

# BADE-PROPERTY; SURVEY AND COMPARISON WITH $\lambda$ -PROPERTY, RUSSO-DYE THEOREM AND EXTREMALLY RICHNESS

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**ABSTRACT.** In this article, we survey a geometric property, called Bade-property, originally introduced by William Bade. First, we review Bade's work in normed linear spaces. Next, we illustrate various interesting results of Bade-property in the spaces of convergent sequences established by Aizpuru. Then, we investigate Bade-property in comparison with some other geometric properties, such as  $\lambda$ -property due to Aron and Lohman, Russo-Dye Theorem and extremally richness in  $C^*$ -algebras,  $JB^*$ -algebras/triples and  $JBW^*$ -triples.

## 1. INTRODUCTION

On his 1971 lecture notes [4], Bade proved an interesting theorem in the extremal structure of the unit ball of spaces of continuous functions; the theorem states that for a compact Hausdorff space  $K$ , the closed unit ball of the space of continuous functions on  $K$  is the closed convex hull of its extreme points if and only if  $K$  is 0-dimensional. This result led him to define the Bade-property, a Banach space  $X$  is said to have the Bade-property if  $B_X = \overline{co(ext(B_X))}$ , where  $B_X$  denotes the closed unit ball of  $X$  and  $ext(B_X)$  is the set of extreme points of  $B_X$ . The definition of Bade motivated Aron and Lohman [1] to introduce a stronger version of Bade-property, called the  $\lambda$ -property. A Banach space is said to have the  $\lambda$ -property if each element of its closed unit ball is a convex combination of an extreme point of the ball with positive weight and a vector of norm at the most one.

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One of the open problems mentioned at the beginning, when Aron and Lohman defined  $\lambda$ -property was the relation between Bade-property and  $\lambda$ -property which has been recently treated in the early nineties.

Aron and Lohman showed that their  $\lambda$ -property is strictly stronger than Bade-property. However, in 1990, Oates [18] proved that Aron and Lohman's  $\lambda$ -property coincides with the Bade-property in the class of spaces of continuous functions.

Later, Aizpuru showed a great admiration for the work of Bade. In his article "The Bade property and the  $\lambda$ -property in spaces of convergent sequences" [2], he summarized all results about the relation between Bade-property and  $\lambda$ -property in the space of convergent sequences. Later he introduced a new property, called the E-property and proved that it lies in between  $\lambda$ -property and Bade-property.

In this survey, we review classical literatures in which Bade-property was introduced and studied by mathematicians named up. In addition, we discuss the relations between the Bade-property and some other geometric properties such as Krein-Milman-property,  $\lambda$ -property, the known Russo-Dye theorem and extremally richness property in  $C^*$ -algebras,  $JB^*$ -algebras/triples and  $JBW^*$ -triples.

In section two of this survey we give various preliminary concepts from theories of Jordan structures (namely, algebras, Jordan triple systems and  $JB^*$ -algebras/triples), Banach algebras and  $C^*$ -algebras along with some of their properties, which we need for the sequel.

Third section contains results of Bade-property found in classical references.

A review of  $\lambda$ -property together with main results about this property in  $C^*$ -algebras,  $JB^*$ -algebras/triples and  $JBW^*$ -triples were included in section 4.

Section 5 and section 6 are devoted for looking into the relation between Bade-property, Krein-Milman property,  $\lambda$ -property and the notion of extremally richness.

## 2. PRELIMINARIES

This section contains the background material. First, we give a brief description of some standard notions such as, extreme points of the closed unit ball, unitary elements, convex hull of extreme points, basic concepts from Jordan structures (namely, Jordan triple systems/algebras and  $JB^*$ -triples/algebras) as well as some related results, which we will need for the sequel. Among other references given in the end, [1, 6, 7, 8, 11, 13, 15, 16, 18, 20, 23, 24, 38, 39] are the main sources of information

for us. The notion of extreme points goes back to H. Minkowski ([16], Vol. II, p. 157-161). An element  $x$  of a convex set  $Y$  in a linear space  $X$  is described as an extreme point of  $Y$  if whenever  $x = (1 - \lambda)y + \lambda z$ , with  $0 < \lambda < 1$  and  $y, z \in Y$ , then  $y = z = x$ .

Thus if  $Y$  is a convex set, then  $x$  is an extreme point of  $Y$  provided  $x$  is not an interior point of any line segment contained in  $Y$ .

Minkowski proved that if  $Y$  is a compact convex set in  $\mathbb{R}^3$ , then each point of  $Y$  can be expressed as a convex combination of extreme points of  $Y$ . In fact, if  $K$  is a finite-dimensional compact convex set in locally convex space, then

$$K = co(ext(K)).$$

This result known in  $\mathbb{R}^n$  as Minkowski's theorem.

In 1940, Krein and Milman extended Minkowski's theorem to infinite dimensional spaces as follows.

**Theorem 2.1** ([13] Krein-Milman, 1940). *Let  $K$  be a nonempty compact convex subset of a Hausdorff locally convex space, then  $K$  is the closed convex hull of  $ext(K)$ . That is,*

$$K = \overline{co(ext(K))}$$

Recall [11] that a subset  $S$  of a linear space is said to be totally disconnected if each pair of its points can be separated by sets that are both open and closed (clopen sets). In case of real scalars, a set  $S$  is totally disconnected if the only connected subsets of  $S$  are the singletons, the situation is quite different for the complex scalars.

