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FIXED POINT RESULTS IN ORTHOGONAL FUZZY METRIC SPACES

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ABSTRACT. In this article, our aim is to discuss the Banach's contraction and Suzuki type contraction in orthogonal fuzzy metric spaces. We furnish our discussion with an example to demonstrate the validity of these results.

1. Introduction and Preliminaries

In order to generalize the well known Banach contraction principle, many authors coined various contractive-type mappings in various type of metric spaces (see [5, 19, 22] and references therein). On the other hand, the notion of fuzzy metric space was coined in different ways by many authors [2, 13] and further to this, the fixed point theory in this kind of spaces has been intensively studied, see for example [3, 4, 9, 10, 11, 14, 20, 21]. Here, we recall as the notion of fuzzy metric space, introduced by Kramosil and Michalek [13] was modified by George and Veeramani [6, 7] that obtained a Hausdorff topology for this class of fuzzy metric spaces. Also, the notion of triangular fuzzy metric space was introduced in [3, 4] and the notion of orthogonal metric space coined in [5]. In this article, our aim is to discuss the Banach's contraction and Suzuki type contraction in orthogonal fuzzy metric spaces. We furnish our discussion with an example to demonstrate the validity of these results.

Now we want to recall some necessary definitions to coherence with the literature.

Definition 1.1. (Schweizer and Sklar [17]) A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following assertions:

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(T1) $\star$ is commutative and associative;
(T2) $\star$ is continuous;
(T3) $a \star 1 = a$ for all $a \in [0,1]$;
(T4) $a \star b \leq c \star d$ when $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Definition 1.2. [6, 7] An ordered triple $(X, M, \star)$ is a fuzzy metric space if X is nonempty, $\star$ is a continuous t-norm, M is a fuzzy set on $X \times X \times (0, +\infty)$ and satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:

(F1) $M(x, y, t) > 0$;
(F2) $M(x, y, t) = 1$ if and only if $x = y$;
(F3) $M(x, y, t) = M(y, x, t)$;
(F4) $M(x, y, t) \star M(y, z, s) \leq M(x, z, t + s)$;
(F5) $M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is continuous.

Definition 1.3. [6, 7] Assume $(X, M, \star)$ be a fuzzy metric space. Hence

(i) a sequence $\{x_n\}$ converges to a point $x \in X$, if and only if $\lim_{n\to+\infty} M(x_n, x, t) = 1$ for every $t > 0$,
(ii) a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for every $\epsilon \in (0, 1)$ and $t > 0$, there exists n_0 whence $M(x_n, x_m, t) > 1 - \epsilon$ for all $m, n \geq n_0$,
(iii) the fuzzy metric space is complete whence every Cauchy sequence converges to some $x \in X$.

Definition 1.4. Suppose $(X, M, *)$ be a fuzzy metric space and $M^{-1}(x, y, t) = \frac{1}{M(x,y,t)}$ for all $x, y \in X$ and $t > 0$. The fuzzy metric $(X, M, *)$ is triangular fuzzy metric space when

$$M^{-1}(x, y, t) - 1 \leq M^{-1}(x, z, t) - 1 + M^{-1}(z, y, t) - 1$$

for all $x, y, \in X$.

Further a fuzzy metric M is called regular if the following condition is satisfied,

$$\text{if } M(x, y, t) = 1 \text{ for some } t > 0 \text{ then } x = y$$

Eshaghi et. [5] coined the notion of orthogonal metric spaces and gave a real generalization of Banach' fixed point theorem in such spaces (Also see [1] , [16]).

Definition 1.5. [5] Assume $X \neq \emptyset$ and $\perp \in X \times X$ be an binary relation. Suppose that there exists $x_0 \in X$ such that $x_0 \perp x$ or $x \perp x_0$ for all $x \in X$. Thus we say that X is an orthogonal set (briefly O-set). Further, we denote orthogonal set by $(X, \perp)$. Also, suppose that $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonal sequence (briefly O-sequence) if $(\forall n; x_n \perp x_{n+1})$ or $(\forall n; x_{n+1} \perp x_n)$.

Definition 1.6. [5] A metric space (X,d) is an orthogonal metric space if $(X,\perp)$ is an O-set. Further, $T:X\to X$ is $\perp-$continuous in $x\in X_\omega$ if for each O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in X if $\lim_{n\to\infty} d(x_n,x)=0$, then, $\lim_{n\to\infty} d(Tx_n,Tx)=0$. Furthermore, T is $\perp$-continuous if T is $\perp$-continuous in each $x\in X$. Also, T is $\perp$-preserving if $Tx\perp Ty$ whence $x\perp y$. Finally, X is orthogonally complete (in brief O-complete) if every Cauchy O-sequence is convergent.

