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DUAL ANNIHILATORS IN BOUNDED BCK-ALGEBRAS

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Abstract. In this paper, for any two subsets A and C of a bounded BCK-algebra X, the concept of dual annihilator of A with respect to C, denoted by $(C : A)^d$, is introduced and some related properties are investigated. It is proved that if A is a dual ideal and C a normal ideal of an involutory BCK-algebra X, then $(C : A)^d$ is the relative pseudocomplement of A with respect to NC. Moreover, applying the concept of dual annihilator, the involutory dual ideal with respect to an ideal is defined, and it is shown that the set of all involutory dual ideals with respect to a normal ideal forms a distributive lattice.

1. Introduction

The notion of BCK-algebras was introduced by Y. Imai and K. Iséki [7] in 1966 as a generalization of set-theoretic difference and propositional calculi. In the same year, K. Iséki introduced the notion of BCI-algebras which is a generalization of BCK-algebras [8]. These algebras are two important classes of logical algebras. The concept of an ideal in a BCK-algebra (and BCI-algebra) was first introduced by K. Iséki and S. Tanaka [9] in 1976 (and by K. Iséki [10] in 1980.) The notion of dual ideals in BCK-algebras was introduced by E.Y. Deeba [5] in 1977. In 1980, E.Y. Deeba [6] introduced the notion of filters and in the setting of bounded implicative BCK-algebra constructed quotient algebra via a filter. But some facts show that the notions

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of dual ideals and filters defined by E.Y. Deeba are not dual to Iséki's ideals, hence J. Meng [13] introduced the notion of dual ideals for bounded BCK-algebras in 1986. In 1991, M. Aslam and A.B. Thaheem [2] introduced the concepts of annihilators and involutory ideals in commutative BCK-algebras and studied their properties. Y.B. Jun, et al. [11] generalized this concept to BCI-algebras and obtained some related results in 1996. M. Kondo [12] showed that the set of involutory ideals in BCK-algebra forms a boolean algebra in 1998. H.A.S. Abujabal, et al. introduced the concepts of generalized annihilators in commutative BCK- algebras and studied their properties [1]. In this paper, for any two subsets A and C of a bounded BCK-algebra X, we introduce the concept of dual annihilator of A with respect to C and investigate some related properties. Using the notation $(C : A)^d$ for the dual annihilator of A with respect to C, we prove that if A is a dual ideal and C a normal ideal of a BCK-algebra then $(C : A)^d$ is the relative pseudocomplement of A with respect to $NC = \{1 * c \mid c \in C\}$. Moreover, we investigate the relationship between $f((C : A)^d)$ and $(f(C) : f(A))^d$ for a BCK-homomorphism f. Finally, applying the concept of dual annihilator, we define the involutory dual ideals of a BCK-algebra and prove that the set of all involutory dual ideals with respect to a normal ideal forms a distributive lattice.

2. PRELIMINARIES

In this section, we review some definitions and results, which will be used in the remaining parts of this paper. The reader is referred to [14, 15] for more details.

Definition 2.1. *By a BCK-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:*

BCK-1: $((x * y) * (x * z)) * (z * y) = 0$,

BCK-2: $(x * (x * y)) * y = 0$,

BCK-3: $x * x = 0$,

BCK-4: $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$*,*

BCK-5: $0 * x = 0$*,*

for all $x, y, z \in X$*. An algebra* $(X, *, 0)$ *of type* $(2, 0)$ *is said to be a* BCI*-algebra if it satisfies the four axioms (BCK-1)-(BCK-4).*

We call the element 0 of X the zero element of X. For brevity, we often write X instead of $(X, *, 0)$ for a BCK-algebra (and BCI-algebra). In any BCK-algebra X (and BCI-algebra), one can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$. A non-empty subset A of X is called a subalgebra of X if $x * y \in A$ for all $x, y \in A$.

In any BCK-algebra X, the following hold: for any $x, y, z \in X$,

(a1) $x * 0 = x$,

(a2) $(x * y) * z = (x * z) * y$,

(a3) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$,

(a4) $(x * z) * (y * z) \leq x * y$,

(a5) $x * (x * (x * y)) = x * y$,

(a6) $x * (x * y) \leq x, y$.

A BCK-algebra X is called commutative if it satisfies the condition: $x * (x * y) = y * (y * x)$ for all $x \in X$. In this case, $x * (x * y)$ (and $y * (y * x)$) is the greatest lower bound of x and y with respect to BCK-order $\leq$, and we will denote it by $x \wedge y$.

A subset A of a BCK-algebra X is called an *ideal* of X if it satisfies (1) $0 \in A$; (2) $x, y * x \in A$ imply $y \in A$ for all $x, y \in X$.
An ideal A of X is called a *normal ideal* if $x * (x * y) \in A$ implies $y * (y * x) \in A$ for all $x, y \in X$. If there is an element 1 in X satisfying $x \leq 1$ for all $x \in X$, then the element 1 is said to be the unit of X. A BCK-algebra with unit is said to be bounded. In a bounded BCK-algebra $1 * x$ is denoted by Nx.