A Banach space in which every point of norm 1 is an extreme point of the closed unit ball is called strictly convex.

The symbols  $l_1(X)$ ,  $l_\infty(X)$  and  $c(X)$  denote the space of all  $X$ -valued sequences  $x = (x_n)$  which are absolutely summable, bounded and convergent, respectively.  $l_1(X)$  is endowed with the norm  $\|x\| = \sum_{n=1}^{\infty} \|x_n\|$ , while the norm in  $l_\infty(X)$  and  $c(X)$  is given by  $\|x\| = \sup_n \|x_n\|$ . If  $T$  is a compact Hausdorff space,  $C_X(T)$  denotes

the space of continuous  $X$ -valued functions on  $T$  endowed with the sup norm.

An algebra  $A$  over a field  $K$  is a vector space  $A$  over  $K$  such that for each ordered pair of elements  $x, y \in A$  a unique product  $xy$  in  $A$  is defined with the properties,

- $x(y + z) = xy + xz$
- $(x + y)z = xz + yz$
- $a(xy) = (ax)y = x(ay)$  for all  $x, y, z \in A$  and scalars  $a$ .

An algebra  $A$  is called an associative (respectively, commutative) algebra if  $(xy)z = x(yz)$  (respectively,  $xy = yx$ ) for all  $x, y, z \in A$ . The associative axiom is frequently incorporated in the definition of an algebra, especially in operator algebras. We call an algebra  $A$  unital if it contains a non-zero element  $e \in A$  that we call a unit satisfying,  $xe = ex = x$  for all  $x \in A$ .

An algebra  $A$  over a field  $K$  with unit  $e$  is said to be a normed algebra when  $A$  is a normed space such that  $\|xy\| \leq \|x\|\|y\|$  for all  $x, y \in A$ , and  $\|e\| = 1$ . If  $A$  is a Banach space relative to this norm,  $A$  is said to be a Banach algebra.

An involutions  $*$  on a real or a complex algebra  $A$ , is a map  $*$  :  $A \rightarrow A$  such that for all  $x, y \in A$  and  $\lambda, \mu \in \mathbb{C}$ ,

- (1)  $(\lambda x + \mu y)^* = \bar{\lambda}x^* + \bar{\mu}y^*$ ;
- (2)  $x^{**} = x$ ;
- (3)  $(x \circ y)^* = x^* \circ y^*$ , where  $x^*$  denotes the image of  $x$  under  $*$ .

An element  $x \in A$  is called self-adjoint (or Hermitian) if  $x^* = x$ .

A  $C^*$ -algebra  $A$  is a Banach Algebra with an involution  $*$  such that for all  $x \in A$ ,  $\|xx^*\| = \|x\|^2$ .

An example of a  $C^*$ -algebra is  $\mathbb{C}$ ; here the algebraic operations are the usual ones, the norm is the modulus  $|\cdot|$ , and the involution is conjugation,  $z \rightarrow \bar{z}$ .

Another example is obtained when  $X$  is a locally compact Hausdorff space, and  $A = C_0(X) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is continuous and vanishes at infinity}\}$ , if we used pointwise operations and the sup-norm given by  $\|f\| = \sup_{x \in X} |f(x)|$  which makes

$X$  into a  $C^*$ -algebra. If  $X$  is compact and Hausdorff, then  $C_0(X)$  is just the space of continuous functions  $f : X \rightarrow \mathbb{C}$ . We write  $C(X)$  for  $C_0(X)$  in this case.

A commutative algebra  $\mathcal{J}$  with product  $\circ$ , over a field of characteristic not 2, is called a Jordan algebra if  $x^2 \circ (x \circ y) = (x^2 \circ y) \circ x$  for all  $x, y \in \mathcal{J}$ , where  $x^2$  means  $x \circ x$ .

Of course, any associative algebra  $\mathcal{A}$ , over a field of characteristic  $\neq 2$ , is a Jordan algebra, denoted by  $\mathcal{A}^+$ , with the same linear space structure but with new binary product  $x \circ y = \frac{1}{2}(xy + yx)$ , called anti-commutator product, where  $xy$  denotes the associative product in the parent algebra (cf. [8]).

The underlying binary product  $\circ$  of a Jordan algebra  $\mathcal{J}$  induces a triple product  $\{xyz\} = (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y$  called the Jordan triple product; this translates as  $\{xyz\} = \frac{1}{2}(xyz + zyx)$  if  $x \circ y = \frac{1}{2}(xy + yx)$  in terms of the associative product  $xy$ .

In any Jordan algebra  $\mathcal{J}$  with unit  $e$ , the Jordan triple product is linear in each of its three variables, symmetric in the outer variables, and it reduces to the original Jordan product (binary) if one of the three variables is the unit  $e$  (cf. [8]).

An element  $x$  in a Jordan algebra  $\mathcal{J}$  with unit  $e$  is called invertible if there exists element  $y \in \mathcal{J}$ , called the inverse of  $x$ , such that  $x \circ y = e$  and  $x^2 \circ y = x$ ; in case of an associative algebra, this invertibility condition is equivalent to  $xy = e = yx$ . Inverse of any element  $x$  in a Jordan algebra is unique if exists, and is symbolized as  $x^{-1}$  (cf. [8]). The set of all invertible elements in Jordan algebra  $\mathcal{J}$  is denoted by  $\mathcal{J}^{-1}$ .

