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ODD VERTEX EQUITABLE EVEN LABELING OF LADDER GRAPHS

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ABSTRACT. Let G be a graph with p vertices and q edges and A={1,3, ...,q} if q is odd or A={1,3, ...,q+1} if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u)+f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \ldots, 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph [2]. In this paper we investigate the odd vertex equitable even labeling behavior of some ladder graphs.

1. Introduction

We consider only simple, finite, connected and undirected graphs and follow the basic notations and terminology of graph theory as in [1]. Let $G(V, E)$ be a graph with p vertices and q edges. The vertex set and the edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. The notion of vertex equitable labeling was due to Lourdusamy and Seenivasan et al.[4]. Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \ldots, \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there

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exists a vertex labeling $f : V(G) \rightarrow A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \ldots, q$ where $v_f(a)$ is the number of vertices v with $f(v) = a$ for $a \in A$. The vertex labeling f is known as vertex equitable labeling. Motivated by the concept of vertex equitable labeling [4], Jeyanthi, Maheswari and Vijayalakshmi extended this concept and introduced a new concept namely odd vertex equitable even labeling in [2] and proved that the graphs, path, $P_n \odot P_m (n, m \geq 1), K_{1,n} \cup K_{1,n-2} (n \geq 3)$, $K_{2,n} (n \geq 1)$, T_p-tree, cycle $C_n (n \equiv 0$ or $1 (\text{mod } 4))$, quadrilateral snake $Q_n (n \geq 1)$, ladder $L_n (n \geq 1)$, $L_n \odot K_1 (n \geq 1)$, arbitrary super subdivision of any path P_n are odd vertex equitable even graphs. They also proved that the graph $K_{1,n}$ is an odd vertex equitable even graph if and only if $n \leq 2$. Let G be a graph with p vertices and q edges and $p \leq \left\lceil \frac{q}{2} \right\rceil + 1$, then G is not an odd vertex equitable even graph. In addition, they proved that if every edge of a graph G is an edge of a triangle, then G is not an odd vertex equitable even graph. In [3], Jeyanthi and Maheswari proved that some cyclic snake related graphs admit odd vertex equitable even labeling.

We use the following definitions and known results in the subsequent section.

Definition 1.1. The graph $\left\langle L_n \hat{O} K_{1,m} \right\rangle$ is the graph obtained from ladder L_n and $2n$ copies of $K_{1,m}$ by identifying a non central vertex of i^{th} copy of $K_{1,m}$ with i^{th} vertex of L_n .

Definition 1.2. The graph $P_n \times P_2$ is called a ladder graph.

Definition 1.3. Let G be a graph. The subdivision graph $S(G)$ is obtained from G by subdividing each edge of G with a vertex.

Definition 1.4. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is defined as a graph obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2.

Definition 1.5. Let G_1 be a graph with p vertices and G_2 be any graph. A graph $G_1\widehat{o}G_2$ is obtained from G_1 and p copies of G_2 by identifying one vertex of i^{th} copy of G_2 with i^{th} vertex of G_1.

Theorem 1.6. [2] *Cycle C_n is an odd vertex equitable even graph if $n \equiv 0$ or $1(mod\ 4)$* .

Theorem 1.7. [2] *$K_{1,n} \cup K_{1,n-2}$ is an odd vertex equitable even graph for any $n \geq 3$.*

2. Main Results

In this section, we prove that $S(L_n)$, $L_m\hat{O}P_n$, $L_n \odot \overline{K_m}$, $\left\langle L_n\hat{O}K_{1,m} \right\rangle$ are odd vertex equitable even graphs.

Theorem 2.1. *The subdivision graph $S(L_n)$ is an odd vertex equitable even graph.*

Proof. Let $V(L_n)) = \{u_i, v_i/1 \leq i \leq n\}$, $E(L_n)) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n\}$. Let v_i' be the newly added vertex between v_i and v_{i+1}, u_i' be the newly added vertex between u_i and u_{i+1}. Let w_i be the newly added vertex between v_i and u_i. Clearly $S(L_n)$ has $5n-2$ vertices and $6n-4$ edges.

Let $A = \left\{ \begin{array}{ll} 1, 3, 5, ..., 5n-2 & \text{if } n \text{ is odd} \\ 1, 3, 5, ..., 5n-1 & \text{if } n \text{ is even} \end{array} \right\}$.

Define the vertex labeling $f : V(S(L_n)) \to A$ as follows:

$f(u_1) = 1$, $f(v_1) = f(v_1') = 3$.