For a bounded BCK-algebra X, if an element x in X satisfies $NNx = x$, then x is called an involution. If any element in X is an involution, then X is called an involutory BCK-algebra. A non-empty subset D of a bounded BCK-algebra X is said to be a dual ideal of X if (1) $1 \in D$; (2) $N(Nx * Ny) \in D$ and $y \in D$ imply $x \in D$ for any $x, y \in X$.

A mapping $f : (X, *, 0) \rightarrow (X', *', 0')$ of a BCK-algebra X into a BCK-algebra X' is called a BCK-homomorphism if $f(x * y) = f(x) *' f(y)$ for all $x, y \in X$. Clearly, $f(0) = 0'$.

Every ideal A of X determines a congruence $\sim$ on X in the sense that $x \sim y$ if and only if $x * y$ and $y * x \in A$ for any $x, y \in X$. We will denote by A_x the equivalence class of an element $x \in X$; and by X/A the quotient algebra $X/\sim$, which is still a BCK-algebra.

If X is a BCK-algebra and A a non-empty subset of X, then the set $A^* := \{x \in X | (\forall a \in A)\ a * (a * x) = 0\}$ is called the annihilator of A (see [12]).

If X is a commutative BCK-algebra and C an ideal of X, then to any subset A of X, the set $(C : A) := \{x \in X \mid x \wedge A \subseteq C\}$, where $x \wedge A = \{x \wedge y \mid y \in A\}$ is called the generalized annihilator of A (relative to C) (see [1]).

In ([3]), A. Banderi, et al. applied the generalized annihilator to BCI-algebras, and for any two subsets A, C of a BCI-algebra X defined $(C : A)$ by the set $\{x \in X \mid a * (a * x) \in C \ for\ all\ a \in A\}$, which is called the relative annihilator of A with respect to C.

In a Lattice L with bottom element 0, for an element $x \in L$, the greatest element x^* satisfieng the condition $x \wedge x^* = 0$, if it exists, is said to be pseudocomplement of x. Let $(L; \wedge, \vee)$ be a lattice. Then for given $x, y \in L$, the relative pseudocomplement of x with respect to y, if it exists, is the (unique) element $(y : x) \in L$ such that: (i) $x \wedge (y : x) \leq y$; (ii) for every $z \in L$ if $x \wedge z \leq y$ then $z \leq (y : x)$ ([4]). It is easy to see that x^* is the relative pseudocomplement of x with respect to 0.

The mapping f is said to be a closure operator on a partially ordered set $(S, \leq)$ if it satisfies the following for any $a \in S$:
(i) $a \leq f(a)$; (ii) $f^2(a) = f(a)$; (iii) $a \leq b$ implies $f(a) \leq f(b)$ ([4]).

3. Dual Annihilators in Bounded BCK-algebras

In this section, we define the dual annihilator in bounded BCK-algebras and investigate some related properties.

Definition 3.1. *Let X be a bounded BCK-algebra and A, C non-empty subsets of X. Then, the set*

$$(C : A)^d = \{x \in X \mid Na * (Na * Nx) \in C \text{ for all } a \in A\}$$

is called the dual annihilator of A with respect to C.

Lemma 3.1. *Let X be a bounded BCK-algebra, C an ideal of X and $A \subseteq X$. Then $NC \subseteq (C : A)^d$, where $NC = \{1 * c \mid c \in C\}$.*

Proof. Let $x \in NC$. Then $x = Nc$ for some $c \in C$, and so $Nx = NNc$. Thus it follows from $NNc \leq c$ that $Nx \in C$, and hence, by using $(a6)$, we get $Na * (Na * Nx) \leq Nx \in C$ for all $a \in A$. It follows that $Na * (Na * Nx) \in C$ for all $a \in A$, and consequently $x \in (C : A)^d$. Therefore $NC \subseteq (C : A)^d$. □

In the following theorem, we give a characterization of $(C : A)^d$.

Theorem 3.1. *Let X be an involutory BCK-algebra, C an ideal of X and $A \subseteq X$. Then the following hold:*

(i) *$(C : A)^d \cap A = NC \cap A$;*

(ii) *if in addition $0 \in A$, then $(C : A)^d = NC$.*

Proof. (i) By Lemma 3.1, we only need to show the inclusion $(C : A)^d \cap A \subseteq NC \cap A$. For this, assume that $x \in (C : A)^d \cap A$. Then from $x \in (C : A)^d$, we get

$$Na * (Na * Nx) \in C \text{ for any } a \in A. \tag{3.1}$$

Putting $a = x$ in (3.1), we obtain $Nx * (Nx * Nx) \in C$, that is, $Nx \in C$ and so $NNx \in NC$. Thus by the involutory property, we get $x \in NC$. Therefore $(C : A)^d \cap A \subseteq NC$, and so the result holds.

