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ON D-METACOMPACTNESS IN BITOPOLOGICAL SPACES

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ABSTRACT. In this paper we define pairwise D-metacompact spaces and study their properties and their relations with other topological spaces. Several examples are discussed and many will known theorems are generalized concerning metacompact spaces.

1. Introduction

In 1963, Kelly [7] introduced the notion of a bitopological space, i.e. a triple (X, τ_1, τ_2) where X is a set and τ_1, τ_2 are two topologies on X. He also defined pairwise regular ($P-$regular), pairwise normal ($P-$normal), and obtained generalization of several standard results such as Urysohn's lemma and Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces, see Kim [8], Fletcher, Hoyle and Patty [5]. In 1969, Fletcher et. al, [5] gave the definitions of $\tau_1\tau_2-$open and $P-$open covers in bitopological spaces. A cover $\tilde{U}$ of the bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2-$open if $\tilde{U} \subset \tau_1 \cup \tau_2$, if in addition, $\tilde{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member

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of τ_2, it is called $P-$open. Also they defined the concept of $P-$compact space as follows: A bitopological space (X, τ_1, τ_2) is called $P-$compact if every $P-$open cover of the space (X, τ_1, τ_2) has a finite subcover. While in 1972, Datta [3] defined $S-$compact space as follows: A bitopological space (X, τ_1, τ_2) is called $S-$compact if every $\tau_1\tau_2-$open cover of the space (X, τ_1, τ_2) has a finite subcover. In 1969, Birsan [1] gave the following definitions: A bitopological space (X, τ_1, τ_2) is called τ_1-compact with respect to τ_2 if for each τ_1-open cover of X, there is a finite τ_2-open subcover. A bitopological space (X, τ_1, τ_2) is called $B-$compact if it is τ_1-compact with respect to τ_2 and τ_2-compact with respect to τ_1. In 1975, Cooke and Reilly [2] discussed the relations between these definitions. In 1983 Fora and Hdieb [6] introduced the definition of $P-$Lindelöf, $S-$Lindelöf, $B-$Lindelöf spaces in analogue manner. They also gave the definitions of certain types of functions as follows : A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called $P-$continuous ($P-$open, $P-$closed, $P-$homeomorphism, respectively), if both functions $f_1 : (X, \tau_1) \longrightarrow (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \longrightarrow (Y, \sigma_2)$ are continuous (open, closed, homeomorphism, respectively).

In this paper we introduce the notion of $D-$metacompact spaces in bitopological spaces, and derive some related results. When (X, τ_1, τ_2) has the property Q this means that both (X, τ_1) and (X, τ_2) have this property. For instance a bitopological space (X, τ_1, τ_2) is called metacompact, if both (X, τ_1) and (X, τ_2) are metacompact spaces.

We will use the letters $P-$, $S-$ to denote the pairwise and semi, respectively, e.g. $P-$metacompact stands for pairwise metacompact, and similarly, one can define $P-$compact, $P-$Lindelöf, ... etc. Also, $S-$metacompact stands for semi metacompact, and similarly, one can define $S-$Lindelöf, ... etc.

Also we will use the letters $P-D-$, $S-D-$ to denote the pairwise$-D$ and semi$-D$, respectively, e.g. $P-D-$metacompact stands for pairwise $D-$metacompact, $S-D-$metacompact stands for semi $D-$metacompact.

τ_i-closure, τ_i-interior of a set A will be denoted by CL_iA, Int_iA respectively. The product of τ_1 and τ_2 will be denoted by $\tau_1 \times \tau_2$.

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ denote the set of all real numbers, integer numbers, natural numbers, and rational numbers, respectively. Let τ_{dis}, τ_{ind}, τ_u, τ_s, τ_{coc}, τ_{cof}, τ_l, τ_r denote the discrete, the indiscrete usual, Sorgenfrey line, cocountable, cofinite, left-ray, and right-ray topologies, respectively. Let ω_0 and ω_1 denote the cardinal numbers of $\mathbb{Z}$ and $\mathbb{R}$, respectively.

2. Pairwise D-Metacompact Spaces

In this section, we will introduce the concept of D-metacompactness in bitopological spaces, and introduce some of their properties, and relate it to other spaces.

Let us recall known definitions which will be used in the sequel.

Definition 2.1. A subset A of topological space (X, τ) is called a $D-$set if there are two open sets U and V such that $U \neq X$ and $A = U - V$. In this case we say that A is a $D-$set generated by U and V.

Observe that every open set U different from X is a $D-$set if $A = U$ and $V = \phi$.

Definition 2.2. A cover $\tilde{D} = \{D_\alpha : \alpha \in \Delta\}$of a topological space (X, τ) is said to be $D-$cover if each D_α is a $D-$set for all $\alpha \in \Delta$.

