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SOLVING THE OPTIMAL CONTROL PROBLEMS WITH CONSTRAINT OF INTEGRAL EQUATIONS VIA MÜNTZ POLYNOMIALS

NEDA NEGARCHI [(1)] AND KAZEM NOURI [(2)]

Abstract. In this study, an efficient numerical scheme is presented for solving a class of optimal control problems governed by the form of the Volterra-Fredholm integral equation. The technique based upon approximating the state and control functions by Müntz polynomials. The numerical integration and new approach utilized to discretize the optimal control problem to a nonlinear programming using the Chebyshev nodes together with the Gauss quadrature rule. Finally, numerical examples illustrate the efficiency of the proposed method.

1. Introduction

The classical theory of optimal control was developed in the last years as a powerful tool to establish optimal solutions for different problems in many aspects of science and technology. Optimal control problem (OCP) received considerable attention during the last four decades because of their applications in many different fields, including aerospace process control, bioengineering, economics, financial mathematics, management science and etc [26]. Also, considering the importance and application of integral equation in various sciences, we consider a main class of OCPs governed by the integral equation.

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There are various techniques for solving this class of OCPs. An overview of these techniques can be studied in the research works provided by Vinokurov [27], Bakke [3], Belbas [4, 5], Pan and Teo [21], Elangar and Kim [12], Carlier and Tahraou [10]. Among all of the numerical techniques for solving OCPs, orthogonal functions and polynomials have attracted a lot of attention. High accuracy and ease of applying them for OCPs are important advantages which have encouraged authors to use them for many problems, see for example, Shih and Wang [24], Chou and Horng [11], Elnagar and Razzaghi [13], Wang and Li [28], Tohidi and Samadi [26], Ross and Fahroo [22], Negarchi and Nouri [20]. In this study, a numerical solution of the OCP governed by the form of nonlinear Volterra-Fredholm integral equation (VFIE) is considered, which is described by the following minimization problem:

Problem 1:

(1.1)

$$\begin{array}{ll} Min & Q(x,y) = \int_0^1 (x(t)-a(t))^2 + (y(t)-b(t))^2 dt, \\ s.t. & x(t) = f(t) + \int_0^t p_1(t,\tau,x(\tau),y(\tau))d\tau + \int_0^1 p_2(t,\tau,x(\tau),y(\tau))d\tau, \quad t\in[0,1]. \end{array}$$

where $f(t)$, $a(t)$ and $b(t)$ are continuous and known functions, $x(t)$ and $y(t)$ are real-valued function and continuous and belong to Sobolev space $W^{r,\infty}$ with $r \geq 2$ (see [1, 9]).

Notice that, a control problem is usually expressed by two types of function, namely the state and control functions ($x(t)$,$y(t)$). The control function directs the evolution of the system from one step to the next, and the state function describes the behavior of the system. In OCP, the state and control functions are both unknown. The purpose of these problems is to determine $x(t)$ and $y(t)$.

The paper is organized as follows: Section 2 contains the basic concepts. In Section 3, a new numerical method for solving problem 1 is proposed. Section 4 presents some numerical examples illustrating the efficiency and accuracy of the proposed method. Finally The conclusion is given in section 5.

2. Preliminaries

In this section, some definitions and basic concepts are expressed.

Definition 2.1. ([17]) A function $\psi : [0,T] \to \mathbb{R}$ belongs to the Sobolev space $W^{z,l}$, if its jth weak derivative $\psi^{(j)}$, lies in $L^l[0,T]$ for all $0 \le j \le z$ with the norm$\|\psi\|_{W^{z,l}} = \sum_{j=0}^{z} \left\|\psi^{(j)}\right\|_{L^l}$, where $\|\psi\|_{L^l}$ denotes the usual Lebesgue norm defined for $1 \le l < \infty$ as follows:

$$\|\psi\|_{L^l} = (\int_0^T |\psi(t)|^l dt)^{\frac{1}{l}}.$$

Lemma 2.1. *Given any function $\psi \in W^{z,\infty}$, $t \in [0,T]$, there is a polynomial $g_N(t)$ of degree N or less, such that*

$$|\psi(t) - g_N(t)| \le CC_0 N^{-z}, \quad \forall t \in [0,1], \tag{2.1}$$

where C is a constant independent of N, z is the order of smoothness of ψ and $C_0 = \|\psi\|_{W^{z,\infty}}$ ($g_N(t)$ with the smallest norm $\|\psi(t) - g_N(t)\|_{L^\infty}$ is called the N th order best polynomial approximation of $\psi(t)$ in the norm of L^∞).