(ii) By Lemma 3.1, we only need to show that $(C : A)^d \subseteq NC$. Let $x \in (C : A)^d$. Then we have

$$Na * (Na * Nx) \in C \text{ for any } a \in A. \tag{3.2}$$

Since $0 \in A$, putting $a = 0$ in (3.2), we get $NNNx \in C$, and so by the involutory property, we obtain $x \in NC$. Therefore $(C : A)^d \subseteq NC$. □

In the following example, we show that both the involutory property of X and $0 \in A$ in Theorem 3.1 are necessary.

Example 3.1. [15] *Let $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, a, b\}$ be two bounded BCK-algebras with the following Cayley tables:*

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	2	1	0	0
4	4	2	1	1	0

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

Then

(i) X is not involutory because $NN3 = 2$. Taking $C := \{0, 1\}$ and $A := \{3\}$, it is routine to check that $NC = \{2, 4\}$ and $(C : A)^d = X$. Therefore $(C : A)^d \cap A \neq NC \cap A$;

(ii) it is easy to see that Y is involutory. Taking $C := \{0\}$ and $A := \{b\}$, it can be checked that $(C : A)^d = Y$ and $NC = \{b\}$. Therefore $(C : A)^d \neq NC$.

The following lemma is an immediate consequence from Definition 3.1.

Lemma 3.2. *Let X be a bounded BCK-algebra. Then for any non-empty subsets A, B, C, D of X, the following hold:*

(i) *if $A \subseteq B$, then $(C : B)^d \subseteq (C : A)^d$;*

(ii) *if* $C \subseteq D$*, then* $(C : A)^d \subseteq (D : A)^d$.

Proposition 3.1. *Let* X *be a bounded* BCK*-algebra,* $\{C_i \mid i \in I\}$ *a family of subsets of* X *and* $A \subseteq X$*. Then the following hold:*

(i) $\bigcap_{i\in I}(C_i : A)^d = (\bigcap_{i\in I} C_i : A)^d$;

(ii) $\bigcup_{i\in I}(C_i : A)^d \subseteq (\bigcup_{i\in I} C_i : A)^d$.

Proof. (i) By Lemma 3.2(*ii*), from $\bigcap_{i\in I} C_i \subseteq C_i$, we get $(\bigcap_{i\in I} C_i : A)^d \subseteq (C_i : A)^d$ for any $i \in I$. It follows that $(\bigcap_{i\in I} C_i : A)^d \subseteq \bigcap_{i\in I}(C_i : A)^d$. To prove the reverse inclusion, assume that $x \in \bigcap_{i\in I}(C_i : A)^d$. Thus for any $i \in I$ and $a \in A$, we have $Na*(Na*Nx) \in C_i$ and consequently $Na*(Na*Nx) \in \bigcap_{i\in I} C_i$ for any $a \in A$. Hence $x \in (\bigcap_{i\in I} C_i : A)^d$ and so $\bigcap_{i\in I}(C_i : A)^d \subseteq (\bigcap_{i\in I} C_i : A)^d$. This completes the proof.

(ii) By the similar argument of (i), we can prove that $\bigcup_{i\in I}(C_i : A)^d \subseteq (\bigcup_{i\in I} C_i : A)^d$. □

The reverse inclusion in Proposition 3.1(ii) is not true in general as seen in the following example.

Example 3.2. *Let* $X = \{0, 1, 2, 3, 4\}$ *be a bounded* BCK*-algebra as in Example 3.1. Taking* $C_0 := \{0\}$*,* $C_1 := \{1\}$*,* $C_2 := \{2\}$ *and* $A := \{1, 2\}$*, it is routine to check that* $(C_0 : A)^d = \{4\}$*,* $(C_1 : A)^d = (C_2 : A)^d = \emptyset$ *and* $(\bigcup_{i=0}^{2} C_i : A)^d = X$*. Therefore* $(\bigcup_{i=0}^{2} C_i : A)^d \not\subseteq \bigcup_{i=0}^{2}(C_i : A)^d$.

Proposition 3.2. *Let* X *be a bounded* BCK*-algebra,* $\{A_i \mid i \in I\}$ *a family of subsets of* X *and* $C \subseteq X$*. Then the following hold:*

(i) $\bigcap_{i\in I}(C : A_i)^d = (C : \bigcup_{i\in I} A_i)^d$;

(ii) $\bigcup_{i\in I}(C : A_i)^d \subseteq (C : \bigcap_{i\in I} A_i)^d$.

Proof. (i) Using Lemma 3.2(i), we get $(C : \bigcup_{i\in I} A_i)^d \subseteq \bigcap_{i\in I}(C : A_i)^d$. To prove the reverse inclusion, assume that $x \in \bigcap_{i\in I}(C : A_i)^d$. Thus $x \in (C : A_i)^d$ for all $i \in I$, and consequently $Na * (Na * Nx) \in C$ for all $a \in A_i$. Hence $Na * (Na * Nx) \in C$ for all $a \in \bigcup_{i\in I} A_i$, and so $x \in (C : \bigcup_{i\in I} A_i)^d$. Therefore $\bigcap_{i\in I}(C : A_i)^d \subseteq (C : \bigcup_{i\in I} A_i)^d$, and so the result holds.

