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ON THE THIRD HANKEL DETERMINANT FOR A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $\mathcal{A}$ denote the class of all normalized analytic function f in the unit disc $\mathbb{U}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The object of this paper is to obtain a bound to the third Hankel determinant denoted by $H_3(1)$ for a subclass of close-to-convex functions.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions defined on the unit disc $\mathbb{U} = \{z : |z| < 1\}$ with the normalization condition $f(0) = 0 = f'(0) - 1$. So $f \in \mathcal{A}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. Let $\mathcal{P}$ denote the class of functions $p(z)$, has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.2}$$

which are regular in the open unit disc $\mathbb{U}$ and satisfy the conditions $p(0) = 1$ and $Re\, p(z) > 0$, for $z \in \mathbb{U}$. Here $p(z)$ is called the Caratheodory function [5]. A function $f \in \mathcal{A}$ is said to be starlike if it satisfies the condition $Re\{\frac{zf'(z)}{f(z)}\} > 0$, for $z \in \mathbb{U}$

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and $\mathcal{S}^*$, be the class of all starlike functions. Further, a function $f \in \mathcal{A}$ is said to be close-to-convex if it satisfies the condition $Re\{\frac{zf'(z)}{\phi(z)}\} > 0$, for $z \in \mathbb{U}$ and for a starlike function $\phi(z)$. $\mathcal{K}$ be the class of all close-to-convex functions which was introduced by Kaplan [9]. In 1977, Chichra [4] introduced a new subclass of $\mathcal{K}$ defined as follows:

Definition 1.1. ([4]) For $\alpha \geq 0$, a function $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ said to be alpha-close-to-convex function if for a starlike function $\phi(z)$, satisfies the condition

$$Re\left\{(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}\right\} > 0, \quad z \in \mathbb{U}$$

and $\mathcal{C}_\alpha$ be the class of all alpha-close-to-convex functions.

For $\alpha = 0$, $\mathcal{C}_\alpha \equiv \mathcal{K}$. For $\phi(z) = f(z)$, the class $\mathcal{C}_\alpha$ is the class of alpha-starlike functions, which was introduced and studied by P.T. Mocanu [15], also studied in [17]. Chichra [4] proved that every alpha-close-to-convex function is close-to-convex. Also established the following theorem:

Theorem 1.1. [4] *Let for $\alpha \geq 0$, $f \in \mathcal{C}_\alpha$. Then*

$$|a_2| \leq \frac{2+\alpha}{1+\alpha}; \quad |a_3| \leq \frac{9+23\alpha+6\alpha^2}{3(1+\alpha)(1+2\alpha)}; \quad |a_4| \leq \frac{4+22\alpha+34\alpha^2+6\alpha^3}{4(1+\alpha)(1+2\alpha)(1+3\alpha)}.$$

The inequalities are sharp.

Later in [1], Babalola derived the sharp upper bounds of the fifth coefficient of the functions in $\mathcal{C}_\alpha$ as follows:

$$|a_5| \leq \frac{25+238\alpha+755\alpha^2+902\alpha^3+120\alpha^4}{5(1+\alpha)(1+2\alpha)(1+3\alpha)(1+4\alpha)}.$$

It is well known that, the coefficient problem in the univalent function theory ever attracts the function theorist. Closely related to the famous Bieberbach conjecture $|a_n| \leq n$ for $f \in \mathcal{S}$, in 1933, Fekete-Szegö obtained the sharp bound for $|a_3 - \mu a_2^2|$, $\mu \in \mathbb{R}$ for $f \in \mathcal{S}$. The functional $|a_3 - \mu a_2^2|$ is known as Fekete-Szegö functional. Many more functionals risen after it, each finding application in certain problems of geometric functions. For $\mu = 1$, a more general coefficient problem of this type, which is the Hankel determinant problem.