The notion of a  $JB^*$ -algebra introduced originally by Irvin Kaplansky [39] in 1976, as a natural generalization of  $C^*$ -algebras and initially called a Jordan  $C^*$ -algebra [39]. A  $JB^*$ -algebra is a complex Banach Jordan algebra  $\mathcal{J}$  with Jordan binary product  $\circ$  equipped with an involution  $*$  satisfying  $\|x \circ y\| \leq \|x\|\|y\|$  and  $\|\{xx^*x\}\| = \|x\|^3$  for all  $x, y \in \mathcal{J}$ .

If, in addition,  $\mathcal{J}$  has a unit  $e$  with  $\|e\| = 1$  then it is called a unital  $JB^*$ -algebra. Of course, any  $C^*$ -algebra with associative product  $ab$  is a  $JB^*$ -algebra under the anti-commutator product  $x \circ y = \frac{1}{2}(xy + yx)$ . An element  $u$  in a unital  $JB^*$ -algebra

$\mathcal{J}$  is unitary if  $u \in \mathcal{J}^{-1}$  and  $u^{-1} = u^*$ ; we denote the set of all unitary elements in  $\mathcal{J}$  by  $\mathcal{U}(\mathcal{J})$ .

A more general notion of Jordan structures is the Jordan triple system (in short, Jordan triple) which is a vector space  $\mathcal{J}$  over a field of characteristic not 2, endowed with a triple product  $\{xyz\}$  which is linear and symmetric in the outer variables  $x, z$ , and linear or anti-linear in the inner variable  $y$  satisfying the Jordan triple identity  $\{xu\{y vz\}\} + \{\{xvy\}uz\} - \{yv\{xuz\}\} = \{x\{uyv\}z\}$  for all  $u, v, x, y, z \in \mathcal{J}$ ; further, if  $\mathcal{J}$  is a Banach space and the triple product is continuous on  $\mathcal{J} \times \mathcal{J} \times \mathcal{J}$ , then  $\mathcal{J}$  is called a Banach Jordan triple (cf. [38]).

Recall ([39], p. 336) that a  $JB^*$ -triple is a complex Banach space  $\mathcal{J}$  together with a continuous, sesquilinear, operator-valued map  $(x, y) \in \mathcal{J} \times \mathcal{J} \mapsto L_{x,y}$  that defines a triple product  $L_{x,y}z := \{xyz\}$  in  $\mathcal{J}$  making it a Jordan triple system such that each  $L_{x,x}$  is a positive hermitian operator on  $\mathcal{J}$  and  $\|\{xx^*x\}\| = \|x\|^3$  for all  $x \in \mathcal{J}$ . The symbol  $Q_x$  will denote the conjugate linear operator on  $\mathcal{J}$  given by  $Q_x(z) := \{xzx\}$ .

A  $JB^*$ -triple  $\mathcal{J}$  is called a  $JBW^*$ -triple if  $\mathcal{J}$ , as a Banach space, is the dual of another Banach space  $\mathcal{J}^*$ , called the predual of  $\mathcal{J}$ . The predual of a  $JBW^*$ -triple is unique. The second dual  $\mathcal{J}^{**}$  of any  $JB^*$ -triple  $\mathcal{J}$  is always  $JBW^*$ -triple.

### 3. BADE-PROPERTY

This section contains a review of the relevant literature for this survey. In every topological space the empty set and the one-point sets are connected. Therefore, a space is *totally disconnected* if the only connected subsets are singletons (one-point subsets). Also, a space is *0-dimensional* if the clopen subsets form a basis for the topology.

Moreover, a 0-dimensional Hausdorff space is necessarily totally disconnected, but the converse fails. However, a locally compact Hausdorff space is 0-dimensional if and only if it is totally disconnected.

**Theorem 3.1.** [4] *Let  $K$  be a compact Hausdorff space. Then the following conditions are equivalent:*

- (1)  $B_{C(K)} = \overline{\text{co}(\text{ext}(B_{C(K)}))}$ ,
- (2)  $K$  is 0-dimensional.

The above result of Bade motivated him to define the Bade-property (cf.[4]).

**Definition 3.1.** [4] (*Bade, 1971*) Let  $X$  be a Banach space. We say that  $X$  has the Bade-property if

$$\overline{\text{co}(\text{ext}(B_X))} = B_X.$$

The following results illustrate celebrated spaces that satisfy Bade-property.

**Theorem 3.2.** [4] Let  $X$  be a compact Hausdorff space. Then  $B_{C(X, \mathbb{R})}$  is the closed convex hull of its extreme points if and only if  $X$  is totally disconnected.

**Theorem 3.3.** [20] If  $X$  is a compact Hausdorff space, then  $B_{C(X, \mathbb{C})}$  is the closed convex hull of its extreme points.

Let  $X$  be a normed space. Aizpuru [2] induced that  $X$  has the Bade-property if and only if

$$(3.1) \quad \sup_{x \in B_X} f(x) = \sup_{x \in \text{ext}(B_X)} f(x)$$

for every continuous linear form  $f : X \rightarrow \mathbb{R}$ .