(ii) Using Lemma 3.2(i), the proof is straightforward. □

The reverse inclusion in Proposition 3.2(ii) is not true in general as seen in the following example.

Example 3.3. *Let $X = \{0, 1, 2, 3, 4\}$ be a bounded BCK-algebra as in Example 3.1. Taking $A_1 := \{1, 4\}$, $A_2 := \{3, 4\}$ and $C := \{0, 4\}$, it is routine to check that $(C : A_1)^d = \{2, 3, 4\}$, $(C : A_2)^d = \{4\}$ and $(C : A_1 \cap A_2)^d = X$. Therefore $X = (C : A_1 \cap A_2)^d \nsubseteq (C : A_1)^d \cup (C : A_2)^d = \{2, 3, 4\}$.*

In the following, we establish an important property of $(C : A)^d$.

Theorem 3.2. *If X is an involutory BCK-algebra with unit 1, then for any ideal C of X and $A \subseteq X$, $(C : A)^d$ is a dual ideal of X.*

Proof. Since $Na * (Na * N1) = 0 \in C$ for all $a \in A$, it follows that $1 \in (C : A)^d$. Now assume that $N(Ny * Nx) \in (C : A)^d$ and $x \in (C : A)^d$ for some $x, y \in X$. Then $Na * (Na * NN(Ny * Nx) \in C$ and $Na * (Na * Nx) \in C$ for all $a \in A$ and so by involutory property, we get

$$Na * (Na * (Ny * Nx) \in C, \ for \ all \ a \in A. \tag{3.3}$$

Using axiom (BCI-1), we obtain

$$(Na * (Na * Ny)) * (Na * (Na * Nx)) \leq (Na * Nx) * (Na * Ny). \tag{3.4}$$

Moreover, we have

$$
\begin{aligned}
&((Na*Nx)*(Na*Ny))*(Na*(Na*(Ny*Nx))) \\
&= ((Na*(Na*(Na*(Ny*Nx))))*(Na*Ny))*Nx && \text{by } (a2) \\
&= ((Na*(Ny*Nx))*(Na*Ny))*Nx && \text{by } (a5) \\
&\leq (Ny*(Ny*Nx))*Nx && \text{by } (\mathrm{BCI}-1) \ \textit{and} \ (a3) \\
&= (Ny*Nx)*(Ny*Nx) && \text{by } (a2) \\
&= 0 \in C && \text{by } (\mathrm{BCI}-3)
\end{aligned}
$$

Thus it follows that $((Na*Nx)*(Na*Ny))*(Na*(Na*(Ny*Nx))) \in C$ and so by (3.3) and (3.4), we get $(Na*(Na*Ny))*(Na*(Na*Nx)) \in C$. Hence from $Na*(Na*Nx) \in C$, we conclude $Na*(Na*Ny) \in C$ for any $a \in A$, and so $y \in (C:A)^d$. Therefore $(C:A)^d$ is a dual ideal of X. □

In the following, we introduce the relation between $(C:A)$ and $(C:A)^d$.

Theorem 3.3. *Let X be an involutory BCK-algebra. Then for any ideal C of X and $A \subseteq X$,*

$$N(C:A) = (C:NA)^d.$$

Proof. Let $z \in N(C:A)$. Then $z = Nx$ for some $x \in (C:A)$. It follows that

$$a*(a*x) \in C \ \textit{for any} \ a \in A. \tag{3.5}$$

Now, let $h := Na \in NA$ be an arbitrary element of NA. Then, by using the involutory property, we get $Nh = a$, and so by (3.5), we conclude $Nh*(Nh*x) \in C$. But $x = Nz$. Thus $Nh*(Nh*Nz) \in C$ for every $h \in NA$. This implies that $z \in (C:NA)^d$ and therefore $N(C:A) \subseteq (C:NA)^d$. To prove the reverse inclusion, let $z \in (C:NA)^d$. Then $Nh*(Nh*Nz) \in C$ for any $h \in NA$. Thus, since $Nh = a$, we get $a*(a*Nz) \in C$ for every $a \in A$ and so $Nz \in (C:A)$. This implies that $z \in N(C:A)$. Therefore $(C:NA)^d \subseteq N(C:A)$, and so the proof is completed. □

In the following, we determine the $(C:A)^d$ for some subsets A of X.

Lemma 3.3. *Let X be a bounded BCK-algebra with unit 1 and C an ideal of X. Then the following hold:*

(i) $(C:\{1\})^d = X$;

(ii) *if in addition X is involutory, then* $(C:NC)^d = X$.

Proof. (i) Since $N1*(N1*Nx) = 0 \in C$ for any $x \in X$, it follows that $X \subseteq (C:\{1\})^d$, and consequently $(C:\{1\})^d = X$.

(ii) Since X is involutory, it follows from Theorem 3.3 that $(C:NC)^d = N(C:C)$. But $(C:C) = X$. Therefore $(C:NC)^d = X$. □

In the following, we will investigate the $(C:A)^d$ in which C is a normal ideal.