Definition 1.2. The q-th Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ is defined by Pommerenke [20] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.3}$$

In the recent years a great deal of attention has been devoted for finding the estimates of Hankel determinants whose elements are the coefficients of the univalent functions for different specific values of q and n. For example, Noonan and Thomas [18] studied about the second Hankel determinant of areally mean p-valent functions. Noor [19] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in $\mathcal{S}$ with a bounded boundary. Ehrenborg [6] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [13]. It is interesting to note that, $H_2(1) = |a_3 - a_2^2|$, the Fekete-Szegö functional for $\mu = 1$.

The Hankel determinant in the case of $q = 2$ and $n = 2$, is known as the second Hankel determinant, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2. \tag{1.4}$$

The bounds of $H_2(2)$ were obtained for various subclasses of univalent and multivalent analytic functions by many authors existed in the literature [3, 8, 11, 14, 16, 21, 22]. Similarly, the third Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by

$H_3(1)$, is defined by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{1.5}$$

For $f \in \mathcal{A}$, $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

and by applying triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{1.6}$$

Recently, Babalola [2], Bansal et.al [3], Prajapat et.al [23] Vamshee Krishna et.al [11], have studied the third Hankel determinant and obtained the bounds of the determinants $|H_2(2)|$ and $|H_3(1)|$ for $\mathcal{K}$, the class of close-to-convex functions. Also in [24], Sahoo obtained the bounds of the determinants $|H_2(2)|$ and $|H_3(1)|$ for a subclass of α-starlike functions. Motivated by the results obtained by Chichra [4], Babalola [2] and Prajapat et.al [23], we obtain an upper bound to $|H_2(2)|$ and $|H_3(1)|$ for the function $f(z)$ in the class $\mathcal{C}_\alpha$.

2. Preliminary Results

The following lemmas are required to prove our main results.

Lemma 2.1. *([20] pp. 41). If $p(z) \in \mathcal{P}$, given by (1.2), then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $p_0(z) = \frac{1+z}{1-z}$.*

Lemma 2.2. *([12]) Let $p(z) \in \mathcal{P}$, given by (1.2), then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.1}$$

for some x, $|x| \leq 1$, and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{2.2}$$

for some x and z such that $|x| \le 1$ and $|z| \le 1$.

Lemma 2.3. [5] *If $f \in \mathcal{S}^*$ be given by (1.1), then $|a_n| \le n$, $n = 2, 3, \ldots$. Equality holds for the rotations of the Koebe function $k(z) = \frac{z}{(1-z)^2}$.*

Lemma 2.4. [10] *If $f \in \mathcal{S}^*$ be given by (1.1), then for any real number μ, we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} 3 - 4\mu, & if \quad 0 \le \mu \le \frac{1}{2}, \\ 1, & if \quad \frac{1}{2} \le \mu < 1, \\ 4 - 3\mu, & if \quad \mu \ge 1, \end{cases}$$

Lemma 2.5. [7] *If $f \in \mathcal{S}^*$ be given by (1.1), then $|a_2 a_4 - a_3^2| \le 1$. Equality holds for Koebe function.*

Lemma 2.6. [2] *If $f \in \mathcal{S}^*$ be given by (1.1), then $|a_2 a_3 - a_4| \le 2$. Equality holds for Koebe function.*

3. Main Results

To obtain our result, we refer to the classical method initiated by Libera and Zlotkiewicz [12].

Theorem 3.1. *For $0 \le \alpha \le 2/3$, let $f \in \mathcal{C}_\alpha$. Then*

$$|H_2(2)| = |a_2 a_4 - a_3^2| \le \frac{85 + 3\alpha[247 + 509\alpha + 397\alpha^2 + 152\alpha^3 + 36\alpha^4]}{36(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}.$$

Proof. Let $f(z)$ given by (1.1), be in the class $\mathcal{C}_\alpha$. Then there exists an analytic function $p \in \mathcal{P}$ given by (1.2) and a starlike function $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$, such that

$$(1-\alpha) z f'(z) \phi'(z) + \alpha \phi(z) (z f'(z))' = p(z) \phi(z) \phi'(z).$$