Now we introduce the notion of orthogonal fuzzy metric spaces by the following methods.

Definition 1.7. Let $(X,M,\star)$ be a fuzzy metric space and $\perp\in X\times X$ be a binary relation. Assume that there exists $x_0\in X$ such that $x_0\perp x$ for all $x\in X$. Then we say that X is an orthogonal orthogonal fuzzy metric space. We denote orthogonal fuzzy metric by $(X,M,\star,\perp)$.

Definition 1.8. Let $(X,M,\star,\perp)$ be an orthogonal fuzzy metric. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called O-sequence if $x_n\perp x_{n+1}$ for all $n\in\mathbb{N}$. Also, $T:X\to X$ is $\perp-$continuous in $x\in X$ if for each O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in X if $\lim_{n\to\infty} M(x_n,x,t)=1$ for all $t>0$, then, $\lim_{n\to\infty} M(Tx_n,Tx,t)=1$ for all $t>0$. Furthermore, T is $\perp$-continuous if T is $\perp$-continuous in each $x\in X$. Also, we say T is $\perp$-preserving if $Tx\perp Ty$ whence $x\perp y$. Finally, X is orthogonally complete (in brief O-complete) if every Cauchy O-sequence is convergent.

2. Banach's Contraction Principle in orthogonal fuzzy metric spaces

Motivated by works of Eshaghi et. [5] we introduce the notion of Banach's contraction principle for nonlinear contraction mappings in the setting of orthogonal fuzzy metric spaces as follows.

Definition 2.1. Let $(X,M,\star,\perp)$ be an orthogonal fuzzy metric space. A map $T:X\to X$ is an $\perp$-contraction if there exits $k\in(0,1)$ such that for every $t>0$ and $x,y\in X$ with $x\perp y$ we have,

$$M(Tx,Ty,kt)\geq M(x,y,t). \tag{1}$$

Now we begin this section with following theorem.

Theorem 2.1. *Assume $(X,M,\star,\perp)$ be an O-complete fuzzy metric space. Let $T:X\to X$ be $\perp-$continuous, $\perp-$contraction and $\perp-$preserving. Thus T has a unique fixed point $x^*\in X$. Furthermore,*

$$\lim_{n\to\infty} M(T^n x,x^*,t)=1$$

for all $x\in X$ and $t>0$.

Proof. Since, $(X, M, \star, \perp)$ is an orthogonal fuzzy metric space, thus there exists $x_0 \in X$ such that,

$$x_0 \perp y \text{ for all } y \in X. \tag{2}$$

This says us, $x_0 \perp Tx_0$. Assume,

$$x_1 := Tx_0, \ x_2 = T^2 x_1, \ \ldots, \ x_n = T^n x_0 = Tx_{n-1}$$

for all $n \in \mathbb{N}$. Since T is $\perp-$preserving, hence $\{x_n\}$ is an O-sequence. Now, since T is an $\perp-$contraction then we can get,

$$M(x_{n+1}, x_n, kt) = M(Tx_n, Tx_{n-1}, kt) \geq M(x_n, x_{n-1}, t)$$

for all $n \in \mathbb{N}$ and $t > 0$. Note that M is increasing on $(0, \infty)$. Therefore by applying the above expression we can deduce,

$$\begin{aligned} M(x_{n+1}, x_n, t) &\geq M(x_{n+1}, x_n, kt) \\ &\geq M(x_n, x_{n-1}, t) \\ &= M(x_n, x_{n-1}, \tfrac{t}{k}k) \\ &\geq M(x_{n-1}, x_{n-2}, \tfrac{t}{k}) \\ &\geq \ldots \\ &\geq M(x_1, x_0, \tfrac{t}{k^n}) \end{aligned} \tag{3}$$

for all $n \in \mathbb{N}$ and $t > 0$. Thus from (3) we have,

$$\begin{aligned} M(x_n, x_{n+p}, t) &= M(x_n, x_{n+p}, \tfrac{t}{p}p) \\ &\geq M(x_n, x_{n+1}, \tfrac{t}{p}) \star \ldots \star M(x_{n+p-1}, x_{n+p}, \tfrac{t}{p}) \\ &\geq M(x_1, x_0, \tfrac{t}{pk^n}) \star \ldots \star M(x_1, x_0, \tfrac{t}{pk^{n+p-1}}) \end{aligned} \tag{4}$$

where p is a arbitrary positive integer. We know that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. So from (4) we get,