Proof. See [9]. □

In this research, the Müntz polynomials and their basic properties recalled. Initially, the orthogonal Müntz systems were introduced by the Armenian mathematicians Badalyan [2] and Taslakyan [25]. Next, they were assessed by Mc Carthy, Sayre and Shawyer [17], and then reassessed by Borwein and Erdélyi [7]. For more details see [18, 19, 20, 23].

Let $A = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ be a complex sequence such that $0 \le \alpha_0 < \alpha_1 < \cdots \to \infty$. The classical Müntz theorem express that the Müntz polynomials of the form:

$$\sum_{k=0}^{n} a_k x^{\alpha_k}, \tag{2.2}$$

real coefficients $\{a_k, k=0,1,\dots,n\}$ are dense in $L^2[0,1]$ if and only if (see [6, 8]),

$$\sum_{k=0}^{\infty} \alpha_k^{-1} = \infty. \tag{2.3}$$

The constant function 1 is also in the system, where $\alpha_0 = 0$, then the result holds for $C[0,1]$ with the uniform norm.

Müntz polynomials define as linear combinations of the Müntz system $\{x^{\alpha_0}, x^{\alpha_1}, \dots, x^{\alpha_n}\}$ and the set of all such polynomials denote with $M_n(A) = span\,\{x^{\alpha_0}, x^{\alpha_1}, \dots, x^{\alpha_n}\}$.

Furthermore, let

$$M(A) := \bigcup_{n=0}^{\infty} M_n(A) = span\,\{x^{\alpha_0}, x^{\alpha_1}, x^{\alpha_2}, \dots\}\,. \tag{2.4}$$

In the following, the Müntz-Legendre polynomials introduced, which are orthogonal on $[0,1]$ with weight function $w(x) = 1$.

Let the complex numbers from the set $A = span\,\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ satisfy the condition $Re(\alpha_k) > -1/2$, then, the Müntz- Legendre polynomial of degree n defined on $[0,1]$ by (see [6, 14, 18]),

$$L_n(x) = \sum_{k=0}^{n} C_{n,k} x^{\alpha_k}, \quad C_{n,k} = \frac{\prod_{v=0}^{n-1} (\alpha_k + \bar{\alpha}_v + 1)}{\prod_{v=0, v\neq k}^{n} (\alpha_k - \alpha_v)}. \tag{2.5}$$

It is shown that they are orthogonal in $L^2[0,1]$ with respect to the Legendre weight function $w(x) = 1$.

For the Müntz-Legendre polynomials the following orthogonality relation holds:

$$(L_n, L_m) = \int_0^1 L_n(x)\, \overline{L_m(x)}\, w(x) dx = \frac{\delta_{nm}}{\alpha_n + \overline{\alpha_n} + 1}. \tag{2.6}$$

Also

$$xL_j'(x) - xL_{j-1}'(x) = \alpha_j L_j(x) + (1 + \bar{\alpha}_{j-1}) L_{j-1}(x), \quad j = 1,2,3,\dots, \tag{2.7}$$

$$xL_j''(x) = (\alpha_j - 1) L_j'(x) + \sum_{k=0}^{j-1} (\alpha_k + \overline{\alpha}_k + 1) L_k'(x), \quad j = 0,1,2,\dots. \tag{2.8}$$

It can easily be proved that:

$$L_n(1) = 1 \quad \text{and} \quad L'_n(1) = \alpha_n + \sum_{k=0}^{n-1} (\alpha_k + \overline{\alpha_k} + 1). \tag{2.9}$$

3. Description of methodology

One of the methods that can be used to solve OCP is a direct method that converts the optimal control into an algebraic optimization problem. This method is based on the approximation of the state and control functions in terms of the basic functions and the orthogonal Müntz-Legendre polynomials. Here, a numerical method provided for solving the OCP. In the proposed method, an approximate is presented for the cost function and integral equation control system.