**Theorem 3.4.** [2] Let  $X$  be a normed space and let  $n \in \mathbb{N}$ . Consider the space  $X^n$ , with the norm

$$(3.2) \quad \|(x_1, \dots, x_n)\| = \max_{1 \leq i \leq n} \|x_i\|.$$

Then:

- (a)  $(x_1, \dots, x_n) \in \text{ext}(B_{X^n})$  if and only if  $x_i \in \text{ext}(B_X)$  for every  $i \in 1, 2, \dots, n$ .
- (b)  $X^n$  has the Bade-property if and only if  $X$  has the Bade-property.

**Theorem 3.5.** [2] Let  $K$  be a compact Hausdorff space and let  $X$  be a normed space,

- (a) If  $C(K)$  and  $X$  have the Bade-property, then  $C(K, X)$  has the Bade-property.
- (b) If  $K$  is non-perfect (has isolated points) and  $C(K, X)$  has the Bade-property, then  $X$  has the Bade-property.

Consequently, we have the following corollary.

**Corollary 3.1.** [2]

Let  $X$  be a normed space. Then  $X$  has the Bade-property if and only if  $c(X)$  has the Bade-property.

**Theorem 3.6.** [3] The space  $l_1(X)$  has the Bade-property if and only if  $X$  has the Bade-property.

Recall that a normed space  $X$  has the Krein-Milman property if every closed and bounded convex subset of  $X$  is the closed convex hull of its set of extreme points which is a stronger version of the Bade-property. On the other hand, the following example is about a Banach space having the Bade-property, while its closed unit ball is not compact.

**Example 3.1** ([24], p. 413). The Banach space  $l_1$  of all absolutely summable real sequences  $x = (x_n)$  with norm  $\|x\| = \sum |x_n|$  satisfies the Bade-property and this doesn't follow from Krein-Milman theorem because  $B_{l_1}$  is not compact; only finite dimensional spaces have compact unit balls. Let  $E = \text{ext}(B_{l_1})$ . To prove that  $B_{l_1}$  is the closure of  $\text{co}(E)$ , we show first that  $E$  consists of sequences  $e_n$ ,  $n = 1, 2, \dots$  and their negatives, where  $e_n$  is the sequence with 1 in the  $n^{\text{th}}$  position and zeros elsewhere. Let us first verify that each  $e_n$  is in  $E$ . Suppose that  $e_n = \frac{1}{2}(x + y)$ , where  $x, y \in B_{l_1}$ . Say  $x = (x_1, x_2, \dots)$  and  $y = (y_1, y_2, \dots)$ . Then  $1 = \frac{1}{2}(x_n + y_n)$  and  $|x_n| \leq 1, |y_n| \leq 1$ . This implies that  $x_n = y_n = 1$ . Since  $x$  and  $y$  have norm at most 1, it follows that  $x_j = y_j = 0$  for  $j \neq n$ . Thus  $x = y = e_n$ . Hence,  $e_n \in E$ . For the reverse inclusion, let  $x = (x_n) \in B_{l_1}$  and suppose that  $x \neq e_n$  for all  $n$ . We may assume that  $\|x\| = 1$  since otherwise  $x$  would clearly not be an extreme point. Then we can write  $x = y + z$ , where  $y = (x_1, x_2, \dots, x_k, 0, 0, \dots)$ ,  $z = (0, 0, \dots, 0, x_{k+1}, x_{k+2}, \dots)$  and neither  $y$  nor  $z$  is the zero sequence. Let  $u = (\frac{1}{\|y\|})y$  and  $v = (\frac{1}{\|z\|})z$ . Then, since  $\|y\| + \|z\| = \|x\| = 1$ , it follows that  $x = \|y\|u + (1 - \|y\|)v$ . Thus  $x \notin E$ . We now show that an arbitrary sequence  $x = (x_i)$  in  $B_{l_1}$  is in the closure of  $\text{co}(E)$ . For each  $n$ , let  $y_n = \sum_{k=1}^n x_k e_k$ . Then  $y_n \in \text{co}(E)$ . To see this, consider the point  $(x_1, x_2, \dots, x_n)$  in the space  $l_1^n$  (which is  $\mathbb{R}^n$  with norm  $\|(x_1, x_2, \dots, x_n)\| = \sum_{i=1}^n |x_i|$ ). Let  $E_n = \text{ext}(B_{l_1^n})$ . Then the members of  $E_n$  are the vectors  $\pm e_n$  of length  $n$ . Hence by the Krein-Milman theorem  $B_{l_1^n} = \text{co}(E_n)$ . Thus,  $(x_1, x_2, \dots, x_n) \in \text{co}(E_n)$  and consequently,  $y_n \in \text{co}(E)$ . Since  $x = \lim_{n \rightarrow \infty} y_n$ , we have the desired result.