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) \geq 1 \star \ldots \star 1 = 1.$$

So, $\{x_n\}$ is Cauchy $O-$sequence. The hypothesis of $O-$completeness of fuzzy metric space $(X, M, \star, \perp)$ ensures that there exists $x^* \in X$ such that $M(x_n, x^*, t) \to 1$ as $n \to +\infty$ for all $t > 0$. Now, since T is an $\perp$-continuous mapping, then $M(x_{n+1}, Tx^*, t) = M(Tx_n, Tx^*, t) \to 1$ as $n \to +\infty$. From

$$M(x^*, Tx^*, t) \geq M(x^*, x_{n+1}, \frac{t}{2}) \star M(x_{n+1}, Tx^*, \frac{t}{2}),$$

taking limit as $n \to +\infty$, we get $M(x^*, Tx^*, t) = 1$ and hence $x^* = Tx^*$.

Now we show the uniqueness of the fixed point of the mapping T. Assume that x^* and y^* are two fixed point of T such that $x^* \neq y^*$.

From (15) we can get,

$$[x_0 \perp x^* \text{ and } x_0 \perp y^*]$$

Since T is $\perp-$preserving, so we can write,

$$[T^n x_0 \perp T^n x^* \text{ and } T^n x_0 \perp T^n y^*]$$

for all $n \in \mathbb{N}$. So from (1) we derive,

$$M(T^n x_0, T^n x^*, t) \geq M(T^n x_0, T^n x^*, kt) \geq M(x_0, x^*, \tfrac{t}{k^n})$$

and

$$M(T^n x_0, T^n y^*, t) \geq M(T^n x_0, T^n y^*, kt) \geq M(x_0, y^*, \tfrac{t}{k^n})$$

and hence we can write,

$$\begin{aligned} M(x^*, y^*, t) &= M(T^n x^*, T^n y^*, t) \\ &\geq M(T^n x_0, T^n x^*, \tfrac{t}{2}) \star M(T^n x_0, T^n y^*, \tfrac{t}{2}) \\ &\geq M(x_0, x^*, \tfrac{t}{2k^n}) \star M(x_0, y^*, \tfrac{t}{2k^n}) \ \rightarrow 1 \text{ as } n \rightarrow \infty. \end{aligned}$$

So, $x^* = y^*$.

Assume x^* be a given fixed point of T. Similarly, we can deduce,

$$[x_0 \perp x^* \text{ and } x_0 \perp x]$$

and

$$[T^n x_0 \perp T^n x^* \text{ and } T^n x_0 \perp T^n x].$$

Again from (1) we have,

$$M(T^n x_0, T^n x^*, t) \geq M(T^n x_0, T^n x^*, tk) \geq M(x_0, x^*, \tfrac{t}{k^n})$$

and

$$M(T^n x_0, T^n x, t) \geq M(T^n x_0, T^n x, tk) \geq M(x_0, x, \tfrac{t}{k^n})$$

and so we can deduce,

$$\begin{aligned} M(x^*, T^n x, t) &= M^{-1}(T^n x^*, T^n x, t) \\ &\geq M(T^n x_0, T^n x^*, \tfrac{t}{2}) \star M(T^n x_0, T^n x, \tfrac{t}{2}) \\ &\geq M(x_0, x^*, \tfrac{t}{2k^n}) \star M(x_0, x, \tfrac{t}{2k^n}) \ \rightarrow 1 \text{ as } n \rightarrow \infty. \end{aligned}$$

□

For $\perp-$contraction that is not $\perp-$continuous we have the following theorem.

Theorem 2.2. *Let $(X, M, \star, \perp)$ be an O-complete fuzzy metric space. Let $T : X \to X$ be $\perp-$contraction and $\perp-$preserving. Also, if $\{x_n\}$ be an O-sequence with $x_n \to x \in X$, then $x \perp x_n$ for all $n \in \mathbb{N}$. Therefore, T has a unique fixed point $x^* \in X$. Furthermore, $\lim_{n\to\infty} M(T^n x, x^*, t) = 1$ for all $x \in X$ and $t > 0$.*