Consider the dynamic system of the OCP as follows:

$$x(t) = f(t) + \int_0^t p_1(t, \tau, x(\tau), y(\tau))d\tau + \int_0^1 p_2(t, \tau, x(\tau), y(\tau))d\tau, \quad t \in [0, T]. \tag{3.1}$$

The dynamic system is approximated using the approximation of state and control functions, $x(t) \approx x_N(t) = \sum\limits_{m=0}^{N} x_N(t_m)L_m(t)$, $y(t) \approx y_N(t) = \sum\limits_{m=0}^{N} y_N(t_m)L_m(t)$, and the set of shifted Chebyshev nodes on $[a, b]$ with $\eta_i = \frac{1}{2}(a+b)+\frac{1}{2}(b-a)\cos(\frac{2i-1}{2N}\pi)$, $i = 0, 1, \dots, N$ as:

$$\begin{aligned} x(\eta_i) = f(\eta_i) &+ \int_0^{\eta_i} p_1(\eta_i, \tau, \sum_{m=0}^{N} x_N(t_m)L_m(\tau), \sum_{m=0}^{N} y_N(t_m)L_m(\tau))d\tau \\ &+ \int_0^1 p_2(\eta_i, \tau, \sum_{m=0}^{N} x_N(t_m)L_m(\tau), \sum_{m=0}^{N} y_N(t_m)L_m(\tau))d\tau, \quad i = 0, 1, \dots, N. \end{aligned} \tag{3.2}$$

In the above equation, the points of the set $\{t_0, t_1 \dots, t_N\}$ correspond with the collocation points $\{\eta_0, \eta_1, \dots, \eta_N\}$. Using linear transformation $\tau = \tilde{\tau}_i(\tau) = \frac{1}{2}(\tau + 1)$ and $\tau = \hat{\tau}_i(\tau) = \frac{\eta_i}{2}(\tau + 1)$ transform the interavels $[0, 1]$ and $[0, \eta_i]$ into $[-1, 1]$. Then, the Gauss-Legendre quadrature rule, with the nodes $\tau_0 = -1, \tau_N = 1$ and τ_j, $j = 1, 2, \dots, N - 1$ (the j-th root of the Legendre polynomial $P_{N-1}(t)$), and the weights $\omega_j = \frac{2}{(1-\tau_j^2)(P'_N(\tau_j))^2}$, $j = 0, 1, \ldots, N$, can be applied as follows:

(3.3)
$$x(\eta_i) = f(\eta_i) + \frac{\eta_i}{2}\sum_{j=0}^{N}\omega_j p_1(\eta_i, , \hat{\tau}_i(\tau_j), \sum_{m=0}^{N} x_N(t_m)L_m(\hat{\tau}_i(\tau_j)), \sum_{m=0}^{N} y_N(t_m)L_m(\hat{\tau}_i(\tau_j)))$$
$$+ \frac{1}{2}\sum_{j=0}^{N}\omega_j p_2(\eta_i, , \tilde{\tau}_i(\tau_j), \sum_{m=0}^{N} x_N(t_m)L_m(\tilde{\tau}_i(\tau_j)), \sum_{m=0}^{N} y_N(t_m)L_m(\tilde{\tau}_i(\tau_j))),$$
$$i = 0, 1, \ldots, N.$$

For convenience, $x(t)$ and $y(t)$ are considered as follows:

$$x(t) \approx x_N(t) = \sum_{m=0}^{N} x_N(t_m)g_m(t), \qquad y(t) \approx y_N(t) = \sum_{m=0}^{N} y_N(t_m)g_m(t). \tag{3.4}$$

With the same process, we approximate the cost functional of problem 1 as:

$$\begin{aligned} &\int_0^1 (x(t) - a(t))^2 + (y(t) - b(t))^2 dt \\ &= \frac{1}{2}\int_{-1}^{1}((x_N(\tfrac{\eta+1}{2}) - a(\tfrac{\eta+1}{2}))^2 + (y_N(\tfrac{\eta+1}{2}) - b(\tfrac{\eta+1}{2}))^2)d\eta \\ &\approx \frac{1}{2}\sum_{j=0}^{N}\omega_j\ ((x_N(\tfrac{\tau_j+1}{2}) - a(\tfrac{\tau_j+1}{2}))^2 + (y_N(\tfrac{\tau_j+1}{2}) - b(\tfrac{\tau_j+1}{2}))^2). \end{aligned} \tag{3.5}$$