On substituting power series expansion of $f(z)$, $p(z)$ and $\phi(z)$, and comparing the coefficient of z^n on both sides we obtain

$$\sum_{k=0}^{n-1} (n-k)[(n-2k-1)\alpha + k + 1] b_{k+1} a_{n-k} = \sum_{k=0}^{n-1} q_{k+1} b_{n-k}, \quad n \ge 2, \tag{3.1}$$

where $a_1 = b_1 = q_1 = 1$ and for $k \geq 2$,

$$q_k = \sum_{j=0}^{k-1} (k-j) c_j b_{k-j}. \tag{3.2}$$

Thus we have,

$$q_2 = 2b_2 + c_1,\ q_3 = 3b_3 + 2b_2c_1 + c_2,\ q_4 = 4b_4 + 3b_3c_1 + 2b_2c_2 + c_3.$$

On substituting the values of $q_2,\ q_3,\ q_4$ in (3.1), and comparing the coefficients of z^2, z^3 and z^4 we get

$$2(1+\alpha)a_2 = (1+\alpha)b_2 + c_1 \tag{3.3}$$

$$3(1+\alpha)(1+2\alpha)a_3 = (1+\alpha)(1+2\alpha)b_3 + (1+3\alpha)b_2c_1 + (1+\alpha)c_2 \tag{3.4}$$

$$\begin{aligned} 4(1+\alpha)(1+2\alpha)(1+3\alpha)a_4 &= (1+\alpha)(1+2\alpha)(1+3\alpha)b_4 \\ &+ (1+2\alpha)(1+5\alpha)b_3c_1 + (1+\alpha)(1+5\alpha)b_2c_2 \\ &+ (1+\alpha)(1+2\alpha)c_3 + \alpha(\alpha-1)b_2^2c_1. \end{aligned} \tag{3.5}$$

Now

$$\begin{aligned} |a_2a_4 - a_3^2| &= \Bigg| \left(\frac{b_2}{2} + \frac{c_1}{2(1+\alpha)}\right) \Bigg[\frac{b_4}{4} + \Bigg\{\frac{1+5\alpha}{4(1+\alpha)(1+3\alpha)} b_3 \\ &\quad - \frac{\alpha(\alpha-1)}{4(1+\alpha)(1+2\alpha)(1+3\alpha)} b_2^2 \Bigg\} c_1 + \frac{1+5\alpha}{4(1+2\alpha)(1+3\alpha)} b_2c_2 \\ &\quad + \frac{1}{4(1+3\alpha)} c_3 \Bigg] - \left(\frac{b_3}{3} + \frac{1+3\alpha}{3(1+\alpha)(1+2\alpha)} b_2c_1 + \frac{1}{3(1+2\alpha)} c_2\right)^2 \Bigg| \\ &= \Bigg| \frac{1}{8}(b_2b_4 - b_3^2) + \frac{1}{8(1+\alpha)}(b_4 - b_2b_3)c_1 + \frac{1+5\alpha}{8(1+\alpha)^2(1+3\alpha)} \times \\ &\quad [b_3 - \mu_1 b_2^2]c_1^2 + \frac{1+6\alpha}{36K(\alpha)} b_2b_3c_1 - \frac{\alpha(1-\alpha)}{8K(\alpha)} b_2^3 c_1 + \frac{1}{72} b_3^2 \end{aligned}$$

$$-\frac{2}{9(1+2\alpha)}[b_3-\mu_2 b_2^2]c_2-\frac{1}{9(1+2\alpha)^2}c_2^2-\frac{7+33\alpha+54\alpha^2}{72K(\alpha)(1+2\alpha)}b_2c_1c_2$$

$$+\frac{(1+2\alpha)}{8K(\alpha)}c_1c_3+\frac{1}{8(1+3\alpha)}b_2c_3\Bigg|, \tag{3.6}$$

where

$$K(\alpha)=(1+\alpha)(1+2\alpha)(1+3\alpha), \tag{3.7}$$

$$\mu_1=\frac{8+81\alpha+208\alpha^2+192\alpha^3}{9(1+5\alpha)(1+2\alpha)^2},\quad \mu_2=\frac{9(1+5\alpha)}{16(1+3\alpha)}. \tag{3.8}$$