4.  $\lambda$ -PROPERTY

This section is dedicated to study the  $\lambda$ -property of normed spaces,  $C^*$ -algebras,  $JB^*$ -algebras/triples and  $JBW^*$ -triples. The original definition of the  $\lambda$ -property due to Aron and Lohman and some examples are given in the following subsection. Then we review this property in other algebraic structures. For details in this area, we refer to standard sources [17, 19, 22, 34]

**4.1.  $\lambda$ -property of Normed spaces.** If  $X$  is a normed space, the closed unit ball, open unit ball and unit sphere will be denoted by  $B_X, U_X$  and  $S_X$  respectively. If  $x, y \in X$ , then  $(x : y)$  denotes  $\{\lambda x + (1 - \lambda)y : 0 < \lambda < 1\}$ ,  $((x : y]$  has the obvious corresponding meaning). The set of extreme points of a convex subset  $A$  of  $X$  is denoted by  $ext(A)$ . Recall that  $X$  is strictly convex if  $ext(B_X) = S_X$ . A convex set  $A$  is called a polyhedron in case  $ext(A)$  is finite and  $A = co(ext(A))$ .

**Definition 4.1.** If  $X$  is a normed space and  $x$  is in the closed unit ball  $B_X$  of  $X$ , a triple  $(e, y, \lambda)$  is said to be amenable to  $x$  in case  $e \in ext(B_X)$ ,  $y \in B_X$ ,  $0 < \lambda \leq 1$  and

$$(4.1) \quad x = \lambda e + (1 - \lambda)y.$$

In this case, the number  $\lambda(x)$  is defined by  $\lambda(x) = \sup \{\lambda : (e, y, \lambda) \text{ is amenable to } x\}$

$X$  is said to have the  $\lambda$ -property if each  $x \in B_X$  admits an amenable triple. If  $X$  has the  $\lambda$ -property and, in addition, satisfies  $\inf \{\lambda(x) : x \in B_X\} > 0$ , we say  $X$  has the uniform  $\lambda$ -property.

The following theorem indicates several facts of amenable vectors and  $\lambda$ -function.

**Theorem 4.1** ([1], Proposition 1.2). *Let  $X$  be a normed space.*

- (1) *If  $e \in ext(B_X)$ , then  $\lambda(e) = 1$ .*
- (2) *If  $(e, y, \lambda)$  is amenable to  $x$  and  $\lambda < 1, \|y\| < 1$ , then there exists  $\lambda' > \lambda$  and  $\lambda' \in S_X$  such that  $y \in (\lambda' : x]$  and  $(e, \lambda', \lambda')$  is amenable to  $x$ .*
- (3) *If  $(e, y, \lambda)$  is amenable to  $x$  and  $0 < \lambda' < \lambda$ , there exists  $y' \in (\lambda' : x)$  such that  $(e, y', \lambda')$  is amenable to  $x$ .*
- (4) *If  $X$  has the  $\lambda$ -property, then  $\lambda(x) \leq \frac{(1+\|x\|)}{2}$  for all  $x \in B_X$ .*
- (5) *If  $X$  is a strictly convex space, then  $\lambda(x) = \frac{(1+\|x\|)}{2}$  for all  $x \in B_X$  and  $\lambda(x)$  is attained.*

- (6) *If  $X$  has the  $\lambda$ -property and  $Y$  is a linear subspace of  $X$  such that  $Y$  has the  $\lambda$ -property, and  $\text{ext}(B_Y) \subset \text{ext}(B_X)$ , then  $\lambda_Y(x) \leq \lambda_X(x)$  for all  $x$  in  $Y$ , where  $\lambda_Y$  and  $\lambda_X$  are the  $\lambda$ -functions defined above in  $B_Y$  and  $B_X$ .*

In ([1], Theorem 2.9), authors showed that  $l_1(X)$  has the  $\lambda$ -property but not the uniform  $\lambda$ -property. Also, they showed that if  $X$  is an infinite-dimensional strictly convex normed space, then  $c(X)$  has the uniform  $\lambda$ -property. Aron and Lohman in ([1], Theorem 1.13) showed that if  $X$  is a strictly convex normed space, then  $l_\infty(X)$  has the uniform  $\lambda$ -property (cf. [22], Theorem 4). Assume that  $X_k$ ,  $k = 1, 2, \dots$ , have the  $\lambda$ -property. If there exists a subset  $N_0$  of  $\mathbb{N}$  with finite complement, such that  $\inf_{k \in N_0} \lambda_k(X_k) > 0$ , then  $X = (\oplus \sum_{k=1}^{\infty} X_k)$  has the  $\lambda$ -property.

Let  $X$  be a normed space having the  $\lambda$ -property (resp. uniform  $\lambda$ -property). Let  $Y$  be a normed space and let  $f : X \longrightarrow Y$  be an isometric isomorphism. Then  $Y$  has the  $\lambda$ -property (resp. uniform  $\lambda$ -property) ([17], Proposition 1.1)

**Theorem 4.2** ([1], Theorem 3.3). *Let  $X$  be a normed space satisfying the  $\lambda$ -property.*

- (1) *If a convex function  $f : B_X \longrightarrow \mathbb{R}$  attains its maximum value, then it attains its maximum value at a member of  $\text{ext}(B_X)$ .*
- (2) *If  $X$  is a Banach space, then  $B_X = \overline{\text{co}(\text{ext}(B_X))}$ .*

Moreover, if  $\mathcal{J}$  has the uniform  $\lambda$ -property, (that is,  $\lambda(x) \geq \epsilon > 0$  for all  $x \in B_{\mathcal{J}}$ ), then  $\mathcal{J}$  has the Krein-Milman-like property,  $(\text{co}(\text{ext}(B_{\mathcal{J}})) \cap \mathcal{J}) = \mathcal{J}$ .