So, problem 1 is discretized to the following nonlinear programming problem:

$$\begin{aligned} &Min \quad Q_N(X, Y) \\ &s.t. \quad I_i(X, Y) = -f(\eta_i), \qquad i = 0, 1, \ldots, N, \end{aligned} \tag{3.6}$$

where $X = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N)$, $Y = (\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_N)$ are the unknown vectors. In other words,

$$\begin{aligned} &Min \quad Q_N(X, Y) \\ &Q_N(X, Y) := \ \frac{1}{2}\sum_{j=0}^{N}\omega_j\ ((x_N(\tfrac{\tau_j+1}{2}) - a(\tfrac{\tau_j+1}{2}))^2 + (y_N(\tfrac{\tau_j+1}{2}) - b(\tfrac{\tau_j+1}{2}))^2), \end{aligned} \tag{3.7}$$

and

(3.8)
$$I_i(X, Y) := \frac{\eta_i}{2}\sum_{j=0}^{N}\omega_j p_1(\eta_i, , \hat{\tau}_i(\tau_j), \sum_{m=0}^{N} x_N(t_m)g_m(\hat{\tau}_i(\tau_j)), \sum_{m=0}^{N} y_N(t_m)g_m(\hat{\tau}_i(\tau_j)))$$
$$+ \frac{1}{2}\sum_{j=0}^{N}\omega_j p_2(\eta_i, , \tilde{\tau}_i(\tau_j), \sum_{m=0}^{N} x_N(t_m)g_m(\tilde{\tau}_i(\tau_j)), \sum_{m=0}^{N} y_N(t_m)g_m(\tilde{\tau}_i(\tau_j))) - x(\eta_i),$$
$$i = 0, 1, \ldots, N.$$

Following [15], problem 1 is approximated by the nonlinear programming problem with object function Eq. (3.8) and down constraint:

$$|I_i(X,Y)+f(\eta_i)| \leq (N-1)^{\frac{3}{2}-z}, \quad i=0,1,\ldots,N. \tag{3.9}$$

Then, the nonlinear programming problem is given by;

Problem 2:

$$\begin{aligned} &Min \quad Q_N(X,Y) \\ &|I_i(X,Y)+f(\eta_i)\ | \leq (N-1)^{\frac{3}{2}-z}, \quad i=0,1,\ldots,N. \end{aligned} \tag{3.10}$$

The feasibility of problem 2 can be proved using the following theorem.

Theorem 3.1. *Given any feasible solution $(x(t),y(t))$ for problem 1, suppose $x(t),y(t)$ belong to $W^{z,\infty}$ with $z\geq 2$. Then, there is a positive integer N_1 such that for any $N>N_1$, problem 2 has a feasible solution $(\hat{x}_i,\hat{y}_i)$ such that, the feasible solution satisfies*

$$|x(t_i)-\hat{x}_i| \leq d_1(N-1)^{1-z},\ |y(t_i)-\hat{y}_i| \leq d_2(N-1)^{1-z}, \qquad i=0,1,\ldots,N, \tag{3.11}$$

where $d_1,d_2>0$ are constants and independents of N.

Proof. See [16]. □

In the next theorem, the convergence of the following sequence is expressed, and thus the convergence of the proposed method for problem 1 is provided,

$$\{(x_N^*(t_i),y_N^*(t_i)),\ 0\leq i\leq N\}_{N=N_1}^{\infty}. \tag{3.12}$$

Theorem 3.2. *Assume that $\{(x_N^*(t_i),y_N^*(t_i)),\ 0\leq i\leq N\}_{N=N_1}^{\infty}$ be a sequence of optimal solutions to problem 2. If the function sequence has a subsequence that uniformly converges to the continuous function $\{(p_1(t),p_2(t))\}$ on interval $[0,1]$ then, $\hat{x}(t)=\int_{t_0}^{t}p_1(\nu)d\nu+\hat{x}_0$ and $\hat{y}(t)=\int_{t_0}^{t}p_2(\nu)d\nu+\hat{y}_0$ are the optimal solution to problem 1.*