Substituting the values of c_2 and c_3 from Lemma 2.2 in the equation (3.6), we have

$$\begin{aligned}|a_2a_4-a_3^2| \;=\; & \Bigg|\frac{1}{8}(b_2b_4-b_3^2)+\frac{1}{8(1+\alpha)}(b_4-b_2b_3)c_1+\frac{1+5\alpha}{8(1+\alpha)^2(1+3\alpha)}[b_3-\mu_1b_2^2]c_1^2\\ & +\frac{1+6\alpha}{36K(\alpha)}b_2b_3c_1-\frac{\alpha(1-\alpha)}{8K(\alpha)}b_2^3c_1+\frac{1}{72}b_3^2-\frac{1}{9(1+2\alpha)}[b_3-\mu_2b_2^2]\times\\ & \times[c_1^2+(4-c_1^2)x]-\frac{1}{36(1+2\alpha)^2}[c_1^2+(4-c_1^2)x]^2-\frac{7+33\alpha+54\alpha^2}{144K(\alpha)(1+2\alpha)}\times\\ & b_2c_1[c_1^2+(4-c_1^2)x]+\frac{1}{32(1+\alpha)(1+3\alpha)}[(1+\alpha)b_2+c_1]\times\\ & [c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z]\Big|\end{aligned}$$

$$\begin{aligned}= \;& \Bigg|\frac{1}{8}(b_2b_4-b_3^2)+\frac{1}{8(1+\alpha)}(b_4-b_2b_3)c_1+\frac{1+6\alpha}{36K(\alpha)}b_2b_3c_1-\frac{\alpha(1-\alpha)}{8K(\alpha)}b_2^3c_1\\ & +\frac{1+4\alpha+12\alpha^2}{288(1+2\alpha)K(\alpha)}c^4+\frac{1}{72}b_3^2+\frac{1+5\alpha}{8(1+\alpha)^2(1+3\alpha)}[b_3-\mu_1b_2^2]c_1^2\\ & -\frac{1}{9(1+2\alpha)}[b_3-\mu_2b_2^2]c_1^2-\frac{1}{9(1+2\alpha)}[b_3-\mu_2b_2^2](4-c_1^2)^2x\\ & -\frac{5+21\alpha+36\alpha^2(1-\alpha)}{288(1+2\alpha)K(\alpha)}b_2c_1^3+\frac{1+6\alpha+9\alpha^2(1+2\alpha)}{72(1+2\alpha)K(\alpha)}b_2c_1(4-c_1^2)x\end{aligned}$$

$$+\frac{1+4\alpha+12\alpha^2}{288(1+2\alpha)K(\alpha)}c_1^2(4-c_1^2)^2x-\frac{c_1^2(4-c_1^2)x^2}{32(1+\alpha)(1+3\alpha)}-\frac{(4-c_1^2)^2x^2}{36(1+2\alpha)^2}$$

$$-\frac{c_1(4-c_1^2)x^2}{32(1+3\alpha)}b_2+\frac{1+2\alpha}{16K(\alpha)}[(1+\alpha)b_2+c_1](4-c_1^2)(1-|x|^2)\Bigg|. \tag{3.9}$$

By Lemma 2.1, we have $|c_1|\leq 2$. For convenience of notation, we take $c_1=c$ and we may assume without loss of generality that $c\in[0,2]$. Applying the triangle inequality with $|x|=\mu$ and using Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.6, we obtain from the above inequality