In a bitopological space (X, τ_1, τ_2) , the $D-$sets generated by open sets in (X, τ_i) are called $\tau_i - D-$sets denoted by D_{τ_i}.

A cover $\tilde{D}$ of the bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2 - D-$cover, if $\tilde{D} \subset D_{\tau_1} \cup D_{\tau_2}$. If in addition, $\tilde{D}$ contains at least one non-empty $\tau_1 - D$ set and at least one non-empty $\tau_2 - D$ set, it is called $P - D-$cover.

A pairwise $D-$cover $\tilde{V}$ of a bitopological space (X, τ_1, τ_2) is called parallel refinement of pairwise $D-$cover $\tilde{U}$ of X if each $\tau_i - D-$set of $\tilde{V}$ is contained in some $\tau_i - D-$set of $\tilde{U}$ $(i = 1, 2)$.

A pairwise $D-$cover $\tilde{U}$ of the bitopological space (X, τ_1, τ_2) is called pairwise point finite if each $x \in X$ is contained in a finite number of $\tau_1 - D-$members of $\tilde{U}$, or it contained in a finite number of of $\tau_2 - D-$members of $\tilde{U}$.

It is clear that every $P-$open cover with proper subsets is a $P-D-$cover, but the converse needs not be true. In the bitopological space $(\mathbb{R}, \tau_{cof}, \tau_{dis})$, $\{\{x\} : x \in \mathbb{R}\}$ is a $P-D-$cover that is not $P-$open cover.

Definition 2.3. A bitopological space (X, τ_1, τ_2) is called $P-D-$metacompact if every $P-D-$cover of the space (X, τ_1, τ_2) has a pairwise point finite parallel refinement.

A bitopological space (X, τ_1, τ_2) is called $S-D-$metacompact if every $\tau_1\tau_2-D-$cover of the space (X, τ_1, τ_2) has a pairwise point finite parallel refinement.

A bitopological space (X, τ_1, τ_2) is called τ_1-D-metacompact with respect to D_{τ_2} if each τ_1-D-cover of X has a point finite τ_2-D-parallel refinement.

A bitopological space (X, τ_1, τ_2) is called $B-D-$metacompact, if it is τ_1-D-metacompact with respect to D_{τ_2} and τ_2-D-metacompact with respect to D_{τ_1}.

Theorem 2.1. *Every $P-D-$metacompact space (X, τ_1, τ_2) is $P-$metacompact.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $P-$open cover of (X, τ_1, τ_2). Then $\tilde{U}$ is a $P-D-$cover, so it has a pairwise point finite parallel refinement. Hence the result. □

Recall that: A space (X, τ) is said to be locally indiscrete if every open set is clopen.

Definition 2.4. A bitopological space (X, τ_1, τ_2) is called locally indiscrete if every τ_i-open set is τ_i-clopen $(i = 1, 2)$.

In a locally indiscrete space (X, τ_1, τ_2), every $\tau_i - D-$set is clopen.

The converse of above theorem is not true as we will see in the following example.

Example 2.1. *Let* $\tau_1 = \{U \subset \mathbb{R} : \mathbb{Q} \subset U\} \cup \{\phi\}$ *and*

$\tau_2 = \{U \subset \mathbb{R} : \mathbb{R} - U \text{ is uncountable}\} \cup \{\phi\}$. Then $(\mathbb{R}, \tau_1, \tau_2)$ is $P-$metacompact which is not $P-D-$metacompact.

Since the $P-D-$cover $\{\{x\} : x \in \mathbb{R} - \mathbb{Q}\} \cup \{(-n, n) : n \in \mathbb{N}\}$ has no point finite parallel refinement.

The following theorem shows that the converse of the above theorem can be true under extra conditions.

Theorem 2.2. *Every locally indiscrete* $P-$*metacompact bitopological space* (X, τ_1, τ_2) *is* $P-D-$*metacompact.*

Proof. Let $\tilde{U}$ be a $P-D-$cover of (X, τ_1, τ_2). Then $\tilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$, where $U_\alpha \in D_{\tau_1}$ for each $\alpha \in \Delta$ and $V_\beta \in D_{\tau_2}$ for each $\beta \in \Gamma$. Since (X, τ_1, τ_2) is locally indiscrete, U_α is clopen set for each $\alpha \in \Delta$ and V_β is clopen set for each $\beta \in \Gamma$. Hence $\tilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ is a $P-$open cover, so it has a pairwise point finite parallel refinement. Hence the result. □

It is clear that the locally indiscrete bitopological space $(\mathbb{R}, \tau_{ind}, \tau_{dis})$ is $P-D-$metacompact, since it is $P-$metacompact.