Proof. See [16, 26]. □

4. Numerical examples

Here, three examples tested using the approach discussed in section 3. Examples are given to demonstrate the ability, punctuality and performance of our presented method. All computations are carried out in Mathematica version 10 software. In order to analyze the errors of the present method, we introduce the notations $\delta_x = \max|x(t_i) - x_N(t_i)|$, $\delta_y = \max|y(t_i) - y_N(t_i)|$, and $\delta_Q = \max|Q(x(t_i), y(t_i)) - Q(x_N(t_i), y_N(t_i))|$ for $i = 0, 1, \ldots, N$.

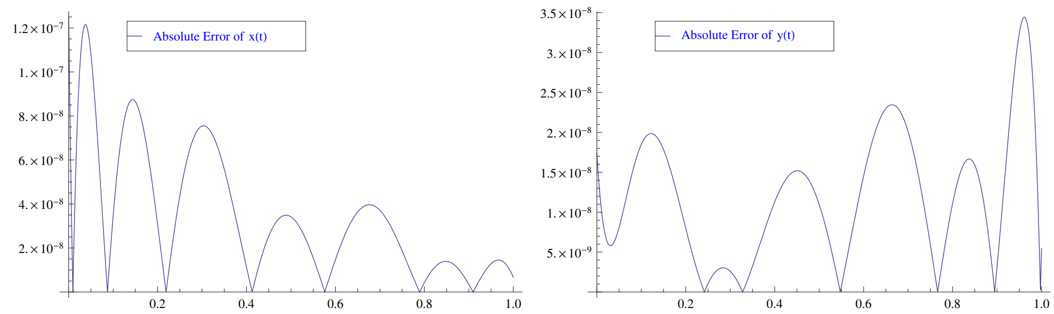
Example 4.1. *Consider the following OCP*

$$\begin{aligned} &Min\ Q(x,y) = \int_0^1 (x(t) - \log(t+1))^2 + (y(t)-t)^2 dt, \\ &s.t. \quad x(t) = f(t) + \int_0^{h(t)} t x(\tau) y(\tau) d\tau + \int_0^1 (t-\tau) x(h(\tau))\, d\tau, \quad 0 \le t \le 1, \end{aligned} \tag{4.1}$$

where $h(t) = \frac{t}{3}$, $f(t) = \log(t+1)+t+\frac{5}{4}+\log(\frac{81}{256})(1+t)-\frac{t}{36}(6t-t^2+\log(\frac{t}{3}+1)(2t^2-18))$. *This equation has exact solutions* $x(t) = \log(t+1)$, $y(t) = t$ *and the optimal value of cost function* $Q = 0$. *The numerical results of solving this example using our method for* $N = 5, 10, 15, 20$ *are demonstrated in Table 1. Also, Figure 2 (a) and (b), show the absolute errors of presented method for the state and control functions with* $N = 15$.

Table 1. Absolute errors of the state, control and cost functions for Example 4.1.

	N			
error	*5*	*10*	*15*	*20*
δ_x	4.35618×10^{-4}	1.65563×10^{-6}	2.71056×10^{-7}	1.77025×10^{-8}
δ_y	5.02916×10^{-5}	7.88033×10^{-6}	4.02253×10^{-8}	4.63509×10^{-9}
δ_Q	2.00442×10^{-7}	8.82044×10^{-10}	3.95540×10^{-14}	6.13608×10^{-17}



(a) (b)

FIGURE 2. Numerical results of Example 4.1 for $N = 15$.