$$\begin{aligned}|a_2a_4-a_3^2| \leq\ & \Bigg[\frac{1}{8}|b_2b_4-b_3^2|+\frac{|b_4-b_2b_3|c}{8(1+\alpha)}+\frac{(1+6\alpha)|b_2||b_3|c}{36K(\alpha)}+\frac{\alpha(1-\alpha)|b_2|^3c}{8K(\alpha)}\\
&+\frac{1+4\alpha+12\alpha^2}{288(1+2\alpha)K(\alpha)}c^4+\frac{1}{72}|b_3|^2+\frac{1+5\alpha}{8(1+\alpha)^2(1+3\alpha)}|b_3-\mu_1b_2^2|c^2\\
&+\frac{1}{9(1+2\alpha)}|b_3-\mu_2b_2^2|c^2+\frac{1}{9(1+2\alpha)}|b_3-\mu_2b_2^2|(4-c^2)^2\mu\\
&+\frac{5+21\alpha+36\alpha^2(1-\alpha)}{288(1+2\alpha)K(\alpha)}|b_2|c^3+\frac{1+6\alpha+9\alpha^2(1+2\alpha)}{72(1+2\alpha)K(\alpha)}|b_2|c(4-c^2)\mu\\
&+\frac{1+4\alpha+12\alpha^2}{144(1+2\alpha)K(\alpha)}c^2(4-c^2)\mu+\frac{c^2(4-c^2)\mu^2}{32(1+\alpha)(1+3\alpha)}+\frac{(4-c^2)^2\mu^2}{36(1+2\alpha)^2}\\
&+\frac{c(4-c^2)\mu^2}{32(1+3\alpha)}|b_2|+\frac{1+2\alpha}{16K(\alpha)}[(1+\alpha)|b_2|+c](4-c^2)(1-\mu^2)\Bigg]\end{aligned}$$

$$\begin{aligned}\leq\ & \Bigg[\frac{3(1+\alpha)}{4(1+3\alpha)}+\frac{A_1(\alpha)}{12k(\alpha)}c+\frac{A_2(\alpha)}{72(1+2\alpha)k(\alpha)}c^2-\frac{A_3(\alpha)}{144(1+2\alpha)k(\alpha)}c^3\\
&+\frac{A_4(\alpha)}{288(1+2\alpha)k(\alpha)}c^4+\frac{(4-c^2)\mu}{144(1+2\alpha)K(\alpha)}\left[B_1(\alpha)+B_2(\alpha)c+B_3(\alpha)c^2\right]\\
&+\frac{(4-c^2)\mu^2}{288(1+2\alpha)K(\alpha)}\left[D_1(\alpha)+D_2(\alpha)c+D_3(\alpha)c^2\right]\Bigg]\end{aligned}$$

$$= F_1(c,\mu), \tag{3.10}$$

where

$$A_1(\alpha) = 8+45\alpha+6\alpha^2,\ A_2(\alpha) = 8+67\alpha+101\alpha^2+6\alpha^3,\ A_3(\alpha) = 4+15\alpha+36\alpha^3, \tag{3.11}$$

$$B_1(\alpha) = 16K(\alpha),\ B_2(\alpha) = 4(1+6\alpha+9\alpha^2+18\alpha^3), \tag{3.12}$$

$$B_3(\alpha) = D_3(\alpha) = A_4(\alpha) = 1+4\alpha+12\alpha^2 \tag{3.13}$$

$$D_1(\alpha) = -4(1+\alpha)(1+12\alpha+36\alpha^2),\ D_2(\alpha) = 18\alpha(1+2\alpha)^2, \tag{3.14}$$

and $K(\alpha)$ defined by (3.7).

Differentiating $F_1(c,\mu)$ with respect to μ, we get

$$\begin{aligned}\frac{\partial F_1}{\partial \mu} &= \frac{(4-c^2)}{144(1+2\alpha)K(\alpha)}\left[4(1-\mu)(1+\alpha)(1+12\alpha+36\alpha^2)+4(1+\alpha)(3+8\alpha-12\alpha^2)\right.\\ &+ \left.2\{2+3(4+3\mu)\alpha+18(1+2\mu)\alpha^2(1+\alpha)+36(1+\mu)\alpha^3\}c+(1+\mu)(1+2\alpha)^2c^2\right],\end{aligned}$$

which shows that $\frac{\partial F_1}{\partial \mu} > 0$ for $0 \le \mu \le 1$ and $0 \le \alpha < 1$. Therefore $F_1(c,\mu)$ is an increasing function of μ for $0 \le \mu \le 1$ and for any fixed c with $c \in [0,2]$. So it attains maximum at $\mu = 1$. Thus

