)\bigcap(B\Gamma A) \subseteq P$. Since P is semiprime, $A\bigcap B \subseteq P$. Since P is strongly irreducible, $A \subseteq P$ or $B \subseteq P$. Hence P is a strongly prime bi-ideal of R. □

Theorem 3.8. *If* P *is a bi-ideal of* R *and* $a \in R$ *such that* $a \notin P$. *Then there exists an irreducible bi-ideal* I *of* R *such that* $P \subseteq I$ *and* $a \notin I$.

Proof. Let $\mathcal{C} = \{B \mid B$ is a bi-ideal of $R, P \subseteq B$ and $a \notin B\}$. Clearly $P \in \mathcal{C}$. Moreover $\mathcal{C}$ is a nonempty partially ordered set with set inclusion of subsets. Let $\{B_i \mid i \in \Delta\}$ be an ascending chain of bi-ideals of $\mathcal{C}$. Take $B' = \bigcup_{i\in\Delta} B_i$. Since $\{B_i \mid i \in \Delta\}$ is an ascending chain of bi-ideals of $\mathcal{C}$, B' is a bi-ideal of R, $P \subseteq B'$ and $a \notin B'$. This implies that $B' \in \mathcal{C}$. Hence B' is an upper bound of $\{B_i \mid i \in \Delta\}$ in $\mathcal{C}$. Then by Zorn's lemma $\mathcal{C}$ has a maximal element, let it be I.

Now we prove that I is an irreducible bi-ideal of R. Let $H,\ K$ be any two bi-ideals of R such that $H\bigcap K = I$. Then $I = H\bigcap K \subseteq H, K$. Suppose if $I \subset H$ and $I \subset K$. Then by the maximality of I, $a \in H$ and $a \in K$ and hence $a \in H\bigcap K = I$, a contradiction. Therefore $H = I$ or $K = I$. Hence I is an irreducible bi-ideal of R such that $P \subseteq I$ and $a \notin I$. □

Theorem 3.9. *Any proper bi-ideal P of R is the intersection of all irreducible bi-ideals of R containing P.*

Proof. Let $\mathcal{C} = \{B_i \mid B_i$ is an irreducible bi-ideal of R and $P \subseteq B_i,\ i \in \Delta\}$. Take $B = \bigcap_{i\in\Delta} B_i$. Then $P \subseteq B$. Suppose that $a \notin P$. Then by Theorem 3.8, there exists an irreducible bi-ideal I of R such that $P \subseteq I$ and $a \notin I$. Then $I \in \mathcal{C}$ and $a \notin B$. Hence $B \subseteq P$, and hence P is the intersection of all irreducible bi-ideals of R containing P. □

Remark 3.10. *If R is a regular complete Γ-so-ring and a be any element of R then $< a >_b = a\Gamma R\Gamma a$.*

Proof. Note that $< a >_b = \{x \in R \mid x \leq \Sigma_n a + \Sigma_j a\alpha_j a + \Sigma_k a\alpha'_k r_k \alpha''_k a,$ where $\alpha_j, \alpha'_k, \alpha''_k \in \Gamma,\ r_k \in R\}$. Let $x \in < a >_b$. Then $x \leq \Sigma_n a + \Sigma_j a\alpha_j a + \Sigma_k a\alpha'_k r_k \alpha''_k a$, where $\alpha_j, \alpha'_k, \alpha''_k \in \Gamma,\ r_k \in R$. Since R is a regular Γ-so-ring, $a \in a\Gamma R\Gamma a$. This implies that $\Sigma_n a \in a\Gamma R\Gamma a$, $\Sigma_j a\alpha_j a \in a\Gamma a\Gamma R\Gamma a \subseteq a\Gamma R\Gamma a$ and $\Sigma_k a\alpha'_k r_k \alpha''_k a \in a\Gamma R\Gamma a$ and hence $< a >_b \subseteq a\Gamma R\Gamma a$. Let $x \in a\Gamma R\Gamma a$. Then $x \leq \Sigma_i a\alpha_i r_i \beta_i a$ for some $\alpha_i,\ \beta_i \in \Gamma$ and $r_i \in R$. $\Rightarrow x \in < a >_b$ and hence $a\Gamma R\Gamma a \subseteq < a >_b$. Hence $< a >_b = a\Gamma R\Gamma a$. □

Definition 3.11. *A Γ-so-ring R is said to be an* intra-regular Γ-so-ring *if for any element a in R, $a \in R\Gamma a\Gamma a\Gamma R$.*

Theorem 3.12. *In a complete Γ-so-ring R with left unity the following statements are equivalent:*

(1) R is regular and intra-regular.