Theorem 2.3. *The bitopological space* (X, τ_1, τ_2) *is* $S-D-$*metacompact if and only if it is* $D-$*metacompact and* $P-D-$*metacompact.*

Proof. $\Longrightarrow$) Assume that (X, τ_1, τ_2) is $S-D-$metacompact. Let $\tilde{U}$ be a $P-D-$cover of X. Then $\tilde{U}$ is a $\tau_1\tau_2-D-$cover of the space (X, τ_1, τ_2). Since (X, τ_1, τ_2) is $S-D-$metacompact, $\tilde{U}$ has a pairwise point finite parallel refinement. Hence (X, τ_1, τ_2) is $P-D-$metacompact. Also any τ_1-D-cover or τ_2-D- cover of (X, τ_1, τ_2) is a $\tau_1\tau_2-D-$cover. Hence (X, τ_1) and (X, τ_2) are $D-$metacompact. So (X, τ_1, τ_2) is $D-$metacompact.

$\Longleftarrow$) Assume that (X, τ_1, τ_2) is $D-$metacompact and $P-D-$metacompact. Let $\tilde{U}$ be a $\tau_1\tau_2 - D-$cover of (X, τ_1, τ_2). If $\tilde{U}$ is a $P-D-$cover, then the result follows. If $\tilde{U}$ is not $P-D-$cover, then it is $\tau_1 - D-$cover or $\tau_2 - D-$cover of (X, τ_1, τ_2). Since (X, τ_1, τ_2) is $D-$metacompact, $\tilde{U}$ has a pairwise point finite parallel refinement. So (X, τ_1, τ_2) is $S-D-$metacompact. □

In a bitopological space (X, τ_1, τ_2), the least upper bound topology of τ_1 and τ_2 is the smallest topology that contains $\tau_1 \cup \tau_2$.

The following theorems can be proved easily.

Theorem 2.4. *The bitopological space (X, τ_1, τ_2) is $S-D-$metacompact if and only if (X, τ) is $D-$metacompact, where τ is the least upper bound topology of τ_1 and τ_2.*

Corollary 2.1. *The bitopological space (X, τ_1, τ_2) is $S-$metacompact if and only if (X, τ) is metacompact, where τ is the least upper bound topology of τ_1 and τ_2.*

Theorem 2.5. *If a bitopological space (X, τ_1, τ_2) is $B-D-$metacompact, then both (X, τ_1) and (X, τ_2) must be $D-$metacompact spaces.*

Corollary 2.2. *If a bitopological space (X, τ_1, τ_2) is $B-$metacompact, then both (X, τ_1) and (X, τ_2) must be metacompact spaces.*

Recall that a space has a hereditary property P, if every subspace of it has this property.

Theorem 2.6. *If a bitopological space (X, τ_1, τ_2) is hereditary $D-$metacompact, then it is $S-D-$metacompact.*

Proof. Let $\tilde{U}$ be a $\tau_1\tau_2-D-$cover of (X, τ_1, τ_2). Then $\tilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$, where U_α is a $\tau_1 - D$ set for each $\alpha \in \Delta$ and V_β is a $\tau_2 - D$ set for each $\beta \in \Gamma$. Let $U = \bigcup\limits_{\alpha \in \Delta} U_\alpha$. If U is finite, we are done. Else write $U_\alpha = K_\alpha - W_\alpha$ where K_α , $W_\alpha \in \tau_1$

and $K_\alpha \neq X$. We can write $U_\alpha = K_\alpha \cap U - W_\alpha \cap U$. Let $\Omega = \{\alpha \in \Delta : K_\alpha \cap U = U\}$. So the sets $\{U_\alpha : \alpha \in \Delta - \Omega\}$ is a family of $D-$sets in U. If $x \in U - \bigcup_{\alpha\in\Delta-\Omega} U_\alpha$. Let $y \in U - \{x\}$. So there exists an open set O_x in τ_1 such that $x \in O_x \cap U$ and $y \notin O_x$. Let $\acute{U}_{\alpha_x} = (O_x \cap K_{\alpha_x} \cap U) - W_{\alpha_x} \cap U \subseteq U_{\alpha_x}$, where $x \in U_{\alpha_x} \in \tilde{U}$. Now; $\{U_\alpha : \alpha \in \Delta - \Gamma\} \cup \left\{\acute{U}_{\alpha_x} : x \in U - \bigcup_{\alpha\in\Delta-\Omega} U_\alpha\right\}$ is a $D-$cover for U . Similarly, we can do the same for the $\tau_2 - D-$cover for $V = \bigcup_{\beta\in\Gamma} V_\beta$.

Since U is $\tau_1 - D-$metacompact, it has a point finite parallel refinement say $\left\{U^*_\alpha : \alpha \in \Delta'\right\}$ and $U = \bigcup_{\alpha\in\Delta'} U^*_\alpha$.

Similarly, since V is $\tau_2 - D-$metacompact, it has a point finite parallel refinement say $\left\{V^*_\beta : \beta \in \Gamma'\right\}$ and $V = \bigcup_{\beta\in\Gamma'} V^*_\beta$.