For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $f(u_{2i}) = 12i-5$,$f(u_{2i-1}') = 12i-3$, $f(v_{2i}) = 12i-7$, $f(w_{2i}) = 12i-5$.

For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, $f(w_{2i-1}) = 12i-11$.

For $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $f(u_{2i+1}) = f(u_{2i}') = 12i-1$, $f(v_{2i}') = 12i+3$, $f(v_{2i+1}) = 12i+1$.

For the vertex labeling f, the induced edge labeling f^* is as follows:

$f^*(u_1w_1) = 2$, $f^*(w_1v_1) = 4$, $f^*(u_1u_1^{'}) = 10$, $f^*(v_1^{'}v_2) = 8$.

For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$ $f^*(u_{2i}u_{2i}^{'}) = 24i-6$, $f^*(w_{2i}v_{2i}) = 24i-12$,

$f^*(v_{2i}v'_{2i}) = 24i - 4$, $f^*(v'_{2i}v_{2i+1}) = 24i + 4$, $f^*(u'_{2i}u_{2i+1}) = 24i - 2$,
$f^*(u_{2i+1}w_{2i+1}) = 24i$, $f^*(w_{2i+1}v_{2i+1}) = 24i + 2$.
For $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$, $f^*(u_{2i+1}u'_{2i+1}) = 24i + 8$, $f^*(v'_{2i+1}v_{2i+2}) = 24i + 10$.
For $1 \le i \le \lfloor \frac{n}{2} \rfloor$, $f^*(v_{2i-1}v'_{2i-1}) = 24i - 18$, $f^*(u'_{2i-1}u_{2i}) = 24i - 8$,$f^*(u_{2i}w_{2i}) = 2(12i-5)$. It can be verified that the induced edge labels of $S(L_n)$ are $2, 4, \ldots, 12n$–8 and $|v_f(i) - v_f(j)| \le 1$ for all $i, j \in A$. Clearly f is an odd vertex equitable even labeling of $S(L_n)$. Thus, $S(L_n)$ is an odd vertex equitable even graph. An odd vertex equitable even labeling of $S(L_6)$ is shown in Figure 1.

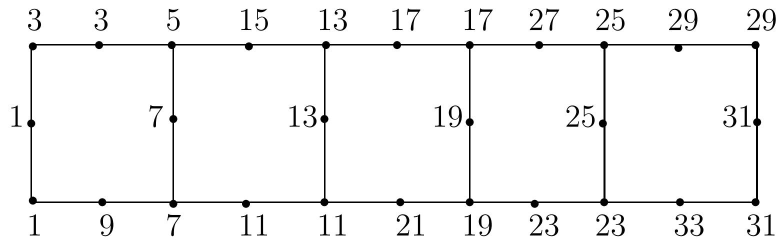

FIGURE 1. An odd vertex equitable even labeling of $S(L_6)$

□

Theorem 2.2. *The graph $L_m\hat{O}P_n$ is an odd vertex equitable even graph.*

Proof. Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_m$ be the vertices of the ladder L_m.
Let v_{ij}, u_{ij} $(1 \le i \le m, 1 \le j \le n)$ be the vertices of m copies of P_n.
Let vertex set $V(L_m\widehat{o}P_n) = \{u_i, v_i, u_{ij}, v_{ij} : 1 \le i \le m, 1 \le j \le n\}$ and edge set $E(L_m\widehat{o}P_n) = \{u_iv_i : 1 \le i \le m\} \bigcup \{u_iu_{i+1}, v_iv_{i+1} : 1 \le i \le m-1\} \bigcup \{u_{ij}u_{ij+1}, v_{ij}v_{ij+1} : 1 \le i \le m, 1 \le j \le n-1$ and $u_{in} = u_i$, $v_{in} = v_i : 1 \le i \le m\}$.
Clearly $L_m\hat{O}P_n$ has $2mn$ vertices and $2mn + m - 2$ edges.

$$\text{Let } A = \left\{ \begin{array}{ll} 1, 3, 5, ..., 2mn + m - 2 & \text{if } m \text{ is odd} \\ 1, 3, 5, ..., 2mn + m - 1 & \text{if } m \text{ is even} \end{array} \right\}.$$

Define the vertex labeling $f : V(L_m\hat{O}P_n) \to A$ as follows:

Case 1. n is odd.