#### 4.2. $\lambda$ -property of $C^*$ -Algebras.

**Definition 4.2** ( $\lambda$ -function). *The  $\lambda$ -function, defined on an element  $T$  of a  $C^*$ -algebra  $\mathcal{J}$ , is the supremum,  $\lambda(T)$ , of numbers  $\lambda$  in  $(0, 1]$  for which there exists a pair  $V, B$  in  $\text{ext}(B_{\mathcal{J}}) \times B_{\mathcal{J}}$ , such that  $T = \lambda V + (1 - \lambda)B$ .*

If  $\mathcal{J}$  is a  $C^*$ -algebra which is closed in weak operator topology on  $B(H)$ , then  $\mathcal{J}$  is a von Neumann algebra. Abstractly, this means that  $\mathcal{J}$  is a dual space. In a von Neumann algebra the unit ball is (weakly) compact, so that the Krein-Milman theorem applies. Also  $\mathcal{J}$  is generated by its projections (in the strong sense that the spectral resolution of every normal operator in  $\mathcal{J}$  belongs to  $\mathcal{J}$ ) and the set of projections in  $\mathcal{J}$  forms a compact lattice (a sublattice of the set of closed subspaces of  $\mathcal{J}$ ).

We shall assume throughout that  $\mathcal{J}$  is unital, so  $I \in \mathcal{J}$ , for the simple reason that otherwise  $\text{ext}(B_{\mathcal{J}}) = \emptyset$ . For a unital  $C^*$ -algebra  $\mathcal{J}$ , the elements in  $\text{ext}(B_{\mathcal{J}})$  were characterized by Kadison [10] in 1951 as follows,

$$\text{ext}(B_{\mathcal{J}}) = \{V \in B_{\mathcal{J}} \mid (I - VV^*)\mathcal{J}(I - V^*V) = 0\}$$

Thus  $V \in \text{ext}(B_{\mathcal{J}})$  if it is a partial isometry (linear map between Hilbert spaces such that it is an isometry on the orthogonal complement of its kernel), such that the two projections  $I - V^*V$  and  $I - VV^*$  (on the kernels of  $V$  and  $V^*$ , respectively) are centrally orthogonal (so that even the two-sided ideals they generate are orthogonal). If  $\mathcal{J}$  is a prime  $C^*$ -algebra (like  $B(H)$ ), or even better, simple (no non-trivial closed ideals), then elements  $V$  in  $\text{ext}(B_{\mathcal{J}})$  are either isometries ( $V^*V = I$ ) or co-isometries ( $VV^* = I$ ).

An important class of extreme points is the set  $U(\mathcal{J})$  of unitary elements in  $\mathcal{J}$  (elements  $U$  such that  $U^*U = UU^* = I$ , or  $U^* = U^{-1}$ ). In contrast to general elements in  $\text{ext}(B_X)$ , the elements in  $U(\mathcal{J})$  are normal operators. We say that the  $C^*$ -algebra is finite, if  $T^*T = I$  implies  $TT^* = I$  for all  $T$  in  $\mathcal{J}$ , that is, every isometry is unitary. In case  $\mathcal{J}$  is a von Neumann algebra, this implies that  $\text{ext}(B_X) = U(X)$ .

Even when non-unitary extreme points exist, the group  $U(\mathcal{J})$  is rich enough to ensure that  $\text{co}(U(\mathcal{J}))$  is dense in  $B_{\mathcal{J}}$ . This fact is the Russo-Dye theorem [25].

**Theorem 4.3** ([19], Corollary 5.5). *A function algebra  $C(X)$ , where  $X$  is a compact Hausdorff space, has the  $\lambda$ -property if and only if the dimension of  $X$  is at most one, in which case  $C(X)$  has the uniform  $\lambda$ -property for  $\lambda = \frac{1}{2}$ .*

**Theorem 4.4** ([19], Theorem 8.3). *A prime  $C^*$ -algebra  $\mathcal{J}$  (is a  $C^*$ -algebra with property the product of any two of its non zero ideals is non zero) has the  $\lambda$ -property if and only if  $\mathcal{J}_l^{-1} \cup \mathcal{J}_r^{-1} = \mathcal{J}$ , in which case it has the uniform  $\lambda$ -property for  $\lambda = \frac{1}{2}$ . Where  $\mathcal{J}_l^{-1} = \{A \in \mathcal{J} \mid \mathcal{J}A = \mathcal{J}\}$ , and  $\mathcal{J}_r^{-1} = \{A \in \mathcal{J} \mid A\mathcal{J} = \mathcal{J}\}$ .*

We recall the following definitions.

**Definition 4.3.** (1) *A  $C^*$ -algebra  $\mathcal{J}$  has real rank zero if and only if the invertible self-adjoint elements of  $\mathcal{J}$  are dense in the set  $\mathcal{J}_+$  of self-adjoint elements in  $\mathcal{J}$  and every non-zero projection is finite.*

(2) A simple  $C^*$ -algebra  $\mathcal{J}$  is said to be purely infinite if it has real rank zero.