Hence, $\left\{U^*_\alpha : \alpha \in \Delta'\right\} \cup \left\{V^*_\beta : \beta \in \Gamma'\right\}$ is a $\tau_1\tau_2 - D-$point finite parallel refinement of $\tilde{U}$. Hence the result. □

Definition 2.5. A bitopological space (X, τ_1, τ_2) is called $P - D-$Lindelöf if every $P - D-$ cover of the space (X, τ_1, τ_2) has a countable subcover.

A bitopological space (X, τ_1, τ_2) is called $S - D-$Lindelöf if every $\tau_1\tau_2 - D -$ cover of the space (X, τ_1, τ_2) has a countable subcover.

A bitopological space (X, τ_1, τ_2) is called $\tau_1 - D-$Lindelöf with respect to D_{τ_2} if for each $\tau_1 - D-$cover of X, there is a countable $\tau_2 - D-$subcover.

A bitopological space (X, τ_1, τ_2) is called $B - D-$Lindelöf

if it is $\tau_1 - D-$Lindelöf with respect to D_{τ_2} and $\tau_2 - D-$Lindelöf with respect to D_{τ_1}.

Example 2.2. *Consider the two topologies τ_1 and τ_2 on $\mathbb{R}$ defined by the basis:*

$\beta_1 = \{(-\infty, a) : a > 0\} \cup \{\{x\} : x > 0\}$

$\beta_2 = \{(a, \infty) : a < 0\} \cup \{\{x\} : x < 0\}$

Then X is $P-$metacompact, $P - D-$metacompact but not a $B-$metacompact, since for the τ_1-open cover $\{(-\infty, 2)\} \cup \{\{x\} : x > 1\}$of $\mathbb{R}$ has no point finite

τ_2-open refinement. So it is not a $B-D-$metacompact. It is clear that both $(\mathbb{R},\tau_1)$ and $(\mathbb{R},\tau_2)$ are $D-$metacompact spaces, so $(\mathbb{R},\tau_1,\tau_2)$ is $D-$metacompact.

Observe that $(\mathbb{R},\tau_1,\tau_2)$ is $S-$metacompact, $S-D-$metacompact. On the other hand we have $(\mathbb{R},\tau_1,\tau_2)$ is countably $P-D-$metacompact. Now we observe that $(\mathbb{R},\tau_1,\tau_2)$ is $P-$Lindelöf, $P-D-$Lindelöf, it is also clear that $(\mathbb{R},\tau_1,\tau_2)$ is not $B-$Lindelöf, and not Lindelöf, so it is not $B-D-$Lindelöf.

Definition 2.6. A subset D of a bitopological space (X,τ_1,τ_2) is called pairwise dense denoted by ($P-$dense) in X , if $CL_{\tau_1}D = CL_{\tau_2}D = X$.

A bitopological space (X,τ_1,τ_2) is called $P-$separable, if it has a $P-$dense countable subset D.

Definition 2.7. A subset A of a topological space (X,τ) is called $D-$dense, if for all $x\in X$ and every $D-$set D_x containing x we have $D_x\cap A\neq\phi$.

It is clear that every $D-$dense set is dense. The converse is not true, since in $(\mathbb{R},\tau_{cof})$ the set of all irrational numbers $\Bbbk=\mathbb{R}-\mathbb{Q}$ is dense but not $D-$dense, since $\{5\}$ is a $D-$set and $\Bbbk\cap\{5\}=\phi$.

Definition 2.8. A subset A of a bitopological space (X,τ_1,τ_2) is called pairwise $D-$dense, if for all $x\in X$ and every τ_i-D-set D_x containing x we have $D_x\cap A\neq\phi$, $(i=1,2)$.

A bitopological space (X,τ_1,τ_2) is called $P-D-$separable, if it has a $P-D-$dense countable subset D.

It is clear that the bitopological space $(\mathbb{Z},\tau_u,\tau_{cof})$ is $P-D-$separable.

Theorem 2.7. *A $P-D-$separable, $P-D-$metacompact space (X,τ_1,τ_2) is $P-D-$Lindelöf.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $P-D-$ cover of X. Assume that $\tilde{U}$ has no countable subcover. Let $\tilde{V} = \{V_\beta : \beta \in \Gamma\}$ be a point finite parallel refinement of $\tilde{U}$. Let D be a countable $D-$dense subset of X. Then $V_\beta \cap D \neq \phi$ for each $\beta \in \Gamma$. Thus $\tilde{V}$ is countable set. So, we may write $\tilde{V} = \{V_i : i \in \mathbb{N}\}$. But for each $i \in \mathbb{N}$, we have $V_i \subseteq U_{\alpha_i}$ for some $\alpha_i \in \Delta$. Thus $X = \bigcup_{i\in\mathbb{N}} V_i \subseteq \bigcup_{i\in\mathbb{N}} U_{\alpha_i} \subseteq X$. Hence $\{U_{\alpha_i} : i \in \mathbb{N}\}$ is a countable subcover of $\tilde{U}$. □

Since every $P-$open cover is $P-D-$cover, The following corollaries are easily proved.