Proposition 3.3. *Let X be an involutory BCK-algebra, C an ideal and $A \subseteq X$. Then the following hold:*

(i) *if* $(C:A)^d = X$, *then* $A \subseteq NC$;

(ii) *if in addition C is normal, then* $(C:A)^d = X$ *if and only if* $A \subseteq NC$.

Proof. (i) Let $(C:A)^d = X$ and $x \in A$. Then $x \in (C:A)^d$ and so $Na*(Na*Nx) \in C$ for every $a \in A$. Thus from $x \in A$, we get $Nx*(Nx*Nx) \in C$, that is, $Nx \in C$. It follows that $NNx \in NC$. But by the involutory property of X, we have $NNx = x$. Therefore $x \in NC$ and so $A \subseteq NC$.

(ii) By (i), we only need to prove the sufficiency. Let $A \subseteq NC$ and let x be an arbitrary element of X. Using axiom BCI-2, we have

$$Nx*(Nx*Na) \leq Na \in NA \; for \; any \; a \in A. \tag{3.6}$$

Now, for any $a \in A$, by hypotheses, there exists $c \in C$ such that $a = Nc$. Thus by (3.6), we get $Nx*(Nx*Na) \leq NNc$. But $NNc \leq c$ for any $c \in C$. Hence $Nx*(Nx*Na) \in C$ and so by the normality of C, we conclude $Na*(Na*Nx) \in$

C, which implies $x \in (C : A)^d$. Therefore $(C : A)^d = X$, and so the proof is completed. □

The following theorem provides a proof for the fact that $(C : A)^d$ is the pseudo-complement of dual ideal A with respect to NC.

Theorem 3.4. *Let X be an involutory BCK-algebra. Then for any normal ideal C and two dual ideals A, B of X,*

$$A \cap B \subseteq NC \;\Leftrightarrow\; A \subseteq (C : B)^d. \tag{3.7}$$

Proof. ($\Rightarrow$) Let $A \cap B \subseteq NC$ and $a \in A$. For any $b \in B$, using $(a6)$, we have $Na * (Na * Nb) \leq Na$. It follows that $NNa \leq N(Na * (Na * Nb))$. But by the involutory property of X, we have $NNa = a$. Thus $a \leq N(Na * (Na * Nb))$ and so, since A is a dual ideal and $a \in A$, we conclude $N(Na*(Na*Nb)) \in A$. By the similar argument, from $Na * (Na * Nb) \leq N(b)$, we can show that $N(Na * (Na * Nb)) \in B$. Therefore $N(Na*(Na*Nb)) \in A\cap B$, hence by hypothesis, we get $N(Na*(Na*Nb)) \in NC$. Then, using the involutory property of X, we obtain $Na * (Na * Nb) \in C$. Thus, by the normality of C, we get $Nb*(Nb*Na) \in C$, which implies $a \in (C : B)^d$. Therefore $A \subseteq (C : B)^d$.

($\Leftarrow$) Let $A \subseteq (C : B)^d$ and let x be an arbitrary element of $A \cap B$. Then $x \in (C : B)^d$, and hence we have

$$Nb * (Nb * Nx) \in C \text{ for every } b \in B. \tag{3.8}$$

Since $x \in B$, putting $b = x$ in (3.8), we get $Nx * (Nx * Nx) \in C$, that is, $Nx \in C$. Thus, by the involutoty property of X, we conclude $x \in NC$. Therefore $A \cap B \subseteq NC$. □

Corollary 3.1. *Let X be an involutory BCK-algebra. Then for any normal ideal C and dual ideal A of X, $(C : A)^d$ is the relative pseudocomplement of A with respect to NC.*

Proof. By Theorem 3.1(i), $(C:A)^d \cap A \subseteq NC$. Now assume that D is a dual ideal of X such that $D \cap A \subseteq NC$. Then by Theorem 3.4, we get $D \subseteq (C:A)^d$. Therefore $(C:A)^d$ is the relative pseudocomplement of A with respect to NC. □

In the following, we establish some other properties of dual annihilators.

Theorem 3.5. *Let X be a bounded BCK-algebra, C a normal ideal of X and $A \subseteq X$. Then the following hold:*

(i) $A \subseteq (C:(C:A)^d)^d$;

(ii) $(C:A)^d = (C:(C:(C:A)^d)^d)^d$.

Proof. (i) Clearly, if $a \in A$, then $Na * (Na * Nx) \in C$ for all $x \in (C:A)^d$. Thus, since C is a normal ideal, we conclude $Nx * (Nx * Na) \in C$, which implies $a \in (C:(C:A)^d)^d$. Therefore $A \subseteq (C:(C:A)^d)^d$.

(ii) By (i), we have $A \subseteq (C:(C:A)^d)^d$ and so by Lemma 3.2(i), we get $(C:(C:(C:A)^d)^d)^d \subseteq (C:A)^d$. On the other hand, using (i), we obtain $(C:A)^d \subseteq (C:(C:(C:A)^d)^d)^d$, and so the proof is completed. □

The reverse inclusion of Proposition 3.5(i) is not true in general as seen in the following example.