For $1 \le j \le \lceil \frac{n}{2} \rceil$ and i is odd, set

$f(u_{i(2j-1)}) = n - (2j-2) + (4n+2)\lfloor \frac{i}{2} \rfloor$,

$f(v_{i(2j-1)}) = n + (2j-2) + (4n+2)\lfloor \frac{i}{2} \rfloor$,

For $1 \le j \le \lfloor \frac{n}{2} \rfloor$ and i is odd, $f(u_{i(2j)}) = n - 2j + (4n+2)\lfloor \frac{i}{2} \rfloor$,

$f(v_{i(2j)}) = n + 2j + (4n+2)\lfloor \frac{i}{2} \rfloor$.

For $1 \le j \le \lceil \frac{n}{2} \rceil$ and i is even, set $f(u_{i(2j-1)}) = 3n + 2 + (2j-2) + (4n+2)\lfloor \frac{i-1}{2} \rfloor$,

$f(v_{i(2j-1)}) = 3n - (2j-2) + (4n+2)\lfloor \frac{i-1}{2} \rfloor$,

For $1 \le j \le \lfloor \frac{n}{2} \rfloor$ and i is even, $f(u_{i(2j)}) = 3n + 2 + (2j-2) + (4n+2)\lfloor \frac{i-1}{2} \rfloor$,

$f(v_{i(2j)}) = 3n - (2j-2) + (4n+2)\lfloor \frac{i-1}{2} \rfloor$.

Case 2. n is even.

For $1 \le j \le \frac{n}{2}$ and i is odd, set

$f(u_{i(2j-1)}) = f(u_{i(2j)}) = n - 2j + 1 + (4n+2)\lfloor \frac{i}{2} \rfloor$,

$f(v_{i(2j-1)}) = f(v_{i(2j)}) = n + 1 + (2j-2) + (4n+2)\lfloor \frac{i}{2} \rfloor$,

For $1 \le j \le \frac{n}{2}$ and i is even, set

$f(u_{i(2j-1)}) = 3n + 1 + (2j-2) + (4n+2)\lfloor \frac{i-1}{2} \rfloor$,

$f(u_{i(2j)}) = 3n + 1 + 2j + (4n+2)\lfloor \frac{i-1}{2} \rfloor$,

$f(v_{i(2j-1)}) = 3n + 1 - (2j-2) + (4n+2)\lfloor \frac{i-1}{2} \rfloor$,

$f(v_{i(2j)}) = (3n-1) - (2j-2) + (4n+2)\lfloor \frac{i-1}{2} \rfloor$.

For the vertex labeling f, the induced edge labeling f^* is as follows:

$f^*(u_i v_i) = (4n+2)(i-1) + 2n \quad if\ 1 \le i \le m$

$f^*(u_i u_{i+1}) = (4n+2)i \quad if\ 1 \le i \le m-1$

$f^*(v_i v_{i+1}) = (4n+2)i - 2 \quad if\ 1 \le i \le m-1.$

For $1 \le i \le m$, $1 \le j \le n-1$

$$f^*(v_{ij} v_{i(j+1)}) = \begin{cases} 2(n + j + (4n+2)\lfloor \frac{i}{2} \rfloor) & if\ i \text{ is odd} \\ 2(3n + (4n+2)\lfloor \frac{i-1}{2} \rfloor) - 2(j-1) & if\ i \text{ is even.} \end{cases}$$

If n is even $f^*(u_{ij}u_{i(j+1)}) = \begin{cases} 2(n-j+(4n+2)\left\lfloor \frac{i}{2} \right\rfloor) & if\ i \text{ is odd} \\ 2(3n+2+(4n+2)\left\lfloor \frac{i-1}{2} \right\rfloor)+2(j-1) & if\ i \text{ is even.} \end{cases}$

If n is odd $f^*(u_{ij}u_{i(j+1)}) = \begin{cases} 2(n-j+(4n+2)\left\lfloor \frac{i}{2} \right\rfloor) & if\ i \text{ is odd} \\ 2(3n+1+(4n+2)\left\lfloor \frac{i-1}{2} \right\rfloor)+2j & if\ i \text{ is even.} \end{cases}$

It can be verified that the induced edge labels of $L_m\hat{O}P_n$ are $2, 4, \ldots, 4mn + 2m - 4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is an odd vertex equitable even labeling of $L_m\hat{O}P_n$. Thus, $L_m\hat{O}P_n$ is an odd vertex equitable even graph. An odd vertex equitable even labeling of $L_5\hat{O}P_6$ is shown in Figure 2.