Corollary 2.3. *A $P-D-$separable, $P-D-$metacompact space (X, τ_1, τ_2) is $P-$Lindelöf.*

Corollary 2.4. *A $P-$separable, $P-$metacompact space (X, τ_1, τ_2) is $P-$Lindelöf.*

The last corollary can be also found in [11].

Example 2.3. *(1) The bitopological space $(\mathbb{N}, \tau_{dis}, \tau_{ind})$ is $P-D-$metacompact, $S-D-$metacompact, not $B-D-$metacompact, $D-$metacompact space since $(\mathbb{N}, \tau_{dis})$ and $(\mathbb{N}, \tau_{ind})$ are $D-$metacompact. It is also countably $P-D-$metacompact, not countably $B-D-$metacompact. On the other hand $(\mathbb{N}, \tau_{dis}, \tau_{ind})$ is $P-D-$Lindelöf, $P-$separable, and so, it is $P-D-$separable.*

(2) The bitopological space $(\mathbb{R}, \tau_{dis}, \tau_u)$ is $P-D-$metacompact, $B-D-$metacompact, $D-$metacompact space since $(\mathbb{R}, \tau_{dis})$ and $(\mathbb{R}, \tau_u)$ are $D-$metacompact. It is also countably $P-D-$metacompact, countably $B-D-$metacompact. It is not $P-$separable, so it is not $P-D-$separable. It is clear that $(\mathbb{R}, \tau_{dis}, \tau_u)$ is neither $P-$Lindelöf nor $P-$compact. It is not $P-$countably compact. So $(\mathbb{R}, \tau_{dis}, \tau_u)$ is neither $P-D-$Lindelöf nor $P-D-$compact. Also it is not $P-D-$countably compact.

Theorem 2.8. *Every $P-D-$Lindelöf countably $P-D-$metacompact space (X, τ_1, τ_2) is $P-D-$metacompact.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $P-D-$cover of X. Since X is $P-D-$Lindelöf, $\tilde{U}$ has a countable subcover $\tilde{V} = \{V_{\alpha_i}\}_{i=1}^{\infty}$. Since X is countably$P-D$ $-$metacompact,$\tilde{V}$ has a point finite parallel refinement$\tilde{W}$ of $\tilde{U}$. Hence (X, τ_1, τ_2) is$P-D$ $-$metacompact. □

Since every $P-$open cover is $P-D-$cover, the following corollaries are easily proved.

Corollary 2.5. *Every $P-D-$Lindelöf countably $P-D-$metacompact space (X, τ_1, τ_2) is $P-$metacompact space.*

Corollary 2.6. *Every $P-$Lindelöf countably $P-D-$metacompact space (X, τ_1, τ_2) is $P-$metacompact.*

Corollary 2.7. *Every $P-$Lindelöf countably $P-$metacompact space (X, τ_1, τ_2) is $P-$metacompact.*

The last corollary can be also found in [11].

It is clear that the bitopological space $(\mathbb{Z}, \tau_{dis}, \tau_{ind})$ is $P-D-$metacompact, since it is countably $P-D-$metacompact and $P-D-$Lindelöf.

Theorem 2.9. *Every $P-D-$metalindelöf, countably $P-D$ $-$metacompact space (X, τ_1, τ_2) is $P-D-$metacompact space.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $P-D-$cover of X. Since X is $P-D-$metaLindelöf, $\tilde{U}$ has a point countable parallel refinement$\tilde{V} = \{V_{\alpha_i}\}_{i=1}^{\infty}$, which is also a $P-D-$cover of (X, τ_1, τ_2). Since X is countably $P-D-$metacompact,$\tilde{V}$ has a point finite parallel refinement$\tilde{W}$ of $\tilde{U}$.

Hence (X, τ_1, τ_2) is $P-D$ $-$metacompact. □

Corollary 2.8. *Every $P-D$ $-$metalindelöf, countably $P-D-$metacompact space (X, τ_1, τ_2) is $P-$metacompact space.*

Corollary 2.9. *Every $P-$metalindelöf, countably $P-D-$metacompact space (X,τ_1,τ_2) is $P-$metacompact space.*

Corollary 2.10. *Every $P-$metalindelöf, countably $P-$metacompact space (X,τ_1,τ_2) is $P-$metacompact space.*

The last corollary can be also found in [11].

The following theorem is easily proved.

