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THE ADJACENCY-JACOBSTHAL SEQUENCE IN FINITE GROUPS

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ABSTRACT. The adjacency-Jacobsthal sequence and the adjacency-Jacobsthal matrix were defined by Deveci and Artun (see [5]). In this work, we consider the cyclic groups which are generated by the multiplicative orders of the adjacency-Jacobsthal matrix when read modulo α $(\alpha > 1)$. Also, we study the adjacency-Jacobsthal sequence modulo α and then we obtain the relationship among the periods of the adjacency-Jacobsthal sequence modulo α and the orders of the cyclic groups obtained. Furthermore, we redefine the adjacency-Jacobsthal sequence by means of the elements of 2-generator groups which is called the adjacency-Jacobsthal orbit. Then we examine the adjacency-Jacobsthal orbit of the finite groups in detail. Finally, we obtain the periods of the adjacency-Jacobsthal orbit of the dihedral group D_{10} as applications of the results obtained.

1. INTRODUCTION

In [5], Deveci and Artun defined the adjacency-Jacobsthal sequence for $k \geq 1$ as follows:

$$J_{m,n}(mn+k) = J_{m,n}(mn-n+k+1) + 2J_{m,n}(k) \tag{1.1}$$

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with initial constants $J_{m,n}(1) = \cdots = J_{m,n}(mn-1) = 0$ and $J_{m,n}(mn) = 1$, where m and n are positive integers such that $m \geq 2$ and $n \geq 4$.

From (1.1), Deveci and Artun given the adjacency-Jacobsthal matrix as shown,

$$C_{m,n} = [c_{i,j}]_{(mn)\times(mn)} = \begin{array}{c} \quad\quad (n-1)\,\text{th} \\ \quad\quad \downarrow \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 2 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Also, by an inductive argument they obtained that

$$(C_{m,n})^{\alpha} = \begin{bmatrix} J_{m,n}^{mn+\alpha} & J_{m,n}^{mn+\alpha+1} & \cdots & J_{m,n}^{mn+\alpha+n-2} & 2J_{m,n}^{\alpha+n-1} & 2J_{m,n}^{\alpha+n} & \cdots & 2J_{m,n}^{\alpha+mn-1} \\ J_{m,n}^{mn+\alpha-1} & J_{m,n}^{mn+\alpha} & \cdots & J_{m,n}^{mn+\alpha+n-3} & 2J_{m,n}^{\alpha+n-2} & 2J_{m,n}^{\alpha+n-1} & \cdots & 2J_{m,n}^{\alpha+mn-2} \\ J_{m,n}^{mn+\alpha-2} & J_{m,n}^{mn+\alpha-1} & \cdots & J_{m,n}^{mn+\alpha+n-4} & 2J_{m,n}^{\alpha+n-3} & 2J_{m,n}^{\alpha+n-2} & \cdots & 2J_{m,n}^{\alpha+mn-3} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ J_{m,n}^{\alpha+1} & J_{m,n}^{\alpha+2} & \cdots & J_{m,n}^{\alpha+n-1} & 2J_{m,n}^{\alpha+n-mn} & 2J_{m,n}^{\alpha+n-mn+1} & \cdots & 2J_{m,n}^{\alpha} \end{bmatrix},$$

where $J_{m,n}(\alpha)$ is denoted by $J_{m,n}^{\alpha}$. It is important to note that
$\det (C_{m,n})^{\alpha} = (-1)^{mn\alpha+\alpha} (2)^{\alpha}$[5].

The study of linear recurrence sequences in groups began with the earlier work of Wall [11] where the ordinary Fibonacci sequences in cyclic groups were investigated. The theory is expanded to 3-step Fibonacci sequence by Ozkan, Aydin and Dikici [10]. Lu and Wang [9] contributed to study of the Wall number for the k-step Fibonacci sequence. Deveci and Karaduman [6] extended the concept to Pell numbers. The concept extended to some special linear recurrence sequences by several authors; see for example, [1, 2, 3, 4, 7, 8, 12]. In this paper, we consider the cyclic groups which are generated by the multiplicative orders of the adjacency-Jacobsthal matrix when read modulo α. Also, we study the adjacency-Jacobsthal sequence modulo α and then we obtain the relationship among the periods of the adjacency-Jacobsthal sequence modulo α and the orders of the cyclic groups obtained. Furthermore, we redefine the adjacency-Jacobsthal sequence by means of the elements of 2-generator groups which is called the adjacency-Jacobsthal orbit. Then we examine the adjacency-Jacobsthal orbit of the finite groups in detail. Finally, we obtain the periods of the adjacency-Jacobsthal orbit of the dihedral group D_{10} as applications of the results obtained.

2. The Adjacency-Jacobsthal Sequence in Finite Groups

Reducing the adjacency-Jacobsthal sequence $\{J_{m,n}(k)\}$ by a modulus α, then we get the repeating sequence, denoted by

$$\left\{J^{\alpha}_{m,n}(k)\right\} = \left\{J^{\alpha}_{m,n}(1), J^{\alpha}_{m,n}(2), \ldots, J^{\alpha}_{m,n}(i), \ldots\right\}$$

where $J^{\alpha}_{m,n}(i) = J_{m,n}(i)\,(mod\alpha)$. It has the same recurrence relation as in (1.1).