$$\max_{0\le\mu\le1} F_1(c,\mu) = F_1(c,1) = G_1(c)\ (say). \tag{3.15}$$

Therefore from (3.10) and (3.15) we have

$$\begin{aligned}G_1(c) &= \frac{3(1+\alpha)}{4(1+3\alpha)} + \frac{A_1(\alpha)}{12k(\alpha)}c + \frac{A_2(\alpha)}{72(1+2\alpha)k(\alpha)}c^2 - \frac{A_3(\alpha)}{144(1+2\alpha)k(\alpha)}c^3\\ &\quad + \frac{B_3(\alpha)}{288(1+2\alpha)k(\alpha)}c^4 + \frac{(4-c^2)}{144(1+2\alpha)K(\alpha)} \times\\ &\quad [(D_1(\alpha)+2B_1(\alpha)) + (D_2(\alpha)+2B_2(\alpha))c + 2B_3(\alpha)c^2],\end{aligned} \tag{3.16}$$

where $A_1(\alpha),\ A_2(\alpha),\ A_3(\alpha),\ B_1(\alpha),\ B_2(\alpha),\ B_3(\alpha)$ and $D_1(\alpha),\ D_2(\alpha)$ defined in (3.11), (3.12), (3.13) and (3.14) respectively, and $K(\alpha)$ defined by (3.7). On differentiating $G_1(c)$ with respect to c, we get

$$\begin{aligned} G_1'(c) &= \frac{1+3\alpha}{144K^2(\alpha)}\big[4(4-c^2)(1+\alpha)\{(1+4\alpha+12\alpha^2)c+6(1+6\alpha+9\alpha^2+18\alpha^3)\} \\ &\quad +3\alpha(7+32\alpha+40\alpha^2)c+8(1+\alpha)(2+36\alpha+72\alpha^2-153\alpha^3)\big], \end{aligned}$$

which shows that $G_1'(c) > 0$ for $0 \le c \le 2$. So $G_1(c)$ is increasing function of c, hence it will attains maximum at $c = 2$. Therefore

$$\max_{0\le c\le 2} G_1(c) = G_1(2) = \frac{85+3\alpha[247+509\alpha+397\alpha^2+152\alpha^3+36\alpha^4]}{36(1+\alpha)^2(1+2\alpha)^2(1+3\alpha)}. \tag{3.17}$$

Hence the upper bound on $|a_2a_4 - a_3^2|$ can be obtained by setting $\mu = 1$ and $c = 2$ in (3.10). Hence the desired result follows from (3.10) and (3.17). □

For $\alpha = 0$, the result was proved in [23].

Theorem 3.2. *For $0 \le \alpha \le \frac{1}{2}$, let $f \in \mathcal{C}_\alpha$. Then*

$$|a_2a_3 - a_4| \le \frac{9+52\alpha+83\alpha^2+37\alpha^3+18\alpha^4}{3(1+\alpha)^2(1+2\alpha)(1+3\alpha)}.$$

Proof. Let $f(z)$ given by (1.1), be in the class $\mathcal{C}_\alpha$. Then substituting the values of $a_2,\ a_3$ and a_4 from (3.3), (3.4) and (3.5) in $|a_2a_3 - a_4|$, we have

$$|a_2a_3 - a_4| = \left|\left(\frac{b_2}{2}+\frac{c_1}{2(1+\alpha)}\right)\left(\frac{b_3}{3}+\frac{1+3\alpha}{3(1+\alpha)(1+2\alpha)}b_2c_1+\frac{1}{3(1+2\alpha)}c_2\right)\right.$$

$$-\left[\frac{b_4}{4}+\left\{\frac{1+5\alpha}{4(1+\alpha)(1+3\alpha)}b_3+\frac{\alpha(\alpha-1)}{4(1+\alpha)(1+2\alpha)(1+3\alpha)}b_2^2\right\}c_1\right.$$
$$\left.+\frac{1+5\alpha}{4(1+2\alpha)(1+3\alpha)}b_2c_2+\frac{1}{4(1+3\alpha)}c_3\right]\Bigg|$$