**Lemma 4.1** ([19], Theorem 10.1). *If  $\mathcal{J}$  is a purely infinite  $C^*$ -algebra, the set of elements  $T$  of the form  $T = V|T|$ , where  $V$  is an isometry or a co-isometry in  $\mathcal{J}$ , is dense in  $\mathcal{J}$ . Thus*

$$\text{ext}(B_{\mathcal{J}_+}) = \mathcal{J}$$

**Theorem 4.5** ([19], Corollary 10.2). *Every purely infinite  $C^*$ -algebra has the uniform  $\lambda$ -property for  $\lambda = \frac{1}{2}$ .*

**Theorem 4.6** ([19], Theorem 10.4). *If  $\mathcal{J}$  is a  $C^*$ -algebra satisfying the uniform  $\lambda$ -property for  $\lambda = \frac{1}{2}$ , then the real rank of  $\mathcal{J}$  is at most one.*

**4.3. The  $\lambda$ -property of  $JB^*$ -algebras/triples and  $JBW^*$ -triples.** As we mentioned in the introduction, the class of  $JB^*$ -algebra was introduced by Kaplansky in 1976 and it includes all  $C^*$ -algebras as a proper subclass. Let us recall that the Bergmann operator associated with a couple of elements  $x, y$  in a  $JB^*$ -algebra (respectively a  $JB^*$ -triple)  $\mathfrak{S}$  is the mapping  $B(x, y) : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathfrak{S}$  defined by  $B(x, y) := I - 2L_{x,y} + Q_x Q_y$ . Tahlawi and Siddiqui introduced in [35], the notion of Brown-Pedersen quasi-invertible elements in a  $JB^*$ -algebras (respectively  $JB^*$ -triples). An element  $a$  in  $\mathfrak{S}$  is Brown-Pedersen quasi-invertible (BP-quasi-invertible for short) if there exists  $b \in \mathfrak{S}$  such that  $B(a, b) = 0$ . We use the notation  $\mathfrak{S}_q^{-1}$  to symbolize the set of BP-quasi invertible elements in  $\mathfrak{S}$ . Let  $m_q(x) = \text{dist}(x, \mathfrak{S} \setminus \mathfrak{S}_q^{-1})$ .

**Theorem 4.7** ([9], Theorem 3.4). *Let  $a$  be a BP-quasi-invertible element in the closed unit ball of a  $JB^*$ -triple  $\mathfrak{S}$ . Then for every  $\lambda \in [\frac{1}{2}, \frac{(1+m_q(a))}{2}]$  there exist  $e, u \in \text{ext}(\mathfrak{S})$  satisfying  $a = \lambda e + (1 - \lambda)u$ .*

**Theorem 4.8** ([9], Corollary 3.5). *Let  $\mathfrak{S}$  be a  $JB^*$ -triple. Let  $a$  be an element in  $B_{\mathfrak{S}}$ . Then  $a \in \mathfrak{S}_q^{-1}$  if and only if  $a = \lambda v_1 + (1 - \lambda)v_2$  for some extreme points  $v_1, v_2 \in \text{ext}(B_{\mathfrak{S}})$  and  $0 \leq \lambda < \frac{1}{2}$ .*

**Definition 4.4 (Extremally rich  $JB^*$ -algebra/triple).** *A  $JB^*$ -algebra (resp.  $JB^*$ -triple),  $\mathfrak{S}$ , is said to be extremally rich if the set of BP-quasi invertible elements  $\mathfrak{S}_q^{-1}$  is dense in  $\mathfrak{S}$ .*

Thus, extremally rich  $JB^*$ -algebra (resp. triple) has a non-empty intersection with  $JB^*$ -algebra (resp. triple) satisfying  $\lambda$ -property.

If we replace  $JB^*$ -triples with  $JBW^*$ -triples, computations are much more simpler on the closed unit ball.

In  $JBW^*$ -triples, the following result due to Siddiqui implies immediately that any  $JBW^*$ -triple satisfies the uniform  $\lambda$ -property.

**Theorem 4.9** ([31], Theorem 5). *If  $\mathfrak{S}$  is a  $JBW^*$ -triple then every element of  $B_{\mathfrak{S}}$  is the average of two extreme points of  $B_{\mathfrak{S}}$ .*

From some deferent perspective, another proof was given by Jamjoom, Tahlawi and Siddiqui to the following result.

**Theorem 4.10** ([9], Corollary 4.3). *Every a  $JBW^*$ -triple  $\mathfrak{S}$  satisfies the uniform  $\lambda$ -property .*

Since a  $JBW^*$ -triples play an analogue role to that played by von Neumann algebra in the class of  $C^*$ -algebras, authors of [9] gave a complete description of the  $\lambda$ -function on the closed unit ball of every  $JBW^*$ -triple .

## 5. BADE-PROPERTY AND $\lambda$ -PROPERTY

The relation between the Bade-property and  $\lambda$ -property of normaed spaces was a natural inquiry that raised and investigated by Aron, Lohman and Pedersen. This relation was the main investigating point of Aizpuru's note, [2] in the space of continuous functions and convergent sequences of 0-dimensional normed spaces. In case of other algebraic structures, we can deduced from Theorem 4.2 the following relation.

**Theorem 5.1.** *If a Banach space  $X$  has the  $\lambda$ -property then it has the Bade-property.*

Consequently, all normed spaces,  $C^*$ -algebras and  $JB^*$ -algebras/triples which satisfy  $\lambda$ -property and mentioned in sections 3 and 4 above, also satisfy the Bade-property.