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. In particular, if the first k elements in the sequence form a repeating subsequence, then the sequence is simply periodic and its period is k.

Theorem 2.1. *The sequence* $\left\{J^{\alpha}_{m,n}(k)\right\}$ *is periodic for every positive integer* α.

Proof. Let us consider the set $U = \{(u_1, u_2, \ldots, u_{mn}) \mid u_i\text{'s are integers such that } 0 \le u_i \le \alpha - 1\}$. Since there are α^{mn} distinct (mn)-tuples of elements of Z_α, at least one of the (mn)-tuples appears twice in the sequence $\left\{J^{\alpha}_{m,n}(k)\right\}$. Thus, the subsequence following this (mn)-tuple repeats; hence, the sequence is periodic. □

Given an integer matrix $A = [a_{ij}]$, $A\,(mod\alpha)$ means that all entries of A are reduced modulo α, that is, $A\,(mod\alpha) = (a_{ij}\,(mod\alpha))$. Let us consider the set $\langle A\rangle_{\alpha} = \{A^{i}\,(mod\alpha) \mid i \geq 0\}$. Since the determinant of the matrix A is different from zero, then the matrix A is nonsingular and so invertible. Also, If $\gcd(\alpha, \det A) \neq 1$, then the matrix A cannot be invertible. Hence, in this case the matrix A is a cyclic group according to modulo α. Otherwise, the matrix A is a semigroup according to modulo α.

Since $\det C_{m,n} = (-1)^{mn+1}2$, it is clear that the set $\langle C_{m,n}\rangle_{\alpha}$ is a cyclic group if α is a positive odd integer and the set $\langle C_{m,n}\rangle_{\alpha}$ is a semigroup if α is a positive even integer.

We next consider the orders of the cyclic groups which are produced by the adjacency-Jacobsthal matrix $C_{m,n}$ according to modulo α. It is important to note that the orders of these cyclic groups are related to the periods of the sequences modulo α.

Theorem 2.2. *Let p be a prime and consider the cyclic group let $\langle C_{m,n}\rangle_{P^{\alpha}}$. If i is the largest positive integer such that $\left|\langle C_{m,n}\rangle_{p}\right| = \left|\langle C_{m,n}\rangle_{p^{i}}\right|$, then $\left|\langle C_{m,n}\rangle_{p^{j}}\right| = p^{j-i}\cdot\left|\langle C_{m,n}\rangle_{p}\right|$ for every $j \geq i$.*

Proof. Let us consider the cyclic group $\langle C_{m,n}\rangle_{P^{\alpha}}$. Suppose that k is a positive integer and $\left|\langle C_{m,n}\rangle_{P^{\alpha}}\right|$ is denoted by $L(p)$. If $(C_{m,n})^{L\left(p^{k+1}\right)} \equiv I\left(modp^{k+1}\right)$, then $(C_{m,n})^{L\left(p^{k+1}\right)} \equiv I\left(modp^{k}\right)$, where I is a $(mn)\times(mn)$ identity matrix. Thus we get that $L\left(p^{k}\right)$ divides $L\left(p^{k+1}\right)$. On the other hand, writing $(C_{m,n})^{L\left(p^{k}\right)} = I + \left(c^{(k)}_{ij}.p^{k}\right)$, by the binomial theorem, we have

$$(C_{m,n})^{L(p^k).p} = \left(I + \left(c_{ij}^{(k)}.p^k\right)\right)^r = \sum_{\beta=0}^{p} \begin{pmatrix} p \\ \beta \end{pmatrix} \left(c_{ij}^{(k)}.p^k\right)^\beta = I \left(mod p^{k+1}\right).$$

So we get that $L\left(p^{k+1}\right)$ divides $L\left(p^k\right).p$. Thus, $L\left(p^{k+1}\right) = L\left(p^k\right)$ or $L\left(p^{k+1}\right) = L\left(p^k\right).p$. It is clear that $L\left(p^{k+1}\right) = L\left(p^k\right).p$ holds if and only if there exists an integer $c_{ij}^{(k)}$ which is not divisible by p. Since i is the largest positive integer such that $L(p) = L(p^i)$, we have $L(p^i) \neq L(p^{i+1})$. Then, there exists an integer $c_{ij}^{(i+1)}$ which is not divisible by p. So we get that $L(p^{i+1}) \neq L(p^{i+2})$. To complete the proof we may use an inductive method on i. □

We next denote the period of the sequence $\left\{J_{m,n}^{\alpha}(k)\right\}$ by $PJ_{m,n}^{\alpha}$.