$$= \left|\frac{1}{4}(b_2b_3-b_4)-\frac{b_2b_3}{12}-\frac{E_2(\alpha)}{12K(\alpha)}(b_3-\mu_3b_2^2)c_1+\frac{1}{12K(\alpha)}\times\right.$$
$$\left.[2(1+3\alpha)c_1-E_1(\alpha)b_2]c_2+\frac{(1+3\alpha)^2}{6(1+\alpha)K(\alpha)}b_2c_1^2-\frac{1}{4(1+3\alpha)}c_3\right|, \tag{3.18}$$

where

$$E_1(\alpha)=(1+9\alpha)(1+\alpha), \quad E_2(\alpha)=(1+9\alpha)(1+2\alpha), \tag{3.19}$$

$$\mu_3=\frac{2+15\alpha(1+\alpha)}{(1+9\alpha)(1+2\alpha)} \tag{3.20}$$

and $K(\alpha)$ defined by (3.7).

Substituting the values of c_2 and c_3 from Lemma 2.2 in the equation (3.18), and on simplification we have,

$$|a_2a_3-a_4| = \left|\frac{b_2b_3-b_4}{4}-\frac{b_2b_3}{12}-\frac{E_2(\alpha)(b_3-\mu_3b_2^2)c_1}{12K(\alpha)}+\frac{(3+13\alpha+17\alpha^2-9\alpha^3)b_2c_1^2}{24(1+\alpha)K(\alpha)}\right.$$
$$-\frac{(1+3\alpha-6\alpha^2)c_1^3}{48K(\alpha)}-\frac{(1+3\alpha+6\alpha^2)c_1(4-c_1^2)x}{24K(\alpha)}-\frac{E_1(\alpha)(4-c_1^2)b_2x}{24K(\alpha)}$$
$$\left.+\frac{(4-c_1^2)c_1x^2}{16(1+3\alpha)}-\frac{(4-c_1^2)(1-|x|^2)z}{8(1+3\alpha)}\right|.$$

By the Lemma 2.1, we have $|c_1|\le 2$. For convenience of notation, we take $c_1=c$ and we may assume without loss of generality that $c\in[0,2]$. Applying the triangle inequality with $|x|=\mu$ and using Lemma 2.3, Lemma 2.4 and Lemma 2.6, we obtain

$$|a_2a_3-a_4| \le \frac{|b_2b_3-b_4|}{4}+\frac{|b_2||b_3|}{12}+\frac{E_2(\alpha)}{12K(\alpha)}|b_3-\mu_3b_2^2|c+\frac{3+13\alpha+17\alpha^2-9\alpha^3}{24(1+\alpha)K(\alpha)}$$

$$\begin{aligned}
&\times|b_2|c^2+\frac{1+3\alpha(1-2\alpha)}{48K(\alpha)}c^3+\frac{1+3\alpha+6\alpha^2}{24K(\alpha)}c(4-c^2)\mu \\
+ \quad & \frac{E_1(\alpha)(4-c^2)\mu}{24K(\alpha)}|b_2|+\frac{c(4-c^2)\mu^2}{16(1+3\alpha)}+\frac{(4-c^2)(1-\mu^2)}{8(1+3\alpha)} \\
\leq \quad & \frac{3(1+2\alpha)}{2(1+3\alpha)}+\frac{5+27\alpha+6\alpha^2}{12K(\alpha)}c+\frac{3+14\alpha+19\alpha^2-24\alpha^3}{24(1+\alpha)K(\alpha)}c^2 \\
+ \quad & \frac{1+3\alpha(1-2\alpha)}{48K(\alpha)}c^3+\frac{(4-c^2)(2(1+\alpha)(1+9\alpha)+(1+3\alpha+6\alpha^2)c)\mu}{24K(\alpha)} \\
+ \quad & \frac{c(4-c^2)(c-2)\mu^2}{16(1+3\alpha)}=F_2(c,\mu). \qquad (3.21)
\end{aligned}$$