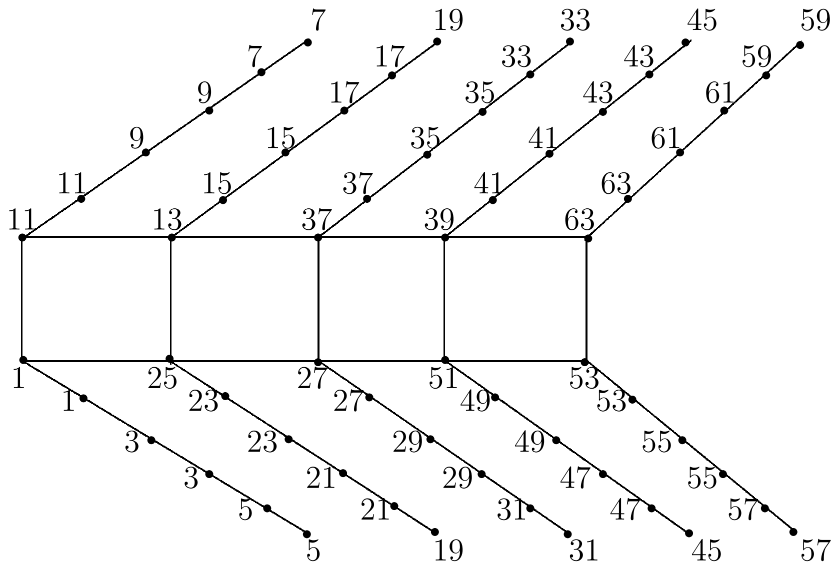

FIGURE 2. An odd vertex equitable even labeling of $L_5\hat{O}P_6$

□

Theorem 2.3. *The graph $L_n \odot \overline{K_m}$ is an odd vertex equitable even graph if $m > 1$.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the ladder L_n.
Let u_{ij}, v_{ij} $(1 \leq i \leq n, 1 \leq j \leq m)$ be the vertices of n copies of $\bar{K_m}$.

Clearly $L_n \odot \overline{K_m}$ has $2n + 2mn$ vertices and $2mn + 3n - 2$ edges.

Let $A = \left\{ \begin{array}{ll} 1, 3, 5, ..., 2mn + 3n - 2 & \text{if } n \text{ is odd} \\ 1, 3, 5, ..., 2mn + 3n - 1 & \text{if } n \text{ is even} \end{array} \right\}.$

Define the vertex labeling $f : V(L_n \odot \overline{K_m}) \rightarrow A$ as follows:

$f(u_{2i-1}) = 4(m + 1)(i - 1) + 2i - 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil$,

$f(u_{2i}) = 4(m + 1)i + 2i - 1$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$f(v_{2i-1}) = 4(m + 1)(i - 1) + 2m + 2i - 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil$,

$f(v_{2i}) = 4(m + 1)(i - 1) + 2m + 2i + 1$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$f(u_{2i-1,j}) = 4(m + 1)(i - 1) + 2j + 2i - 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil$ $1 \leq j \leq m$,

$f(v_{2i-1,j}) = 4(m + 1)(i - 1) + 2j + 2i - 1$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ $1 \leq j \leq m$,

$f(u_{2i,j}) = f(v_{2i,j}) = 4(m + 1)i + 2j + 2i - 2m - 3$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ $1 \leq j \leq m$.

For the vertex labeling f, the induced edge labeling f^* is as follows:

For $1 \leq i \leq n$ $f^*(u_i v_i) = 2((2m + 3)(i - 1) + (m + 1))$,

For $1 \leq i \leq n - 1$ $f^*(u_i u_{i+1}) = 2(2m + 3)i$, $f^*(v_i v_{i+1}) = 2(2m + 3)i - 2$,

For $1 \leq i \leq n$, $1 \leq i \leq m$

$$f^*(u_i u_{ij}) = \begin{cases} 2(2m + 3)(i - 1) + 2j & if\ i \text{ is odd} \\ 2((2m + 3)(i - 1) + m + 1) + 2j & if\ i \text{ is even.} \end{cases}$$

$$f^*(v_i v_{ij}) = \begin{cases} 2((2m + 3)(i - 1) + m + 1) + 2j & if\ i \text{ is odd} \\ 2((2m + 3)(i - 1)) + 2j & if\ i \text{ is even.} \end{cases}$$

It can be verified that the induced edge labels of $L_n \odot \overline{K_m}$ are $2, 4, \ldots, 4mn + 6n - 4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is an odd vertex equitable even labeling of $L_n \odot \overline{K_m}$. Thus, $L_n \odot \overline{K_m}$ is an odd vertex equitable even graph. An odd vertex equitable even labeling of $L_6 \odot \overline{K_4}$ is shown in Figure 3.