Example 3.4. [15] *Let $(X = \{0,1,2,3\}, *, 0)$ be a BCK-algebra with the following Cayley table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Taking $C := \{0,1\}$ and $A = \{2\}$, it is routine to check that C is a normal ideal and $(C:(C:A)^d)^d = \{2,3\}$. Therefore $(C:(C:A)^d)^d \nsubseteq A$.

In the following, we introduce other property of dual annihilators.

Proposition 3.4. *Let X be an involutory BCK-algebra. Then for any two subsets A and B and an ideal C of X, $((C : A)^d : B)^d \subseteq (C : (A \cap B))^d$.*

Proof. Let $x \in ((C : A)^d : B)^d$. Then $Nb * (Nb * Nx) \in (C : A)^d$ for every $b \in B$, and hence $Na * (Na * (Nb * (Nb * Nx))) \in C$ for every $a \in A$ and $b \in B$. Consequently, it follows that $Nt * (Nt * (Nt * (Nt * Nx))) \in C$ for every $t \in A \cap B$, and so by $(a5)$, we get $Nt * (Nt * Nx) \in C$ for every $t \in A \cap B$. This implies $x \in (C : (A \cap B))^d$, and therefore the proof is completed. □

In the following, we investigate the relationship between $f(C : A)^d$ and $(f(C) : f(A))^d$ for a BCK-homomorphism f.

Theorem 3.6. *Let $f : X \longrightarrow Y$ be a BCK-epimomorphism of bounded BCK-algebras and $A, C \subseteq X$. Then the following hold:*

(i) $f((C : A)^d) \subseteq (f(C) : f(A))^d$;

(ii) *if f is a BCK-isomorphism, then $f((C : A)^d) = (f(C) : f(A))^d$.*

Proof. (i) Let $y \in f((C : A)^d)$, then $y = f(x)$ for some $x \in (C : A)^d$. It follows that $Na * (Na * Nx) \in C$ for all $a \in A$, and so $f(Na * (Na * Nx)) \in f(C)$. Therefore

$$f(Na) * (f(Na) * f(Nx)) \in f(C) \ for \ all \ a \ \in A. \tag{3.9}$$

Clearly, since f is epimorphism, $f(1) = 1$, and so from (3.9), we conclude $Nf(a) * (Nf(a) * Nf(x)) \in f(C)$. This implies $y = f(x) \in (f(C) : f(A))^d$ for all $f(a) \in f(A)$. Therefore $f((C : A)^d) \subseteq (f(C) : f(A))^d$.

(ii) By (i), we only need to show that $(f(C) : f(A))^d \subseteq f((C : A)^d)$. Assume that $y \in (f(C) : f(A))^d$. Thus,

$$Nf(a) * (Nf(a) * Ny) \in f(C) \text{ for all } a \in A. \tag{3.10}$$

Since f is a BCK-isomorphism, we have $f(1) = 1$ and $y = f(x)$ for some $x \in X$, and so from (3.10), we get $f(Na * (Na * Nx)) \in f(C)$. It follows that $Na * (Na * Nx) \in$

$f^{-1}(f(C))$. By the injectivity of f, we have $f^{-1}(f(C)) = C$. Thus $Na * (Na * Nx) \in C$ for all $a \in A$, and so $x \in (C : A)^d$. Therefore $y \in f((C : A)^d)$, and hence $(f(C) : f(A))^d \subseteq f((C : A)^d)$. This completes the proof. □

In the following example, we show that both injective and surjective conditions of f in Theorem 3.6(ii) are necessary.

Example 3.5. [15] (i) *Let* $(X = \{0, 1, 2, 3\}, *, 0)$ *and* $(Y = \{0, a, b\}, *', 0)$ *be two bounded* BCK*-algebras with the following Cayley tables:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

$*'$	0	a	b
0	0	0	0
a	a	0	0
b	b	b	0

Define $f : X \longrightarrow Y$ *by* $f(0) = f(1) = 0$, $f(2) = a$ *and* $f(3) = b$. *It can be checked that* f *is a* BCK*-epimorphism but not injective. Taking* $C := \{1, 3\}$ *and* $A := \{2\}$, *it is routine to check that* $f((C : A)^d) = \{0, a\}$ *and* $(f(C) : f(A))^d = \{0, a, b\}$. *Therefore* $f((C : A)^d) \neq (f(C) : f(A))^d$.

(ii) [15] *Let* $(X = \{0, 1\}, *, 0)$ *and* $(Y = \{0, a, b\}, *', 0)$ *be two* BCK*-algebras with the following Cayley tables:*

$*$	0	1
0	0	0
1	1	0

$*'$	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

Define $f : X \longrightarrow Y$ *by* $f(0) = 0$ *and* $f(1) = b$. *It can be checked that* f *is a* BCK*-monomorphism but not surjective. Taking* $C := \{0\}$ *and* $A := \{1\}$, *it is* *routine to check that* $f((C:A)^d)=\{0,b\}$ *and* $(f(C):f(A))^d=\{0,a,b\}$. *Therefore* $f((C:A)^d)\neq(f(C):f(A))^d$.