(2) For any bi-ideal B of R, $B\Gamma B = B$.

(3) For any bi-ideals $A,\ B$ of R, $A\bigcap B = (A\Gamma B)\bigcap(B\Gamma A)$.

(4) Every bi-ideal of R is semiprime.

(5) Each proper bi-ideal of R is the intersection of irreducible semiprime bi-ideals of R which contain it.

Proof. We prove the equivalence of the statements as (1)⇔(2)⇒(3)⇒(4)⇒(5)⇒(2).

(1)⇒(2): Suppose R is regular and intra-regular. Let B be any bi-ideal of R. Since B is a Γ-subso-ring of R, $B\Gamma B \subseteq B$. Let $a \in B$. Since R is regular and intra-regular, $a \in a\Gamma R\Gamma a$ and $a \in R\Gamma a\Gamma a\Gamma R$. Now $a \in a\Gamma R\Gamma a \subseteq (a\Gamma R)\Gamma(a\Gamma R\Gamma a)$ $\subseteq (a\Gamma R)\Gamma(R\Gamma a\Gamma a\Gamma R)\Gamma R\Gamma a = (a\Gamma(R\Gamma R)\Gamma a)\Gamma(a\Gamma(R\Gamma R)\Gamma a) \subseteq (a\Gamma R\Gamma a)\Gamma(a\Gamma R\Gamma a)$ $\subseteq (B\Gamma R\Gamma B)\Gamma(B\Gamma R\Gamma B)$ (since $a \in B$) $\subseteq B\Gamma B$ (Since B is a bi-ideal of R). This implies that $B \subseteq B\Gamma B$ and hence $B\Gamma B = B$.

(2)⇒(1): Suppose $B\Gamma B = B$ for any bi-ideal B of R. To prove R is regular Γ-so-ring, by Lemma 2.20, we prove that $A\Gamma B = A\bigcap B$ for any right ideal A and left ideal B of R. So, let H be a right ideal and K be a left ideal of R. It is clear that $H\Gamma K \subseteq H\bigcap K$ and $H\bigcap K$ is a bi-ideal of R. Then by assumption $(H\bigcap K)\Gamma(H\bigcap K) = H\bigcap K$. This implies $H\bigcap K = (H\bigcap K)\Gamma(H\bigcap K) \subseteq H\Gamma K$. This implies $\Rightarrow H\Gamma K = H\bigcap K$ and hence by Lemma 2.20, R is a regular Γ-so-ring. To prove R is intra-regular, let $a \in R$. Then by Remark 3.10, $< a >_b = a\Gamma R\Gamma a$. Since $< a >_b$ is a bi-ideal of R, by assumption $< a >_b = < a >_b \Gamma < a >_b$. Then $< a >_b = < a >_b \Gamma < a >_b = (a\Gamma R\Gamma a)\Gamma(a\Gamma R\Gamma a) = (a\Gamma R)\Gamma a\Gamma a\Gamma(R\Gamma a) \subseteq R\Gamma a\Gamma a\Gamma R$. Since $a \in < a >_b$, $a \in R\Gamma a\Gamma a\Gamma R$. Hence R is a intra-regular Γ-so-ring.