□

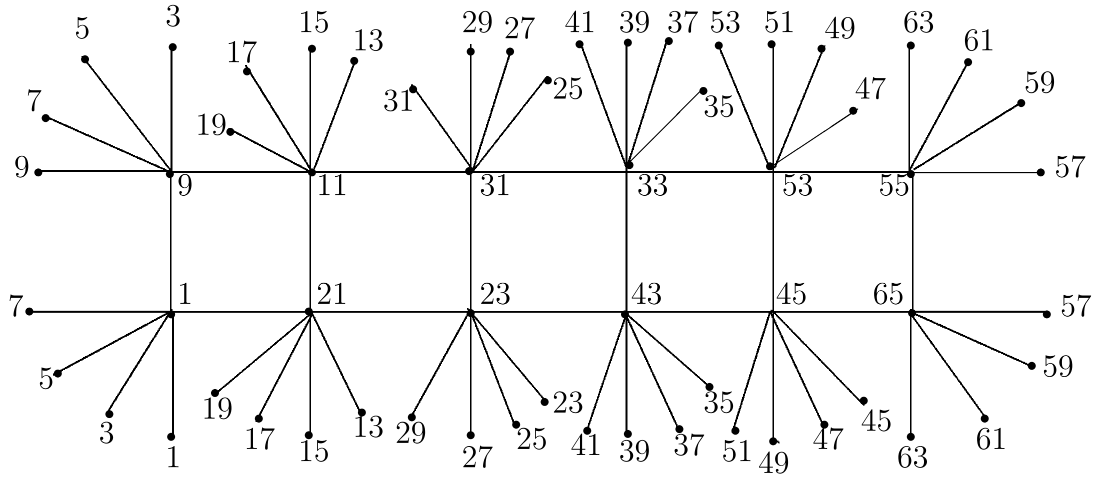

FIGURE 3. An odd vertex equitable even labeling of $L_6 \odot \overline{K_4}$

Theorem 2.4. *The graph* $\left\langle L_n \hat{O} K_{1,m} \right\rangle$ *is an odd vertex equitable even graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the ladder L_n.
Let vertex set $V(\left\langle L_n \hat{O} K_{1,m} \right\rangle) = \{u_i, v_i, u_{i0}, v_{i0}, v_{ij}, u_{ij} : 1 \le i \le n, 1 \le j \le m\}$ and edge set $E(\left\langle L_n \hat{O} K_{1,m} \right\rangle) = \{u_i v_i : 1 \le i \le n\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \le i \le n-1\} \cup \{u_i u_{i0}, v_i v_{i0} : 1 \le i \le n\} \cup \{u_{i0} u_{ij}, v_{i0} v_{ij} : 1 \le i \le n, 1 \le j \le m$ and $u_{im} = u_i, v_{im} = v_i : 1 \le i \le n\}$. Clearly $\left\langle L_n \hat{O} K_{1,m} \right\rangle$ has $2n + 2mn$ vertices and $2mn + 3n - 2$ edges.

Let $A = \left\{ \begin{array}{ll} 1, 3, 5, ..., 2mn + 3n - 2 & \text{if } n \text{ is odd} \\ 1, 3, 5, ..., 2mn + 3n - 1 & \text{if } n \text{ is even} \end{array} \right\}.$

Define the vertex labeling $f : V(\left\langle L_n \hat{O} K_{1,m} \right\rangle) \to A$ as follows:

For $1 \le i \le \lceil \frac{n}{2} \rceil$ $f(u_{(2i-1)0}) = f(u_{2i-1}) = 4(m+1)(i-1) + 2i - 1$,

$f(v_{(2i-1)0}) = f(v_{2i-1}) = 4(m+1)(i-1) + 2m + 2i - 1$,

$f(u_{(2i-1)j}) = f(v_{2i-1,j}) = 4(m+1)(i-1) + 2j + 2i - 1$ if $1 \le j \le m$,

For $1 \le i \le \lfloor \frac{n}{2} \rfloor$ $f(u_{(2i)0}) = 4(m+1)i + 2i - 3$, $f(v_{(2i)0}) = 4(m+1)(i-1) + 2m + 2i + 3$,

$f(u_{2i}) = 4(m+1)i + 2i - 1$, $f(v_{2i}) = 4(m+1)(i-1) + 2m + 2i + 1$,

$f(v_{2i,j}) = 4(m+1)i - 2j + 2i - 3$ if $1 \leq j \leq m$,

$f(u_{2i,j}) = 4(m+1)i - 2j + 2i - 1$ if $1 \leq j \leq m$.