Differentiating $F_2(c,\mu)$ with respect to c, we have

$$\begin{aligned}
\frac{\partial F_2}{\partial c} \quad = \quad & \frac{(5+27\alpha+6\alpha^2)-(1+3\alpha+6\alpha^2)\mu c^2}{12K(\alpha)} \\
+ \quad & \frac{(3-\mu)(1+5\alpha)+24\alpha^2(1-\alpha)-2(1+\alpha)(1+9\alpha)\mu}{12(1+\alpha)K(\alpha)} \\
+ \quad & \frac{(4-c^2)(1+3\alpha+6\alpha^2)}{24K(\alpha)}+\frac{1+3\alpha(1-2\alpha)}{26K(\alpha)}c^2+\frac{(2-c)(2+3c)}{16(1+3\alpha)}\mu^2,
\end{aligned}$$

which shows that $\frac{\partial F_2}{\partial c}>0$ for $0\leq c\leq 2$. So $F_2(c,\mu)$ is increasing function of c, hence it will attains maximum at $c=2$. Therefore

$$\max_{0\leq c\leq 2}F_2(c,\mu)=F_2(2,\mu)=G_2(\mu)\ (say). \qquad (3.22)$$

From (3.21) and (3.22), we get

(3.23)

$$G_2(\mu)=\frac{3(1+2\alpha)}{2(1+3\alpha)}+\frac{5+27\alpha+6\alpha^2}{6K(\alpha)}+\frac{3+14\alpha+19\alpha^2-24\alpha^3}{6(1+\alpha)K(\alpha)}+\frac{1+3\alpha(1-2\alpha)}{6K(\alpha)},$$

which is independent of μ. Hence the sharp upper bound of the functional $|a_2a_3-a_4|$ is $F_2(2,\mu)=G_2(\mu)$. Thus the desired result follows from (3.21) and (3.23). □

Theorem 3.3. *For* $0\leq\alpha\leq\frac{1}{2}$, *let* $f\in\mathcal{C}_\alpha$. *Then*

$$|a_3-a_2^2|\leq\frac{4(1+2\alpha+4\alpha^2+2\alpha^3)}{3(1+2\alpha)(1+2\alpha+4\alpha^2)}.$$

Proof. The proof is similar to the proof of the Theorem 3.1 and Theorem 3.2. □

Theorem 3.4. *If for* $0 \leq \alpha \leq \frac{1}{2}$, $f \in \mathcal{C}_\alpha$, *then*

$$H_3(1) \leq \frac{1}{3(1+2\alpha)K(\alpha)}\left[\frac{P_1(\alpha)}{72(1+\alpha)^2(1+2\alpha)} + \frac{P_2(\alpha)}{(1+\alpha)^2(1+3\alpha)} + \frac{P_3(\alpha)}{5(1+4\alpha)(1+2\alpha+4\alpha^2)}\right],$$

where $K(\alpha)$ *defined by* (3.7) *and*

$$P_1(\alpha) = [9 + 23\alpha + 6\alpha^2][85 + 3(247 + 509\alpha + 397\alpha^2 + 152\alpha^3 + 36\alpha^4)],$$

$$P_2(\alpha) = [2 + 11\alpha + 17\alpha^2 + 3\alpha^3][9 + 52\alpha + 83\alpha^2 + 37\alpha^3 + 18\alpha^4],$$

$$P_3(\alpha) = 4[1 + 2\alpha + 4\alpha^2 + 2\alpha^3][25 + 238\alpha + 755\alpha^2 + 902\alpha^3 + 120\alpha^4].$$

Proof. Let for $0 \leq \alpha \leq \frac{1}{2}$, $f \in \mathcal{C}_\alpha$. Then from (1.6) we have,

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

By using the bounds of $|a_2a_4 - a_3^2|$, $|a_2a_3 - a_4|$, $|a_3 - a_2^2|$ from Theorem 3.1, Theorem 3.2, Theorem 3.3 respectively, and the bounds $|a_3|$, $|a_4|$ due to Chichra[4], and the bound of $|a_5|$ due to Babalola[1] we get the desired result. □

Remark: Finding the function for which the upper bound for $|H_2(2)|$ and $|H_3(1)|$ are to be sharp is an open problem. The nature of the function to be maximised much more complicated even for $\alpha = 0$, that is, for the functions in $\mathcal{K}$.

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