In the following, we establish a property of dual annihilators in quotient BCK-algebras.

Proposition 3.5. *Let* X *be a bounded* BCK*-algebra. Then for any ideal* A *and subsets* I, J *of* X *containing* A,

$$(\frac{J}{A}:\frac{I}{A})^d=\frac{(J:I)^d}{A}. \tag{3.11}$$

Proof. We have

$$\begin{aligned} A_x\in(\frac{J}{A}:\frac{I}{A})^d &\Leftrightarrow NA_y*(NA_y*NA_x)\in\frac{J}{A}\ for\ all\ A_y\in\frac{I}{A}\\ &\Leftrightarrow Ny*(Ny*Nx)\in J\ for\ all\ y\in I\\ &\Leftrightarrow x\in(J:I)^d\\ &\Leftrightarrow A_x\in\frac{(J:I)^d}{A}. \end{aligned}$$

This completes the proof. □

For any X, we denote by $\mathcal{I}(X)$ the set of all dual ideals of X, and show that the notion of dual annihilator induces a closure operator as follows.

Theorem 3.7. *Let* X *be a bounded* BCK*-algebra and* C *a normal ideal of* X. *Then, the mapping* $f_C:\mathcal{I}(X)\to\mathcal{I}(X)$ *defined by*

$$f_C(A)=(C:(C:A)^d)^d \text{ for any } A\in\mathcal{I}(X), \tag{3.12}$$

is a closure operator on $(\mathcal{I}(X),\subseteq)$.

Proof. By Lemma 3.2(i) and Theorem 3.5, the result holds. □

Now, we introduce a class of dual ideals that is connected to the notion of dual annihilator, and show that it can be endowed with a lattice structure.

Definition 3.2. *Let X be a bounded BCK-algebra and C an ideal of X. Then a dual ideal A of X is called an involutory dual ideal with respect to C if $A = (C : (C : A)^d)^d$. We denote the set of all involutory dual ideals with respect to C by $S^d_C(X)$.*

Example 3.6. [15] *Let $(X = \{0, 1, 2, 3, 4\}, *, 0)$ be a BCK-algebra with the following Cayley table:*

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	2	0	0
4	4	3	2	1	0

Taking $A := \{3, 4\}$ and $C := \{0, 1\}$, we can check that C is an ideal, A is a dual ideal of X and $(C : (C : A)^d)^d = \{3, 4\}$ and so $A = (C : (C : A)^d)^d$. Therefore A is an involutory dual ideal with respect to C.

Lemma 3.4. *Let X be an involutory BCK-algebra. Then for any ideal C of X, NC is a dual ideal of X.*

Proof. Since $0 \in C$, $N0 = 1 \in NC$. Now let $N(Nx * Ny) \in NC$ and $y \in NC$. Then by the involutory property of X, we get $Nx * Ny \in C$ and $Ny \in C$. Thus, since C is an ideal, it follows that $Nx \in C$, and hence $x \in NC$. Therefore NC is a dual ideal. □

In the following, we introduce some involutory dual ideals.

Proposition 3.6. *Let X be an involutory BCK-algebra. Then the following hold:*

(i) *if C is an ideal of X, then $NC \in S^d_C(X)$;*
(ii) *if C is a normal ideal of X, then $(C : A)^d \in S^d_C(X)$ for any $A \subseteq X$.*

Proof. (i) By Lemma 3.4, NC is a dual ideal of X. Using Theorem 3.3, we get $(C : NC)^d = N(C : C)$. Note that $(C : C) = X$, and so by the involutory property of

X, we have $N(C:C) = NX = X$, and hence $(C:NC)^d = X$. Now, using Theorem 3.1(ii), we conclude $(C:(C:NC)^d)^d = (C:X)^d = NC$. Therefore $NC \in S_C^d(X)$.

(ii) It is clear by Theorem 3.2 and Theorem 3.5. □

Note that if C is a normal ideal of a bounded BCK-algebra X, then by Theorem 3.5(i), $X = (C:(C:X)^d)^d$. Therefore $S_C^d(X) \neq \emptyset$.

In the following theorem, we show that if C is a normal ideal, then $S_C^d(X)$ is a distributive lattice.