For the vertex labeling f, the induced edge labeling f^* is as follows:

For $1 \leq i \leq n$ $f^*(u_i v_i) = (4(m+1)+2)(i-1) + 2(m+1)$

$$f^*(u_i u_{i0}) = \begin{cases} (4(m+1)+2)(i-1) + 2 & if\ i\ \ is\ odd, \\ (4(m+1)+2)(i-2) + 8(m+1) & if\ i\ \ is\ even. \end{cases}$$

$$f^*(v_i v_{i0}) = \begin{cases} (4(m+1)+2)(i-1) + 2(2m+1) & if\ i\ \ is\ odd, \\ (4(m+1)+2)(i-2) + 4m + 8 & if\ i\ \ is\ even. \end{cases}$$

For $1 \leq i \leq n-1$,

$f^*(u_i u_{i+1}) = (4(m+1)+2)i$, $f^*(v_i v_{i+1}) = (4(m+1)+2)(i-1) + 4(m+1)$

For $1 \leq i \leq n$, $1 \leq j \leq m-1$,

$$f^*(u_{i0} u_{ij}) = \begin{cases} (4(m+1)+2)(i-1) + 2(j+1) & if\ i\ \ is\ odd, \\ (4(m+1)+2)(i-2) + 2(j+1) + 6(m+1) & if\ i\ \ is\ even. \end{cases}$$

$$f^*(v_i v_{i0}) = \begin{cases} (4(m+1)+2)(i-1) + 2(j+1) + 2m & if\ i\ \ is\ odd, \\ (4(m+1)+2)(i-2) + 2(j+1) + 4m + 6 & if\ i\ \ is\ even. \end{cases}$$

It can be verified that the induced edge labels of $\left\langle L_n \hat{O} K_{1,m} \right\rangle$ are $2, 4, \ldots, 4mn+6n-4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$.

Clearly f is an odd vertex equitable even labeling of $\left\langle L_n \hat{O} K_{1,m} \right\rangle$.

Thus, $\left\langle L_n \hat{O} K_{1,m} \right\rangle$ is an odd vertex equitable even graph.

An odd vertex equitable even labeling of $\left\langle L_4 \hat{O} K_{1,5} \right\rangle$ is shown in Figure 4.

□

Theorem 2.5. *The cycle C_n is an odd vertex equitable even graph if and only if $n \equiv$ 0 or 1 (mod 4).*

Proof. The necessary condition is already proved in [2]. Conversely assume that $n \equiv 2$ or $3(mod\ 4)$. Let f be an odd vertex equitable even labeling of the cycle C_n .Then

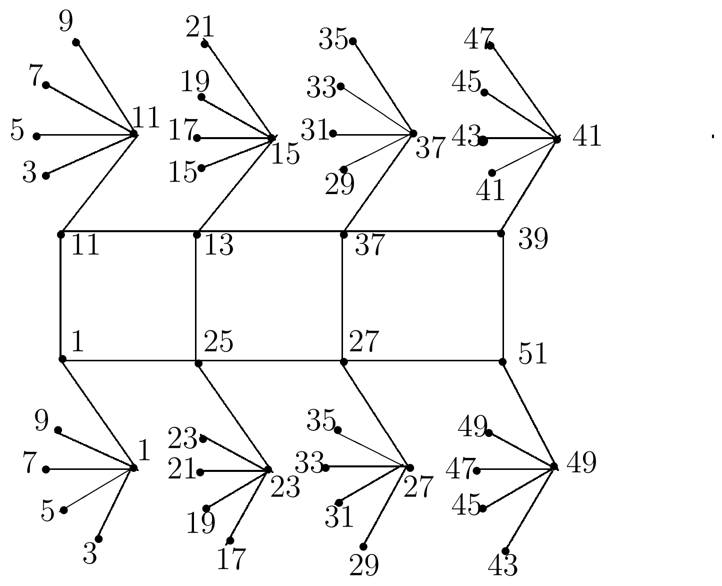

FIGURE 4. An odd vertex equitable even labeling of $L_4\hat{O}K_{1,5}$

$2\sum_{u\in V} f(u) = n(n+1)$. ....(1)

Case (i): $n \equiv 2(mod\ 4)$.