Proof. Since, $(X, M, \star, \perp)$ is an $O-$set, then there exists $x_0 \in X$ such that,

$$x_0 \perp y \text{ for all } y \in \mathcal{A}. \tag{5}$$

As in the proof of Theorem 2.1, we derive that an O-sequence $\{x_n\}$ starting at x_0 is Cauchy and so converges to a point $x^* \in X$. Hence, $x^* \perp x_n$ for all $n \in \mathbb{N}$. Therefore from (1) we get,

$$M(Tx^*, x_{n+1}, t) = M(Tx^*, Tx_n, t) \geq M(Tx^*, Tx_n, tk) \geq M(x^*, x_n, t)$$

and so,

$$\lim_{n\to\infty} M(Tx^*, x_{n+1}, t) = 1.$$

Then we can write,

$$M(Tx^*, x^*, t) \geq M(Tx^*, x_{n+1}, \frac{t}{2}) \star M(x_{n+1}, x^*, \frac{t}{2}) \to 1 \text{ as } n \to \infty.$$

That is, x^* is fixed point of T. The other statements follow as in the proof of Theorem 2.1. □

3. Suzuki Type Fixed point Results

In this section we obtain some Suzuki type results for $\Theta-$contraction in the setting of orthogonal triangular fuzzy metric spaces.

Consistent with Jleli et al. [12], we denote by Δ_Θ the set of all functions $\Theta : (0, +\infty) \to (1, +\infty)$ satisfying the following conditions:

(Θ_1) Θ is increasing;

(Θ_2) for all sequence $\{\alpha_n\} \subseteq (0, +\infty)$, $\lim_{n\to+\infty} \alpha_n = 0$ if and only if $\lim_{n\to+\infty} \Theta(\alpha_n) = 1$;

(Θ_3) there exist $0 < r < 1$ and $\ell \in (0, +\infty]$ such that $\lim_{t\to 0^+} \frac{\Theta(t)-1}{t^r} = \ell$.

For Suzuki type $\Theta-$contraction mapping that is $\perp-$continuous we have the following theorem in O-complete triangular fuzzy metric space.

Theorem 3.1. *Let $(X, M, \star, \perp)$ be an O-complete triangular fuzzy metric space with M regular and let $T : X \to X$ be an $\perp-$continuous and $\perp-$preserving self-mapping. Assume that there exist a real number $r \in [0, 1)$ and a function $\Theta \in \Delta_\Theta$ such that for all $t > 0$ and $x, y \in X$ with $x \perp y$, $\frac{1}{2}[M^{-1}(x, Tx, t) - 1] \leq M^{-1}(x, y, t) - 1$ and $M^{-1}(Tx, Ty, t) > 1$, we have*

$$\Theta\big(M^{-1}(Tx, Ty, t) - M_T^\alpha\big) \leq \Big[\Theta\big(M^{-1}(x, y, t) - M_T^\alpha\big)\Big]^r \tag{6}$$

where $M_T^\alpha = [M(y, Tx, t)]^\alpha$ and $\alpha \geq 0$. Then T has a fixed point.

Proof. Since, $(X, M, \star, \perp)$ is an orthogonal fuzzy metric space, then there exists $x_0 \in X$ such that,

$$x_0 \perp y \text{ for all } y \in X. \tag{7}$$

This implies, $x_0 \perp Tx_0$. Assume,

$$x_1 := Tx_0,\ x_2 = T^2 x_1,\ \ldots,\ x_n = T^n x_0 = Tx_{n-1}$$

for all $n \in \mathbb{N}$. Since T is $\perp-$preserving, then $\{x_n\}$ is an O-sequence. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is a fixed point of T and we have nothing to prove. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Now, assume that there exists $n_0 \in \mathbb{N}$ such that $M(Tx_{n_0-1}, Tx_{n_0}, t) = 1$ for some $t > 0$. Then, O regularity implies, $x_{n_0} = Tx_{n_0-1} = Tx_{n_0} = x_{n_0+1}$, which is a contradiction. Hence, $M^{-1}(Tx_{n-1}, Tx_n, t) > 1$ for all $n \in \mathbb{N}$ and $t > 0$. Also, evidently,

$$\frac{1}{2}[M^{-1}(x_{n-1}, Tx_{n-1}, t) - 1] \leq M^{-1}(x_{n-1}, x_n, t) - 1$$

for all $n \in \mathbb{N}$ and $t > 0$. So from (6) we can derive

$$\Theta\big(M^{-1}(Tx_{n-1}, Tx_n, t) - M_T^{\alpha}\big) \leq \Theta\big(M^{-1}(x_{n-1}, x_n, t) - M_T^{\alpha}\big)^k$$

where $M_T^{\alpha} = [M(x_n, x_n, t)]^{\alpha} = 1^{\alpha} = 1$. This implies that

$$\Theta\big(M^{-1}(x_n, x_{n+1}, t) - 1\big) \leq \Theta\big(M^{-1}(x_{n-1}, x_n, t) - 1\big)^k. \tag{8}$$