Theorem 2.10. *Every $D-$metaLindelöf, countably $D-$metacompact space (X,τ_1,τ_2) is $D-$metacompact space.*

Theorem 2.11. *Every $P-D-$compact space (X,τ_1,τ_2) is $P-$compact.*

Proof. Let $\tilde{U}=\{U_\alpha:\alpha\in\Delta\}$ be a $P-$open cover of (X,τ_1,τ_2). Then $\tilde{U}$ is a $P-D-$cover, and so it has a finite subcover. Hence the result. □

The converse of above theorem is not true as we will see in the following example.

Example 2.4. *Let $X=\mathbb{R}$, $\tau_1=\{\phi,\ X\ ,\ \{1\}\ ,\ \{1,2\}\}$, $\tau_2=\tau_{cof}$. Then $(\mathbb{R},\tau_1,\tau_2)$ is $P-$compact but not $P-D-$metacompact, for the $P-D-$cover $\{\{x\}:x\in\mathbb{R}\}$ of $\mathbb{R}$ has no finite subcover.*

Example 2.5. *Let $X=\mathbb{R}$, $ß_1=\{X\ \}\cup\{x\}:x\in X-\{0\}\}$, $ß_2=\{X\}\cup\{x\}:x\in X-\{1\}\}$. Let τ_1 and τ_2 be the topologies on X which are generated by the bases $ß_1$ and $ß_2$, respectively. Then $(\mathbb{R},\tau_1,\tau_2)$ is $P-D-$metacompact and countably $P-D-$metacompact. On the other hand, $(\mathbb{R},\tau_1,\tau_2)$ is not $P-$Lindelöf, since the $P-$open cover $\{\{x\}\ :x\in X\}$ of X has no countable subcover. It is clear that $(\mathbb{R},\tau_1,\tau_2)$ is not $P-$compact, since the $P-$open cover $\{\{x\}\ :x\in X\}$ of X has no finite subcover, it is not compact space. So $(\mathbb{R},\tau_1,\tau_2)$ is not $P-D-$Lindelöf nor $P-D-$compact.*

The following definitions can be found in [10] and [6].

Example 2.6. *(1) The bitopological space* $(\mathbb{R}, \tau_{dis}, \tau_{coc})$ *is* $P-$*metacompact, not* $B-$*metacompact, since the* τ_1-*open cover* $\{\{x\} : x \in \mathbb{R}\}$*of* $\mathbb{R}$ *has no point finite* τ_2-*open refinement. Also it is* $P-D-$*metacompact*

and $B-D-$*metacompact. It is also countably* $P-$*metacompact. It is clear that* $(\mathbb{R}, \tau_{dis}, \tau_{coc})$ *is not* $P-D-$*Lindelöf space which is neither* $P-$*countably compact nor* $P-$*compact. So* $(\mathbb{R}, \tau_{dis}, \tau_{coc})$ *is* $P-$*Lindelöf space which is neither* $P-D-$*countably compact nor* $P-D-$*compact.*

(2) Let τ_s *denotes the Sorgenfrey line topology on* $\mathbb{R}$*.Then the bitopological space* $(\mathbb{R}, \tau_s, \tau_u)$ *is* $S-D-$*Lindelöf, so it is* $P-D-$*Lindelöf and* $D-$*Lindelöf. Also it is* $P-D-$*separable, so* $(\mathbb{R}, \tau_s, \tau_u)$ *is* $P-D-$*metacompact. It is clear that* $(\mathbb{R}, \tau_s, \tau_u)$ *is not* $B-$*metacompact, since the* τ_u *open cover* $\{(-n, n) : n \in \mathbb{N}\}$ *of* $\mathbb{R}$ *has no point finite* τ_s-*open refinement, because* $\tau_s \subsetneq \tau_u$*. It is also clear that* $(\mathbb{R}, \tau_s, \tau_u)$ *is neither* $S-D-$*compact nor* $S-D-$*countably compact.*

Remark 1. It is clear that every $P-D-$paracompact $(S-D-$paracompact$)$ space is $P-D-$metacompact $(S-D-$metacompact$)$.

Theorem 2.12. *If the bitopological space* (X, τ_1, τ_2) *is* $P-D-$*metacompact, then each* τ_1-*closed subset of* X *is* τ_2-D-*metacompact relative to* X*, and each* τ_2-*closed subset of* X *is* τ_1-D-*metacompact relative to* X*.*

Proof. Let $K \neq \phi$ be a τ_1-closed subset of X and $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a τ_2-D-cover of K. Then $\tilde{O} = \{X - K\} \cup \{U_\alpha : \alpha \in \Delta\}$ is a $P-D-$cover of X. Since (X, τ_1, τ_2) is $P-D-$metacompact, $\tilde{O}$ has a pairwise point finite parallel refinement, say $\{V_\beta : \beta \in \Gamma\} \cup \{U^*_\alpha : \alpha \in \Delta\}$, where V_β is a τ_1-D-sets for each $\beta \in \Gamma$, and U^*_α is a $\tau_2-D-$for each $\alpha \in \Delta$. Thus $\{U^*_\alpha : \alpha \in \Delta\}$ is a point finite parallel refinement of $\tilde{U}$. Hence K is a τ_2-D-metacompact relative to X. The proof of other case is similar. □