Theorem 2.3. *Let α be a positive integer and suppose that $\alpha = \prod_{i=1}^{k} (p_i)^{e_i}$ $(k \geq 1)$, where p_i's are distinct primes and $e_i \geq 1$. Then $PJ_{m,n}^{\alpha}$ equals the least common multiple of the $PJ_{m,n}^{(p_i)^{e_i}}$'s.*

Proof. It is clear that the sequence $\left\{J_{m,n}^{(p_i)^{e_i}}\right\}$ repeats only after blocks of length $\lambda.PJ_{m,n}^{(p_i)^{e_i}}$ where λ is a natural number. Since $PJ_{m,n}^{\alpha}$ is the period of the sequence $\left\{J_{m,n}^{\alpha}(k)\right\}$, the sequence $\left\{J_{m,n}^{\alpha}(p_i^{e_i})\right\}$ repeats after $PJ_{m,n}^{\alpha}$ terms for all values i. Thus, we easily see that $PJ_{m,n}^{\alpha}$ is of the form $\lambda.PJ_{m,n}^{(p_i)^{e_i}}$ for all values of i, and since any such number gives a period of $PJ_{m,n}^{\alpha}$. Therefore, we conclude that

$$PJ_{m,n}^{\alpha} = lcm\left[PJ_{m,n}^{(p_1)^{e_1}}, \ldots, PJ_{m,n}^{(p_k)^{e_k}}\right].$$

□

Let G be a finite j-generator group and let

$$X = \left\{(x_1, x_2, \ldots, x_j) \in \underbrace{G \times G \times \cdots \times G}_{j} \mid \langle\{x_1, x_2, \ldots, x_j\}\rangle = G\right\}.$$

We call $(x_1, x_2, \ldots, x_j)$ a generating j-tuple for G.

Definition 2.1. Let G be a 2-generator group and let (x, y) be a generating pair of G such that $|x| = n$ and $|y| = m$, where $m \geq 2, n \geq 4$. Then, for a generating pair (x, y), we define the adjacency-Jacobsthal orbit $AJ_{(m,n)}(G : x, y)$ as follows:

$$x_{m,n}(mn + k) = (x_{m,n}(k))^2 (x_{m,n}(mn - n + k + 1)) \quad (k \geq 1)$$

with initial constants $x_{m,n}(1) = x$, $x_{m,n}(2) = \cdots = x_{m,n}(mn - n + 1) = e$, $x_{m,n}(mn - n + 2) = y$ and $x_{m,n}(mn - n + 3) = \cdots = x_{m,n}(mn) = e$.

Theorem 2.4. *A adjacency-Jacobsthal orbit $AJ_{(m,n)}(G : x, y)$ of a finite group G is a periodic sequence of group elements.*

Proof. Suppose that q is the order of the group G. Since there are q^{mn} distinct mn-tuples of elements of G, at least one of the mn-tuples appears twice in a adjacency-Jacobsthal orbit $AJ_{(m,n)}(G : x, y)$. Thus, the subsequence following this mn-tuples repeats. Because of the repeating, adjacency-Jacobsthal orbit of the group G is periodic. □

We denote the length of the period of the orbit $AJ_{(m,n)}(G : x, y)$ by $LAJ_{(m,n)}(G : x, y)$.

It is well-known that the dihedral group D_{2n} of order $2n$ is defined by the presentation

$$D_{2n} = \langle x, y \mid x^n = y^2 = (xy)^2 = e \rangle.$$

Example 2.1. *For $n = 5$, we consider the length of the period of the adjacency-Jacobsthal orbit of the dihedral group D_{10}. First note that $|x| = 5$ and $|y| = 2$. The* *sequence* $AJ_{(2,5)}(D_{10} : x, y)$ *is*

$$
\begin{aligned}
x_{2,5}(1) &= x, x_{2,5}(2) = \cdots = x_{2,5}(6) = e, x_{2,5}(7) = y, x_{2,5}(8) = \cdots = x_{2,5}(10) = e,\\
x_{2,5}(11) &= x^2y, x_{2,5}(12) = \cdots = x_{2,5}(14) = e, x_{2,5}(15) = x^2y,\\
x_{2,5}(16) &= \cdots = x_{2,5}(18) = e, x_{2,5}(19) = x^2y, x_{2,5}(20) = e,\\
x_{2,5}(21) &= e, x_{2,5}(22) = e, x_{2,5}(23) = x^2y, x_{2,5}(24) = \cdots = (26) = e,\\
x_{2,5}(27) &= x^2y, x_{2,5}(28) = \cdots = x_{2,5}(30) = e,\\
x_{2,5}(31) &= x^2y, x_{2,5}(32) = \cdots = x_{2,5}(34) = e, x_{2,5}(35) = x^2y,\\
x_{2,5}(36) &= \cdots = x_{2,5}(38) = e, x_{2,5}(39) = x^2y, x_{2,5}(40) = e, \ldots.
\end{aligned}
$$

Since $x_{2,5}(11) = x_{2,5}(31), x_{2,5}(12) = x_{2,5}(32), x_{2,5}(13) = x_{2,5}(33), x_{2,5}(14) = x_{2,5}(34), x_{2,5}(15) = x_{2,5}(35), x_{2,5}(16) = x_{2,5}(36), x_{2,5}(17) = x_{2,5}(37), x_{2,5}(18) = x_{2,5}(38), x_{2,5}(19) = x_{2,5}(39)$ *and* $x_{2,5}(20) = x_{2,5}(40)$, $LAJ_{(2,5)}(D_{10} : x, y) = 20$.

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