Therefore,

$$\begin{aligned} 1 &< \Theta\big(M^{-1}(x_n, x_{n+1}, t) - 1\big) \leq \Theta\big(M^{-1}(x_{n-1}, x_n, t) - 1\big)^k \\ &\leq \Theta\big(M^{-1}(x_{n-2}, x_{n-1}, t) - 1\big)^{k^2} \leq \cdots \leq \Theta(M^{-1}(x_0, x_1, t) - 1)^{k^n}. \end{aligned} \tag{9}$$

Taking the limit as $n \to +\infty$ in (9), we get

$$\lim_{n \to +\infty} \Theta\big(M^{-1}(x_n, x_{n+1}, t) - 1\big) = 1$$

and since $\Theta \in \Delta_\Theta$, we obtain

$$\lim_{n \to +\infty} [M^{-1}(x_n, x_{n+1}, t) - 1] = 0. \tag{10}$$

Thus there exist $0 < r < 1$ and $0 < \ell \leq +\infty$ such that

$$\lim_{n \to +\infty} \frac{\Theta\big(M^{-1}(x_n, x_{n+1}, t)\big) - 1}{[M^{-1}(x_n, x_{n+1}, t) - 1]^r} = \ell. \tag{11}$$

Now, let $B^{-1} \in (0, \ell)$. So, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\Theta\big(M^{-1}(x_n, x_{n+1}, t)\big) - 1}{[M^{-1}(x_n, x_{n+1}, t) - 1]^r} \geq B^{-1} \quad \text{for all} \quad n \geq n_0$$

and so

$$n[M^{-1}(x_n,x_{n+1},t)-1]^r \leq nB[\Theta\big(M^{-1}(x_n,x_{n+1},t)-1\big)-1] \quad \text{for all} \quad n \geq n_0.$$

From (4), we can derive

$$n[M^{-1}(x_n,x_{n+1},t)-1]^r \leq nB[\Theta(M^{-1}(x_0,x_1,t)-1)^{k^n}-1] \quad \text{for all} \quad n \geq n_0.$$

Taking the limit as $n \to +\infty$ in the above inequality, we have

$$\lim_{n\to+\infty} n[M^{-1}(x_n,x_{n+1},t)-1]^r = 0. \tag{12}$$

From (12), it follows that there exists $N_0 \in \mathbb{N}$ such that

$$n[M^{-1}(x_n,x_{n+1},t)-1]^r \leq 1 \quad \text{for all} \quad n \geq N_0.$$

Thus

$$M^{-1}(x_n,x_{n+1},t)-1 \leq \frac{1}{n^{1/r}} \quad \text{for all} \quad n \geq N_0. \tag{13}$$

Now, for $n \geq N_0$ and a arbitrary positive integer p, by (13), we get

$$\begin{aligned} M^{-1}(x_n,x_{n+p},t)-1 &= M^{-1}(x_n,x_{n+p},t)-1 \\ &\leq \sum_{i=n}^{n+p-1}[M^{-1}(x_i,x_{i+1},t)-1] \\ &\leq \sum_{i=n}^{n+p-1}\frac{1}{i^{1/r}}. \end{aligned}$$

Since $0<r<1$, then

$$\lim_{n\to+\infty}\sum_{i=n}^{\infty}\frac{1}{i^{1/r}}=0$$

and hence $\{x_n\}$ is a Cauchy $O-$sequence. The hypothesis of $O-$completeness of triangular fuzzy metric space $(X,M,\star,\perp)$ ensures that there exists $x^* \in X$ such that $M(x_n,x^*,t) \to 1$ as $n \to +\infty$. Now, since T is an $\perp$-continuous mapping, then $M(x_{n+1},Tx^*,t) = M(Tx_n,Tx^*,1) \to 1$ as $n \to +\infty$. From

$$M^{-1}(x^*,Tx^*,t)-1 \leq M^{-1}(x^*,x_{n+1},t)-1+M^{-1}(x_{n+1},Tx^*,t)-1,$$

taking limit as $n \to +\infty$, we get $M^{-1}(x^*,Tx^*,t)=1$ and hence $x^* = Tx^*$. □

If we take $\Theta(t) = e^{rt}$, $r \in (0,1))$, in Theorem 3.1 we have the following:

Corollary 3.1. *Let* $(X, M, \star, \perp)$ *be an O-complete triangular fuzzy metric space with* M *regular and let* $T : X \to X$ *be an* $\perp-$*continuous and* $\perp-$*preserving self-mapping. Assume that there exist a real number* $r \in [0, 1)$ *such that for all* $t > 0$ *and* $x, y \in X$ *with* $x \perp y$,

$$\frac{1}{2}[M^{-1}(x, Tx, t) - 1] \leq M^{-1}(x, y, t) - 1$$

and $M^{-1}(Tx, Ty, t) > 1$, *we have*

$$M^{-1}(Tx, Ty, t) \leq rM^{-1}(x, y, t) + (1 - r)M_T^{\alpha}$$

where $M_T^{\alpha} = [M(y, Tx, t)]^{\alpha}$ *and* $\alpha \geq 0$. *Then* T *has a fixed point.*

For Suzuki type $\Theta-$contraction mapping that is not $\perp-$continuous we have the following theorem.

Theorem 3.2. *Let* $(X, M, \star, \perp)$ *be an O-complete fuzzy metric space with* M *regular and let* $T : X \to X$ *be a* $\perp-$*preserving self-mapping. Also, if* $\{x_n\}_{n\in\mathbb{N}}$ *be a O-sequence with* $x_n \to x \in X$, *then* $x \perp x_n$ *for all* $n \in \mathbb{N}$. *Assume that there exist* $0 < r < 1$ *and a function* $\Theta \in \Delta_{\Theta}$ *such that for all* $t > 0$ *and* $x, y \in X$ *with* $x \perp y$, $\frac{1}{2}[M^{-1}(x, Tx, t) - 1] \leq M^{-1}(x, y, t) - 1$ *and* $M^{-1}(Tx, Ty, t) > 1$, *we have*

$$\Theta\big(M^{-1}(Tx, Ty, t) - M_T^{\alpha}\big) \leq \Big[\Theta\big(M^{-1}(x, y, t) - M_T^{\alpha}\big)\Big]^r. \tag{14}$$

where $M_T^{\alpha} = [M(y, Tx, t)]^{\alpha}$ *and* $\alpha \geq 0$. *Then* T *has a fixed point.*

Proof. Since, $(X, M, \star, \perp)$ is an orthogonal fuzzy metric space, then there exists $x_0 \in X$ such that,

$$x_0 \perp y \text{ for all } y \in X. \tag{15}$$

As in the proof of Theorem 2.1, we deduce that an O-sequence $\{x_n\}$ starting at x_0 is Cauchy and so converges to a point $x^* \in X$. Hence, $x^* \perp x_n$ for all $n \in \mathbb{N}$. Also from (8) we know that,

$$\Theta\big(M^{-1}(x_n, x_{n+1}, t) - M_T^{\alpha}\big) \leq \Theta\big(M^{-1}(x_{n-1}, x_n, t) - M_T^{\alpha}\big)^k$$

where $M_T^{\alpha} = [M(x_n, x_n, t)]^{\alpha} = 1^{\alpha} = 1$. So we can write,

$$\begin{aligned}\Theta\big(M^{-1}(x_n, x_{n+1}, t) - 1\big) &\leq \Theta\big(M^{-1}(x_{n-1}, x_n, t) - 1\big)^k \\ &\leq \Theta\big(M^{-1}(x_{n-1}, x_n, t) - 1\big).\end{aligned}$$

This implies

$$M^{-1}(x_n, x_{n+1}, t) \leq M^{-1}(x_{n-1}, x_n, t). \tag{16}$$

First assume that, for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $M^{-1}(x_{k_n+1}, Tx^*, t) = 1$ and $k_n > k_{n-1}$ where $k_0 = 1$. Note that,

$$M^{-1}(x^*,Tx^*,t)-1 \leq M^{-1}(x^*,x_{k_n+1},t)-1+M^{-1}(x_{k_n+1},Tx^*,t)-1$$

and so we get, $M^{-1}(x^*,Tx^*,t)=1$. That is, x^* is a fixed point of T. Next we assume, $M^{-1}(x_{n+1},Tx^*)>1$.