Theorem 3.8. *Let X be a bounded BCK-algebra and C a normal ideal of X. Then in the poset $(S_C^d(X), \subseteq)$, the following hold: for any $A, B \in S_C^d(X)$,*

(i) *$inf\{A, B\} = A \cap B$;*

(ii) *$sup\{A, B\} = (C:(C:A\cup B)^d)^d$;*

(iii) *$(S_C^d(X), \wedge, \vee)$ is a distributive lattice, where $A \wedge B = A \cap B$ and $A \vee B = (C:(C:A\cup B)^d)^d$.*

Proof. (i) Let $A, B \in S_C^d(X)$. Then by Theorem 3.5(i), we have $A \cap B \subseteq (C:(C:(A\cap B))^d)^d$. Also by Proposition 3.2(i), it follows from $A \cap B \subseteq A, B$ that $(C:(C:A\cap B)^d)^d \subseteq (C:(C:A)^d)^d \cap (C:(C:B)^d)^d$. But $A, B \in S_C^d(X)$. Thus $(C:(C:A)^d)^d \cap (C:(C:B)^d)^d = A \cap B$, and so $A \cap B = (C:(C:A\cap B)^d)^d$. It follows that $A \cap B \in S_C^d(X)$. Moreover, clearly $A \cap B$ is the biggest involutory dual ideal contained in A, B. This implies $inf\{A, B\} = A \cap B$.

(ii) By Proposition 3.6(ii), $(C:(C:A\cup B)^d)^d \in S_C^d(X)$, and from Theorem 3.5($i$), we get $A, B \subseteq A\cup B \subseteq (C:(C:A\cup B)^d)^d$. Now assume that $M \in S_C^d(X)$ such that $A, B \subseteq M$. By Lemma 3.2(i), we obtain $(C:M)^d \subseteq (C:A)^d \cap (C:B)^d$. But by Proposition 3.2(i), $(C:A)^d \cap (C:B)^d = (C:A\cup B)^d$. Hence $(C:M)^d \subseteq (C:A\cup B)^d$, and consequently, by Proposition 3.2(i), we get $(C:(C:(A\cup B))^d)^d \subseteq (C:(C:M)^d)^d = M$. It follows that $(C:(C:(A\cup B))^d)^d$ is the least involutory dual ideal containing A, B. Therefore, $sup\{A, B\} = (C:(C:(A\cup B))^d)^d$.

(iii) By (i) and (ii), $(S^d_C(X), \wedge, \vee)$ is a lattice. To prove the distributivity of $S^d_C(X)$, let $A, B, D \in S^d_C(X)$. It is well known that in any lattice, $(A \wedge B) \vee (A \wedge D) \subseteq A \wedge (B \vee D)$. Thus, it suffices to prove that $A \wedge (B \vee D) \subseteq (A \wedge B) \vee (A \wedge D)$. For brevity, we put $T := (A \wedge B) \vee (A \wedge D)$. Since $T \in S^d_C(X)$, it follows from Definition 3.2 that $T = (C : (C : T)^d)^d$. But $A \cap B \subseteq T$. Hence, we have

$$A \cap B \subseteq (C : (C : T)^d)^d. \tag{3.13}$$

Thus, if we intersect the two sides of (3.13) with $(C : T)^d$, we have

$$(A \cap B) \cap (C : T)^d \subseteq (C : T)^d \cap (C : (C : T)^d)^d. \tag{3.14}$$

On the other hand, by Theorem 3.1, we have

$$(C : T)^d \cap (C : (C : T)^d)^d = NC \cap (C : T)^d \tag{3.15}$$

It follows from (3.14) and (3.15) that $B \cap (A \cap (C : T)^d) \subseteq NC$. But by Corollary 3.1, $(C : B)^d$ is the pseudocomplement of B with respect to NC. Thus $A \cap (C : T)^d \subseteq (C : B)^d$. By the similar argument for D, we have $A \cap (C : T)^d \subseteq (C : D)^d$. Thus $A \cap (C : T)^d \subseteq (C : B)^d \cap (C : D)^d$, and hence from $(C : B)^d \cap (C : D)^d \in S^d_C(X)$, we get

$$A \cap (C : T)^d \subseteq (C : (C : ((C : B)^d \cap (C : D)^d))^d)^d \tag{3.16}$$

For brevity, we put $S := (C : ((C : B)^d \cap (C : D))^d)^d$. Thus, if we intersect the two sides of (3.16) with S and using Theorem 3.1(i), we have

$$(A \cap (C : T)^d) \cap S \subseteq (C : S)^d \cap S = NC \cap S, \tag{3.17}$$

and hence $(C : T)^d \cap (A \cap S) \subseteq NC$. But by Corollary 3.1, $(C : (C : T)^d)^d (= T)$ is the pseudocomplement of $(C : T)^d$ with respect to NC. Thus $A \cap S \subseteq T$. Also, by Proposition 3.2(i), we get $S = (C : (C : B \cup D)^d)^d$, and so $A \cap S = A \wedge (B \vee D)$. Therefore $A \wedge (B \vee D) \subseteq T = (A \wedge B) \vee (A \wedge D)$, and so the proof is completed. □

4. Conclusion

In this paper, we introduced the notion of dual annihilator in bounded BCK-algebras and investigated some related properties. We gave a characterization of the dual annihilators, and established the relationship between the relative annihilators and the dual annihilators. Also, using the above-mentioned notion, we characterized the relative pseudocomplement of a dual ideal with respect to a normal ideal. Finally, we defined the involutory dual ideal in bounded BCK-algebras, and showed that the set of all involutory dual ideals with respect to a normal ideal forms a distributive lattice.

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