Corollary 2.11. *If the bitopological space* (X, τ_1, τ_2) *is* $P-$*metacompact, then each* τ_1-*closed subset of* X *is* $\tau_2 - D-$*metacompact relative to* X*, and each* τ_2-*closed subset of* X *is* $\tau_1 - D-$*metacompact relative to* X*.*

3. Product of D-metacompact Bitopological Spaces

Definition 3.1. A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called

$P-$continuous ($P-$ open, $P-$closed, $P-$homeomorphism, respectively) if the functions $f : (X, \tau_1) \longrightarrow (Y, \sigma_1)$ and $f : (X, \tau_2) \longrightarrow (Y, \sigma_2)$ are continuous (open, closed, homeomorphism, respectively).

Definition 3.2. A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called

$P-$perfect, if the function f is $P-$continuous, $P-$closed, and for all $y \in Y$, the set $f^{-1}(y)$ is $P-$compact.

Theorem 3.1. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *be a* $P-$*perfect function. If* X *is locally indiscrete space, then* X *is* $P - D-$*metacompact space if* Y *is so.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be any $P - D-$cover of X, where $\{U_\alpha : \alpha \in \Delta\}$ is a set of $\tau_1 - D-$members of $\tilde{U}$ and $\{V_\beta : \beta \in \Gamma\}$ is a set of $\tau_2 - D-$members of $\tilde{U}$.

Now, since f is $P-$perfect, for every $y \in Y$ we have $f^{-1}(y)$ is $P-$compact subset of X. So there exist finite subsets Δ_1 and Δ_2 of Δ and Γ respectively such that $f^{-1}(y) \subseteq \{\bigcup U_{\alpha_i} : \alpha_i \in \Delta_1\} \bigcup \{\bigcup V_{\beta_i} : \beta_i \in \Delta_2\}$. Now;

$O_{y_1} = Y - f(X - \bigcup U_{\alpha_i} : \alpha_i \in \Delta_1)$ is a τ_1-open subset of Y and $f^{-1}(O_{y_1}) \subseteq \{\bigcup U_{\alpha_i} : \alpha_i \in \Delta_1\}$.

$O_{y_2} = Y - f(X - \bigcup V_{\beta_i} : \beta_i \in \Delta_2)$ is a τ_2-open subset of Y and $f^{-1}(O_{y_2}) \subseteq \{\bigcup V_{\beta_i} : \beta_i \in \Delta_2\}$.

$y \in O_{y_1} \cup O_{y_2}$. So, $\tilde{O} = \{O_{y_1} : y \in Y\} \cup \{O_{y_2} : y \in Y\}$ is a $P-$open cover of Y. Since Y is $P - D-$metacompact, $\tilde{O}$ has a pairwise point finite parallel refinement

$\tilde{O}^* = \{O^*_{y_1} : y \in Y\} \cup \{O^*_{y_2} : y \in Y\}$.

Now, $O^*_{y_1}$ is a $\tau_1 - D-$open subset of X and $O^*_{y_2}$ is a $\tau_2 - D-$open subset of X.

Since f is perfect, the set $\{f^{-1}(O^*_{y_1}) : y \in Y\} \cup \{f^{-1}(O^*_{y_2}) : y \in Y\}$ is a pairwise point finite parallel refinement of X. So, X is $P - D-$metacompact. □

Corollary 3.1. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a $P-$perfect function. Then X is $P-$metacompact space if Y is so.*

The last corollary can be also found in [11].

Definition 3.3. Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Then the Cartesian product of (X, τ_1, τ_2) and (Y, σ_1, σ_2) is the bitopological space $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$.

Lemma 3.1. *If A is a compact subset of a topological space $(X\ , \tau)$ and B is a compact subset of a topological space $(Y\ , \sigma)$ and $A \times B \subseteq W$; where W is an open subset of $X \times Y$, then there exist open sets U and V in X and$Y$ respectively such that $A \times B \subseteq U \times V \subseteq W$.*

Theorem 3.2. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. If X is a Hausdorff compact, then the projection function $P : X \times Y \longrightarrow Y$ is $P-$closed .*