Take $n = 4k+2$. Then $A = \{1, 3, \ldots, n+1\}$ and $|A| = \frac{n}{2}+1$. Since f is an odd vertex equitable even labeling and $p = q = n$, we must have $v_f(a) = 2$ for $\frac{n}{2} - 1$ elements a of A and $v_f(b) = 1$ for remaining two elements b of A. Let $a_1, a_2, \ldots a_{(\frac{n}{2}-1)}$ be the elements of A such that $v_f(a_i) = 2$ for $1 \leq i \leq \frac{n}{2} - 1$ and $v_f(a_{\frac{n}{2}}) = v_f(a_{\frac{n}{2}+1}) = 1$. Let v_i^1 and v_i^2 be the vertices of C_n so that $f(v_i^1) = f(v_i^2) = a_i$ for $1 \leq i \leq \frac{n}{2} - 1$, $f(v^1_{\frac{n}{2}}) = a_{\frac{n}{2}}$ and $f(v^1_{\frac{n}{2}+1}) = a_{\frac{n}{2}+1}$. Then the vertex set V of C_n can be written as $V = V_1 \cup V_2$ where $V_1 = \{v_i^1 : 1 \leq i \leq \frac{n}{2} + 1\}$ and $V_2 = \{v_i^2 : 1 \leq i \leq \frac{n}{2} - 1\}$. Hence (1) can be written as $2\sum_{u\in V_1} f(u) + 2\sum_{u\in V_2} f(u) = n(n+1)$.

Since $2\sum_{u\in V_1} f(u) = 2(1+3+...+(n+1)) = 2(\frac{(n+1)(n+2)}{2} - 2(1+2+...+\frac{n}{2}))=\frac{(n+2)^2}{2}$.

Then $2\sum_{u\in V_2} f(u) = n(n+1) - \frac{(n+2)^2}{2}$ which implies that $2\sum_{u\in V_2} f(u) = \frac{n^2-2n-4}{2}$.

Thus we have $\sum_{u\in V_2} f(u) = 2(2k^2+k) - 1$.

Now, $|V_2| = \frac{n}{2} - 1 = 2k$. Hence $\sum\limits_{u\in V_2} f(u)$ is the sum $2k$ odd numbers and hence it is an even number. But $2(2k^2+k)-1$ is always an odd number which is a contradiction.

Case (ii): $n \equiv 3(mod\ 4)$.

Take $n = 4k+3$. Then $A = \{1, 3, \ldots, n\}$ and $|A| = \frac{n+1}{2}$. Since f is an odd vertex equitable even labeling and $p = q = n$, we must have $v_f(a) = 2$ for $\frac{n-1}{2}$ elements a of A and $v_f(b) = 1$ for remaining two elements b of A. Let $a_1, a_2, \ldots a_{\frac{n-1}{2}}$ be the elements of A such that $v_f(a_i) = 2$ for $1 \leq i \leq \frac{n-1}{2}$ and $v_f(a_{\frac{n+1}{2}}) = 1$. Let v_i^1 and v_i^2 be the vertices of C_n so that $f(v_i^1) = f(v_i^2) = a_i$ for $1 \leq i \leq \frac{n-1}{2}$, and $f(v^1_{\frac{n+1}{2}}) = a_{\frac{n+1}{2}}$. Then the vertex set V of C_n can be written as $V = V_1 \cup V_2$ where $V_1 = \{v_i^1 : 1 \leq i \leq \frac{n+1}{2}\}$ and $V_2 = \{v_i^2 : 1 \leq i \leq \frac{n-1}{2}\}$. Hence (1) can be written as $2\sum\limits_{u\in V_1} f(u) + 2\sum\limits_{u\in V_2} f(u) = n(n+1)$.

Since $2\sum\limits_{u\in V_1} f(u) = 2(1+3+...+n) = 2(\frac{n(n+1)}{2} - 2(1+2+...+\frac{n-1}{2})) = \frac{(n+1)^2}{2}$.

Then $2\sum\limits_{u\in V_2} f(u) = n(n+1) - \frac{(n+1)^2}{2}$ which implies that $2\sum\limits_{u\in V_2} f(u) = \frac{n^2-1}{2}$. Thus we have $\sum\limits_{u\in V_2} f(u) = 2(k+1)(2k+1)$.

Now, $|V_2| = \frac{n-1}{2} = 2k+1$. Hence $\sum\limits_{u\in V_2} f(u)$ is the sum $2k+1$ odd numbers and hence it is an odd number. But $2(k+1)(2k+1)$ is always an even number which is a contradiction.