(2)⇒(3): Suppose $B\Gamma B = B$ for any bi-ideal B of R. Let $A,\ B$ be any bi-ideals of R. Then $A\bigcap B$ is a bi-ideal of R. By assumption $(A\bigcap B)\Gamma(A\bigcap B) = A\bigcap B$. Since $(A\bigcap B)\Gamma(A\bigcap B) \subseteq A\Gamma B$ and $(A\bigcap B)\Gamma(A\bigcap B) \subseteq B\Gamma A$, we have $A\bigcap B \subseteq (A\Gamma B)\bigcap(B\Gamma A)$. We know that if $A,\ B$ are bi-ideals of R then $A\Gamma B$ and $B\Gamma A$ are bi-ideals of R. This implies that $(A\Gamma B)\bigcap(B\Gamma A)$ is a bi-ideal of R. Then by assumption $(A\Gamma B)\bigcap(B\Gamma A)$

$= [(A\Gamma B)\bigcap(B\Gamma A)]\Gamma[(A\Gamma B)\bigcap(B\Gamma A)] \subseteq (A\Gamma B)\Gamma(B\Gamma A) \subseteq A\Gamma R\Gamma A \subseteq A$ (since A is bi-ideal). Similarly we can prove that $(A\Gamma B)\bigcap(B\Gamma A) \subseteq B$. Therefore $(A\Gamma B)\bigcap(B\Gamma A) \subseteq A\bigcap B$. Hence $(A\Gamma B)\bigcap(B\Gamma A) = A\bigcap B$.

(3)⇒(4): Suppose that $A\bigcap B = (A\Gamma B)\bigcap(B\Gamma A)$ for any bi-ideals $A,\ B$ of R. Let P be a bi-ideal of R. To prove P is a semiprime bi-ideal of R, let A be any bi-ideal of R such that $A\Gamma A \subseteq P$. By Assumption $A = A\bigcap A = (A\Gamma A)\bigcap(A\Gamma A) = A\Gamma A \subseteq P$. Hence P is a semiprime bi-ideal of R.

(4)⇒(5): Suppose that each bi-ideal of R is semiprime. Let P be a proper bi-ideal of R. Then P is a semiprime bi-ideal of R. Also by Theorem 3.9, P is the intersection of all irreducible bi-ideals of R containing P. Hence each proper bi-ideal of R is the intersection of irreducible semiprime bi-ideals of R which contain it.

(5)⇒(2): Suppose that each proper bi-ideal of R is the intersection of irreducible semiprime bi-ideals of R which contain it. Let B be a bi-ideal of R. Suppose that $B\Gamma B = R$. Since B is a Γ-subso-ring of R, $B\Gamma B \subseteq B$. Then $R \subseteq B$. This implies that $B = R$ and hence $B\Gamma B = B$. Suppose $B\Gamma B \neq R$. Then $B\Gamma B$ is a proper bi-ideal of R. By assumption, $B\Gamma B$ is the intersection of irreducible semiprime bi-ideals of R which contain $B\Gamma B$. That is $B\Gamma B = \bigcap\{B_i \mid B_i$ is an irreducible semiprime bi-ideal of R and $B\Gamma B \subseteq B_i,\ i \in \Delta\}$. Since each B_i is semiprime bi-ideal and $B\Gamma B \subseteq B_i\ \forall\ i \in \Delta$, we have $B \subseteq B_i\ \forall\ i \in \Delta$, and thus $B \subseteq B\Gamma B$. Hence $B\Gamma B = B$. Hence the theorem. □

Theorem 3.13. *Let R be a regular and intra-regular complete Γ-so-ring with left unity. Then for any bi-ideal B of R, B is strongly irreducible bi-ideal if and only if B is strongly prime bi-ideal.*

Proof. Suppose B is a strongly irreducible bi-ideal of R. Let $H,\ K$ be any bi-ideals of R such that $(H\Gamma K)\bigcap(K\Gamma H) \subseteq B$. Then By the Theorem 3.12, $H\bigcap K =$

$(H\Gamma K)\bigcap(K\Gamma H) \subseteq B$. Since B is strongly irreducible bi-ideal, $H \subseteq B$ or $K \subseteq B$. Hence B is a strongly prime bi-ideal of R.

Conversely, suppose that B is a strongly prime bi-ideal of R. Let H, K be any bi-ideals of R such that $H\bigcap K \subseteq B$. By the Theorem 3.12, $(H\Gamma K)\bigcap(K\Gamma H) = H\bigcap K \subseteq B$. Since B is strongly prime bi-ideal, $H \subseteq B$ or $K \subseteq B$. Hence B is a strongly irreducible bi-ideal of R. □

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