Suppose that for some $m\in\mathbb{N}$, we have

$$\frac{1}{2}[M^{-1}(x_{m-1},x_m,t)-1] > M^{-1}(x_{m-1},x^*,t)-1$$

and

$$\frac{1}{2}[M^{-1}(x_m,x_{m+1},t)-1] > M^{-1}(x_m,x^*,t)-1.$$

Therefore from (16) and the above inequalities we get,

$$\begin{aligned}
&M^{-1}(x_{m-1},x_m,t)-1\\
&\leq M^{-1}(x_{m-1},x^*,t)-1+M^{-1}(x_m,x^*,t)-1\\
&< \tfrac{1}{2}[M^{-1}(x_{m-1},x_m,t)-1]+\tfrac{1}{2}[M^{-1}(x_m,x_{m+1},t)-1]\\
&\leq \tfrac{1}{2}[M^{-1}(x_{m-1},x_m,t)-1]+\tfrac{1}{2}[M^{-1}(x_{m-1},x_m,t)-1]\\
&= M^{-1}(x_{m-1},x_m,t)-1,
\end{aligned}$$

which is a contradiction. Hence, either

$$\frac{1}{2}[M^{-1}(x_{n-1},x_n,t)-1] \leq M^{-1}(x_{n-1},x^*,t)-1$$

or

$$\frac{1}{2}[M^{-1}(x_n,x_{n+1},t)-1] \leq M^{-1}(x_n,x^*,t)-1$$

holds for all $n\in\mathbb{N}$.

Let, $\frac{1}{2}[M^{-1}(x_{n-1},x_n,t)-1] \leq M^{-1}((x_{n-1},x^*,t)-1$. Than from (14) we get,

$$\begin{aligned}
\Theta\big(M^{-1}(Tx_n,Tx^*,t)-M_T^\alpha\big) &\leq \Theta\big(M^{-1}(x_n,x^*,t)-M_T^\alpha\big)^r\\
&\leq \Theta\big(M^{-1}(x_n,x^*,t)-M_T^\alpha\big)
\end{aligned}$$

where $M_T^\alpha = M(x^*,x_{n+1})$. This implies

$$M^{-1}(x_{n+1},Tx^*,t)-M_T^\alpha \leq M^{-1}(x_n,x^*,t)-M_T^\alpha.$$

Then

$$\lim_{n\to+\infty} M^{-1}(x_{n+1},Tx^*,t)=1$$

and hence

$$\begin{aligned}
&M^{-1}(x^*,Tx^*,t)-1\\
&\leq \lim_{n\to+\infty}[M^{-1}(x^*,x_{n+1},t)-1+M^{-1}(x_{n+1},Tx^*,t)-1]=0.
\end{aligned}$$

Thus, we get $x^*=Tx^*$. Similarly x^* is fixed point of T whence $\frac{1}{2}[M^{-1}(x_n,x_{n+1},t)-1] \leq M^{-1}(x_n,x^*,t)-1$. Therefore, T has a fixed point. □

Corollary 3.2. *Let $(X, M, \star, \perp)$ be an O-complete fuzzy metric space with O regular and let $T : X \to X$ be a $\perp-$preserving self-mapping. Also, if $\{x_n\}_{n\in\mathbb{N}}$ be a O-sequence with $x_n \to x \in X$, then $x\perp x_n$ for all $n \in \mathbb{N}$. Assume that there exists $0 < r < 1$ such that for all $t > 0$ and $x, y \in X$ with $x\perp y$, $\frac{1}{2}[M^{-1}(x, Tx, t) - 1] \leq M^{-1}(x, y, t) - 1$ and $M^{-1}(Tx, Ty, t) > 1$, we have*

$$M^{-1}(Tx, Ty, t) \leq rM^{-1}(x, y, t) + (1 - r)M_T^\alpha$$

where $M_T^\alpha = [M(y, Tx, t)]^\alpha$ and $\alpha \geq 0$. Then T has a fixed point.

Example 3.1. Let $X = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$. We define a binary relation $\perp$ by

$$(x, y)\perp(u, v) \Leftrightarrow x \leq u \quad \text{and} \quad y \leq v.$$

Clearly, by putting $x_0 = (0, 0)(X, \perp)$ is an O-set. And define fuzzy metric M (whence $a \star b = \min\{a, b\}$) by

$$M((x_1, x_2), (y_1, y_2), t) = \frac{t}{t + |x_1 - y_1| + |x_2 - y_2|}.$$

Evidently, $(X, M, \star, \perp)$ is an $O-$complete triangular fuzzy metric space. Define $T : X \to X$ by

$$T(x_1, x_2) = \begin{cases} (x_1, 0), & \text{if} \quad x_1 \leq x_2 \\ (0, x_2) & \text{if} x_1 > x_2 \end{cases}$$

If $(0, 0)\perp(u, v)$, then $T(0, 0)\perp T(u, v)$. Assume that,

$$(4, 0)\perp(4, 5),\ (4, 0)\perp(5, 4),\ (0, 4)\perp(4, 5),\ (0, 4)\perp(5, 4),$$

and so

$$T(4, 0) = (0, 0)\perp(4, 0) = T(4, 5),\ T(4, 0) = (0, 0)\perp(0, 4) = T(5, 4),$$

$$T(0, 4) = (0, 0)\perp(4, 0) = T(4, 5),\ T(0, 4) = (0, 0)\perp(0, 4) = T(5, 4).$$

That is, T is an $\perp-$preserving mapping.