In both cases we get a contradiction. Hence f can not be an odd vertex equitable even labeling of C_n if $n \equiv 2$ or $3(mod\ 4)$. □

Theorem 2.6. *The graph $G = K_{1,n+k} \cup K_{1,n}$ is an odd vertex equitable even graph if and only if $k = 1, 2$.*

Proof. Let $V(G) = \{u, v, u_j, v_i : 1 \leq j \leq n+k$ and $1 \leq i \leq n\}$ and $E(G) = \{uu_j, vv_i : 1 \leq j \leq n+k$ and $1 \leq i \leq n\}$. Then G has $2n+k+2$ vertices and $2n+k$ edges. Let $A = \{1, 3, 5, \ldots 2n+k$ or $2n+k+1\}$ according as k is odd or even. Let f be a an odd vertex equitable even labeling of the graph $G = K_{1,n+k} \cup K_{1,n}$.

To get an edge label 2, there must be two adjacent vertices with vertex labels 1 and

1. So we can take $f(u) = 1$ and $f(u_1) = 1$. To get an edge label 4, there must be two adjacent vertices with vertex labels 1 and 3 and so we have $f(u_2) = 3$. Since all the edge labels are distinct, the pendent vertices $u_1, u_2, u_3, u_4, \ldots, u_{n+k}$ should receive the distinct labels from the set A. So we have $f(u_3) = 5, f(u_4) = 7, \ldots, f(u_{n+k}) = 2n + 2k - 1$.

If k is odd then the maximum of A is $2n+k$. Hence $2n+2k-1 \leq 2n+k$ which implies $k \leq 1$. If k is even then the maximum of A is $2n+k+1$. Hence $2n+2k-1 \leq 2n+k+1$ which implies $k \leq 2$.

If $k = 0$ then $q = 2n$ and $A = 1, 3, 5, \ldots, 2n + 1$. Since $|A| = n + 1, p = 2n + 2$ and f is odd vertex equitable even labeling, $v_f(a) = 2$ for all $a \in A$. Hence, the pendent vertices of the first component receive the labels $1, 3, 5, \ldots, 2n–1$, centre vertex receives the label 1 and the pendent vertices of the second component receive the labels $3, 5, \ldots, 2n–1, 2n + 1$.

To get an edge label $4n$, there must be two adjacent vertices with vertex labels $2n+1$ and $2n - 1$. So we can take $f(v) = 2n + 1$ or $2n - 1$.

If $f(v) = 2n - 1$ then $v_f(2n - 1) = 3$ and $v_f(2n + 1) = 1$. If $f(v) = 2n + 1$ then the edge vv_n receives the label $4n + 2$. In both cases we get a contradiction. Thus, if $k = 0$ then G is not an odd vertex equitable even graph.

If $k = 1$ then $A = \{1, 3, 5, \ldots, 2n+1\}$. The vertex labeling $f : V(G) \to A$ is defined as follows: $f(u) = 1; f(u_j) = 2j - 1$ for $1 \leq j \leq n+1$, $f(v) = 2n+1$ and for $1 \leq i \leq n$, $f(v_i) = 2i+1$. Hence, $f(V(G)) = \{1, 1, 3, 5, \ldots, 2n+1\} \cup \{2n+1, 3, 5, \ldots, 2n+1\}$ and also $f^*(E(G)) = \{2, 4, 6, \ldots, 2n + 2\} \cup \{2n + 4, 2n + 6, \ldots, 4n + 2\}$. Hence $v_f(i) = 2$ for $i = 1, 3, 5, \ldots, 2n–1$ and $v_f(2n + 1) = 3$. Thus we have $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Hence, $G = K_{1,n+k} \cup K_{1,n}$ is an odd vertex equitable even graph.

If $k = 2$ then $A = \{1, 3, 5, \ldots, 2n + 3\}$. The vertex labeling $f : V(G) \to A$ is defined as follows: $f(u) = 1$; $f(u_j) = 2j - 1$ for $1 \leq j \leq n + 2$, $f(v) = 2n + 3$ and for $1 \leq i \leq n$, $f(v_i) = 2i + 1$.

$f(V(G)) = \{1, 1, 3, 5, \ldots, 2n+1, 2n+3\} \cup \{2n+3, 3, 5, \ldots, 2n+1\}$ and also $f(E(G)) = \{2, 4, 6, \ldots, 2n+4\} \cup \{2n+6, 2n+8, \ldots, 4n+4\}$. Hence $v_f(i) = 2$ for all $i \in A$. Thus we have $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Hence, $G = K_{1,n+k} \cup K_{1,n}$ is an odd vertex equitable even graph.

If $k = 2$ the proof follows from Theorem 1.7 by replacing n by $m - 2$. Hence $G = K_{1,n+k} \cup K_{1,n}$ is an odd vertex equitable even graph. □

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