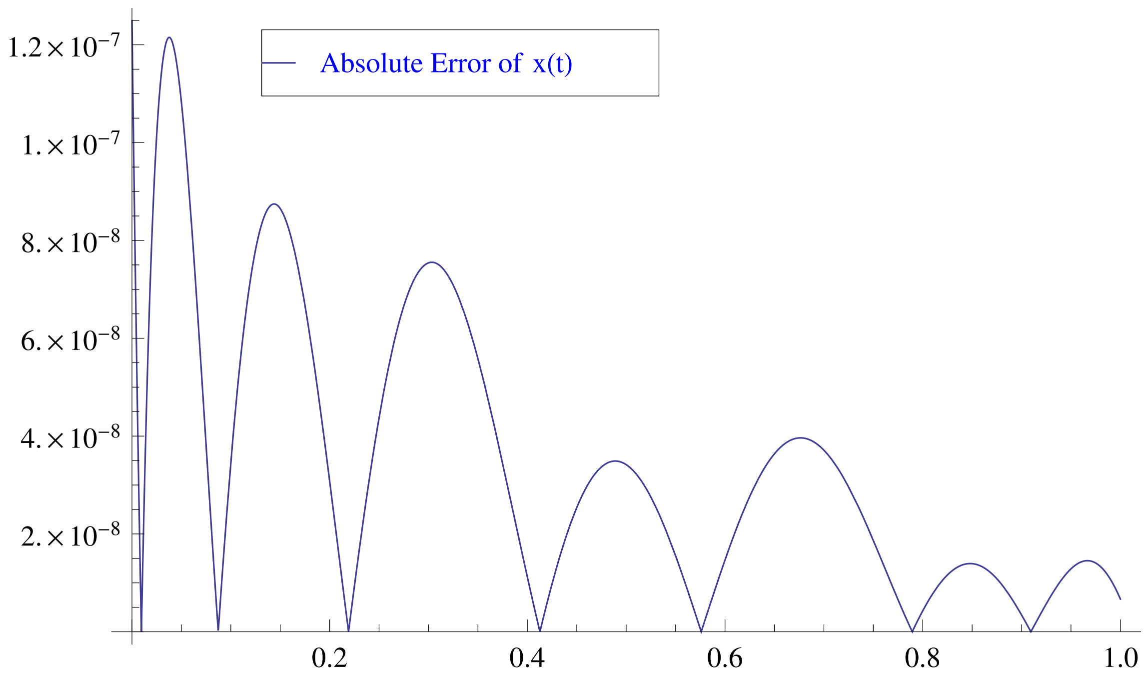
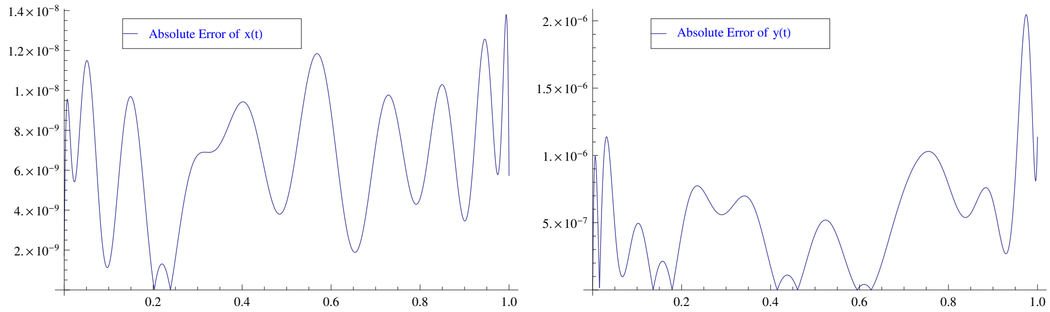


FIGURE 1. Graph between n, α, h, k and TIC for Model I

Example 4.2. *Consider the following OCP*

$$(4.2)\qquad \begin{aligned} &Min\ Q(x,y) = \int_0^1 (x(t) - t^2)^2 + (y(t) - e^t)^2 dt, \\ &s.t. \quad x(t) = f(t) + \int_0^t e^t \frac{x(\tau)}{y(\tau)} d\tau - \int_0^1 e^t x(\tau) y(\tau) d\tau, \quad 0 \le t \le 1, \end{aligned}$$



(a) (b)