Proof. To show that the projection function $P : X \times Y \longrightarrow Y$ is $P-$closed, we show that the projection functions $P_1 : (X \times Y, \tau_1 \times \sigma_1) \longrightarrow (Y, \sigma_1)$ and $P_2 : (X \times Y, \tau_2 \times \sigma_2) \longrightarrow (Y, \sigma_2)$ are closed. Let $y \in Y$ and let U be an open set in $(X \times Y, \tau_1 \times \sigma_1)$ such that $P_1^{-1}(\{y\}) \subseteq U$. So by (Wallace lemma), there exists a σ_1-open set in$Y$ say V_y such that $P_1^{-1}(\{y\}) = X \times \{y\} \subseteq X \times V_y \subseteq U$. So, $y \in V_y$ and $P_1^{-1}(V_y) = X \times V_y \subseteq U$. So, $P_1 : (X \times Y, \tau_1 \times \sigma_1) \longrightarrow (Y, \sigma_1)$ is closed function. Similarly, we have $P_2 : (X \times Y, \tau_2 \times \sigma_2) \longrightarrow (Y, \sigma_2)$ is closed function. Hence, the projection function $P : X \times Y \longrightarrow Y$ is $P-$closed. □

Theorem 3.3. *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be bitopological spaces such that* X *is a compact Hausdorff space and* Y *a* $P-D-$*metacompact space. Then* $X \times Y$ *is a* $P-D-$*metacompact.*

Proof. First, we know that the projection function $P : X \times Y \longrightarrow Y$ is $P-$continuous and for all $y \in Y$, we have $P^{-1}\{y\} = X \times \{y\} \simeq X$ is $P-D-$compact. Then $P : X \times Y \longrightarrow Y$ is $P-$perfect function. Since Y is $P-D-$metacompact, then $X \times Y$ is $P-D-$metacompact by Theorem 3.2. □

Corollary 3.2. *The product of a compact Hausdorff bitopological space and a* $P-$*metacompact bitopological space is* $P-$*metacompact.*

The last two corollaries can be also found in [11].

Example 3.1. *The bitopological space* $(\mathbb{R}, \tau_{cof}, \tau_{dis})$ *is* $P-$*compact, so it is* $P-$*metacompact but not* $P-D-$*compact. Also it is* $B-D-$*metacompact,* $D-$*metacompact space since both* $(\mathbb{R}, \tau_{cof})$ *and* $(\mathbb{R}, \tau_{dis})$ *are* $D-$*metacompact.*

The space $(\mathbb{R}^2, \tau_{cof} \times \tau_{cof}, \tau_{dis} \times \tau_{dis})$ *is* $P-D-$*metacompact, but not* $P-$*compact nor* $P-$*Lindelöf, since the* $P-$*open cover*

$\{\mathbb{R} \times (\mathbb{R} - \{0\})\} \cup \{(x\ ,0) : x \in \mathbb{R}\}$ *for* $\mathbb{R}^2$ *has no countable subcover. Hence the space* $(\mathbb{R}^2, \tau_{cof} \times \tau_{cof}, \tau_{dis} \times \tau_{dis})$ *is not* $P-D-$*compact nor* $P-D-$*Lindelöf.*

Lemma 3.2. *Let* $f : (X, \tau) \longrightarrow (Y, \sigma)$ *be a continuous, onto function. If* $\tilde{A} = \{A_\alpha : \alpha \in \Delta\}$ *is a point finite family subset of* X*, then* $\{f(A_\alpha) : \alpha \in \Delta\}$ *is a point finite family subset of* Y*.*

Theorem 3.4. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *be a* $P-$*continuous,* $P-$*closed, onto function and* Y *is locally indiscrete space. Then* Y *is* $P-D-$*metacompact, if* X *is so.*

Proof. Let $\tilde{V} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be any $P-D-$cover of Y, where $\{U_\alpha : \alpha \in \Delta\}$ are $\sigma_1 - D-$members of $\tilde{V}$ and $\{V_\beta : \beta \in \Gamma\}$ are $\sigma_2 - D-$members of $\tilde{V}$. Since f is $P-$continuous, onto function, the set

$\tilde{U} = \{f^{-1}(U_\alpha) : \alpha \in \Delta\} \cup \{f^{-1}(V_\beta) : \beta \in \Gamma\}$ is a $P-$open cover of X.

Since X is $P-D-$metacompact space, there exists a pairwise point finite open parallel refinement of $\tilde{U}$, say $\tilde{U}^* = \{f^{-1}(U_\alpha^*) : \alpha \in \Delta\} \cup \left\{f^{-1}\left(V_\beta^*\right) : \beta \in \Gamma\right\}$. Thus, $\tilde{V}^* = \{U_\alpha^* : \alpha \in \Delta\} \cup \left\{V_\beta^* : \beta \in \Gamma\right\}$ is a pairwise point finite parallel refinement of $\tilde{V}$. So, Y is $P-D-$metacompact. □

Corollary 3.3. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a $P-$continuous, $P-$closed, onto function. Then Y is $P-D-$metacompact, if X is $P-$metacompact .*

Corollary 3.4. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a $P-$continuous, $P-$closed, onto function. Then Y is $P-$metacompact, if X is so.*

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