Let (x_n, y_n) be an $O-$sequence with $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Equivalently, $x_n \leq x_{n+1}$, $y_n \leq y_{n+1}$, $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then we have, $x_n \leq x$ and $y_n \leq y$. That is, $(x_n, y_n)\perp(x, y)$.

Let, $x\perp y$, $\frac{1}{2}[M^{-1}(x, Tx, t)-1] \leq M^{-1}(x, y, t)-1$ and $M^{-1}(Tx, Ty, t) > 1$.

If, $x\perp y$ and $M^{-1}(Tx, Ty, t) > 1$, then

$$(x, y) \in \Big\{ \big((0, 0), (4, 5)\big), \big((0, 0), (5, 4)\big), \big((4, 0), (4, 5)\big), \big((4, 0), (5, 4)\big), \big((0, 4), (4, 5)\big), \big((0, 4), (5, 4)\big)\Big\}$$

Now since,

$$\frac{1}{2}[M^{-1}\big((0,0),T(0,0),t\big)-1]=0\leq\frac{9}{t}=M^{-1}\big((0,0),(4,5),t\big)-1,$$

$$\frac{1}{2}[M^{-1}\big((0,0),T(0,0),t\big)-1]=0\leq\frac{9}{t}=M^{-1}\big((0,0),(5,4),t\big)-1,$$

$$\frac{1}{2}[M^{-1}\big((4,0),T(4,0),t\big)-1]=\frac{2}{t}\leq\frac{5}{t}=M^{-1}\big((4,0),(4,5),t\big)-1,$$

$$\frac{1}{2}[M^{-1}\big((4,0),T(4,0),t\big)-1]=\frac{2}{t}\leq\frac{5}{t}=M^{-1}\big((4,0),(5,4),t\big)-1,$$

$$\frac{1}{2}[M^{-1}\big((0,4),T(0,4),t\big)-1]=\frac{2}{t}\leq\frac{10}{t}=M^{-1}\big((0,4),(4,5),t\big)-1$$

and

$$\frac{1}{2}[M^{-1}\big((0,4),T(0,4),t\big)-1]=\frac{2}{t}\leq\frac{10}{t}=M^{-1}\big((0,4),(5,4),t\big)-1,$$

then we have the following cases:

- if $(x,y)=\big((0,0),(4,5)\big)$, then,

$$M^{-1}(T(0,0),T(4,5),t)-1=\frac{4}{t}\leq\frac{7.38}{t}=0.82[M^{-1}((0,0),(4,5),t)-1]$$

- if $(x,y)=\big((0,0),(5,4)\big)$, then,

$$M^{-1}(T(0,0),T(5,4),t)-1=\frac{4}{t}\leq\frac{7.38}{t}=0.82[M^{-1}((0,0),(5,4),t)-1]$$

- if $(x,y)=\big((4,0),(4,5)\big)$, then,

$$M^{-1}(T(4,0),T(4,5),t)-1=\frac{4}{t}\leq\frac{4.1}{t}=0.82[M^{-1}((4,0),(4,5),t)-1]$$

- if $(x,y)=\big((4,0),(5,4)\big)$, then,

$$M^{-1}(T(4,0),T(5,4),t)-1=\frac{4}{t}\leq\frac{4.1}{t}=0.82[M^{-1}((4,0),(5,4),t)-1]$$

- if $(x,y)=\big((0,4),(4,5)\big)$, then,

$$M^{-1}(T(0,4),T(4,5),t)-1=\frac{4}{t}\leq\frac{4.1}{t}=0.82[M^{-1}((0,4),(4,5),t)-1]$$

- if $(x,y)=\big((0,4),(5,4)\big)$, then,

$$M^{-1}(T(0,4),T(5,4),t)-1=\frac{4}{t}\leq\frac{4.1}{t}=0.82[M^{-1}((0,4),(5,4),t)-1]$$

and so we can write,

$$M^{-1}(Tx, Ty, t) \leq 0.82M^{-1}(x, y, t) + (1 - 0.82)M_T^0$$

where $M_T^0 = [M(y, Tx, t)]^0 = 1$. Therefore all conditions of Theorem 3.2 hold and T has a fixed point.

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