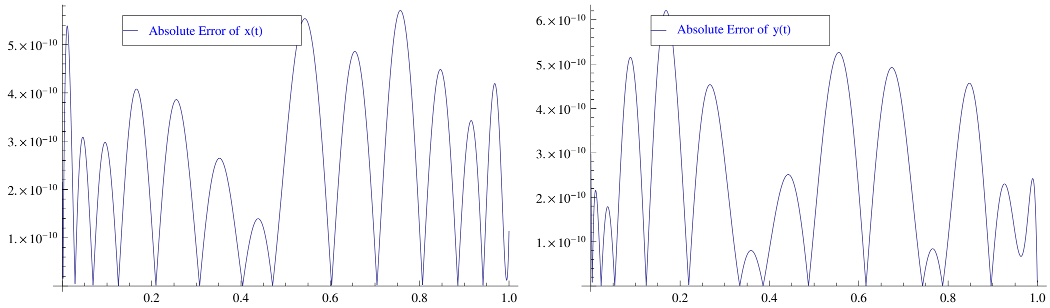
FIGURE 3. Numerical results of Example 4.2 for $N = 12$.

where $f(t) = (-4+e)e^t + 2t^2 + 2t + 2$*. Trivially, the optimal value of the cost function is* $Q = 0$ *and the optimal state and control functions are* $x(t) = t^2$ *and* $y(t) = e^t$*. Table 2, reports the numerical results of the proposed method for this example for* $N = 4, 8, 12, 16$*. Figure 3 (a) and (b), show respectively the absolute errors of state and control functions using mentioned method for* $N = 12$.

TABLE 2. Absolute errors of the state, control and cost functions for Example 4.2.

	N			
error	*4*	*8*	*12*	*16*
δ_x	4.14720×10^{-5}	9.44506×10^{-8}	3.21064×10^{-8}	6.64294×10^{-9}
δ_y	3.22347×10^{-4}	2.02631×10^{-5}	3.89277×10^{-6}	8.76667×10^{-8}
δ_Q	7.63302×10^{-8}	4.71962×10^{-11}	5.46642×10^{-14}	6.72935×10^{-16}

Example 4.3. *Consider the main OCP* (1.1) *with* $p_1(t, \tau, x(\tau), y(\tau)) = \sin t x(\tau) y(\tau)$, $p_2(t, \tau, x(\tau), y(\tau)) = e^t x(\tau) y(\tau)$, $a(t) = \cos t$, $b(t) = 1 - 2t^2$ *and* $f(t) = \cos t + e^t(4cos1 - 3\sin 1) + \sin t\,(4t\cos t + (2t^2 - 5)\sin t)$*. For this case the exact solutions are* $x(t) = \cos t$, $y(t) = 1 - 2t^2$ *and the optimal value of the cost function is* $Q = 0$.



(a) (b)

FIGURE 4. Numerical results of Example 4.3 for $N = 20$.

Table 3 denotes the numerical results obtained by our method for $N = 8, 16, 24, 32$. *Figure 4 (a) and (b), display the absolute errors of state and control functions using the proposed method in this study for* $N = 20$.

TABLE 3. Absolute errors of the state, control and cost functions for Example 4.3.

	N			
error	*8*	*16*	*24*	*32*
δ_x	5.22704×10^{-5}	2.76613×10^{-8}	1.32447×10^{-10}	7.96510×10^{-11}
δ_y	5.80417×10^{-5}	4.01463×10^{-8}	2.15508×10^{-10}	9.11381×10^{-11}
δ_Q	3.43719×10^{-12}	5.77193×10^{-16}	8.91048×10^{-19}	1.22039×10^{-19}

5. CONCLUSION

In this paper, a new approach has posed to solve a class of OCP including the form of VFIE by a direct method of solution based upon orthogonal Müntz polynomials together with the shifted Chebyshev points as the collocation nodes. The proposed method is based on converting the OCP into a finite dimensional mathematical program problem. Illustrative examples have been presented to demonstrate effectiveness of our method. It is noteworthy that with increasing N, the approximations of the the state, control and cost functions are improved, so the presented method has stable properties, as N increases, the error reduces and finally stabilizes. The obtained rapid convergence shows this method can successfully solve the OCP. In future research, the method can extend for solving the OCP governed by a system of partial differential equation. Also, the mentioned method can be develop for solving the OCP with control and state functions of the vectors.

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(1) (corresponding Author) Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P. O. Box 35195-363, Semnan, Iran

E-mail address: N.negarchi@semnan.ac.ir; N.negarchi@gmail.com

(2) Department of Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, Semnan University, P. O. Box 35195-363, Semnan, Iran

E-mail address: knouri@semnan.ac.ir; knouri.h@gmail.com