We note that the converse of the theorem above is not true, in fact if  $X$  is a finite-dimensional strictly convex space, then  $C_X(T)$  may fail to have the  $\lambda$ -property, while of course it satisfies the Bade-property by Minkowsk's theorem.

**Remark 5.1.** Aron and Lohman([1], Example 3.1.1) proved that if  $K$  is a compact metric space and  $X$  is a strictly convex normed space then  $C(K, X)$  has the uniform  $\lambda$ -property (and, hence, the Bade-property). As a consequence, it may happen that  $C(K, X)$  has the Bade-property but  $C(K)$  does not have it (this occurs, for instance, when  $K = [0, 1]$ ). Theorem 3.5 (a) gives us a sufficient condition for  $C(K, X)$  to have the Bade-property, if  $C(K)$  and  $X$  have it (in this case  $K$  is 0-dimensional). We do not know if there exist spaces  $C(K, X)$  with the Bade-property such that neither  $C(K)$  nor  $X$  have the property. Theorem 3.5 (b) tells us that this cannot occur if  $K$  is non-perfect.

Aizpuru proved the following result which was crucial to give clear description of the relation between Bade-property,  $\lambda$ -property and uniform  $\lambda$ -property (as shown in the Theorem 5.2). The proof of Lemma 5.1 is technical and we refer the reader to see it in [2].

**Lemma 5.1** ([2], Proposition 3.3). *Let  $K$  be a 0-dimensional Hausdorff compact space and let  $X = C(K)$ . Then  $c(X)$  has the  $\lambda$ -property and  $\lambda(c(X)) = \frac{1}{2}$ .*

**Theorem 5.2** ([2], Corollary 3.5). *Let  $K$  be a Hausdorff compact space, then the following statements are equivalent:*

- (a)  $K$  is 0-dimensional.
- (b)  $C(K)$  has the Bade-property.
- (c)  $c(C(K))$  has the Bade-property.
- (d)  $C(K)$  has the  $\lambda$ -property.
- (e)  $C(K)$  has the uniform  $\lambda$ -property and  $\lambda(C(K)) = \frac{1}{2}$ .
- (f)  $c(C(K))$  has the  $\lambda$ -property.
- (g)  $c(C(K))$  has the uniform  $\lambda$ -property and  $\lambda(c(C(K))) = \frac{1}{2}$ .

*Proof.* (a)  $\Leftrightarrow$  (b) from Theorem 3.1.

(b)  $\Leftrightarrow$  (c) from Corollary 3.1.

(c)  $\Rightarrow$  (d) from Theorem 4.2.

(d)  $\Rightarrow$  (e) from [[1], Theorem 1.15].

(e)  $\Rightarrow$  (f) it is clearly from Lemma 5.1.

(f)  $\Rightarrow$  (g) from Lemma 5.1 and [[1], Theorem 1.15].

(g)  $\Rightarrow$  (c) from Theorem 4.2.

□

## 6. BADE-PROPERTY AND EXTREMALLY RICHNESS

Throughout this section,  $\mathcal{J}$  is a unital  $C^*$ -algebra with unitary group  $U(\mathcal{J})$ , and invertible group  $\mathcal{J}^{-1}$ . We denote by  $B_{\mathcal{J}}^{\circ}$  the interior of the closed unit ball  $B_{\mathcal{J}}$  of  $\mathcal{J}$ . Russo and Dye ([25]; p. 414) show that the convex hull of the set of unitaries in a  $C^*$ -algebra  $\mathcal{J}$ ,  $co(U(\mathcal{J}))$ , contains the open ball about  $\frac{1}{2}$ . Robertson [23] gave an improved version of this result as follows.

**Theorem 6.1** ([23], Proposition 1). *If  $\mathcal{J}$  is a unital  $C^*$ -algebra, then  $B_{\mathcal{J}}^{\circ} \subset co(U(\mathcal{J}))$ .*

Consequently, the closed unit ball of  $\mathcal{J}$  is contained in the closed convex hull of the unitaries,  $\overline{co(U(\mathcal{J}))}$  which is well known as the Russo-Dye Theorem. The converse inclusion  $\overline{co(U(\mathcal{J}))} \subset B_{\mathcal{J}}$  follows immediately from the fact that unitaries are contained in the unit ball of  $\mathcal{J}$ , and  $B_{\mathcal{J}}$  is closed. Hence  $B_{\mathcal{J}} = \overline{co(U(\mathcal{J}))}$ .

Originally,  $C^*$ -algebras of topological stable rank 1 is due to M. Reiffel [21] and it is defined as follows.

**Definition 6.1.** *A  $C^*$ -algebra  $\mathcal{J}$  is of topological stable rank 1 if and only if its invertible elements are norm dense in  $\mathcal{J}$ . We symbolize this by  $tsr\ 1$ .*

**Theorem 6.2.** *Every unital  $C^*$ -algebra of topological stable rank 1 satisfies the Bade-property.*

*Proof.* Recall that if  $\mathcal{J}$  is of  $tsr\ 1$ , then  $U(\mathcal{J}) = ext(\mathcal{J})$  (see for example [29], Corollary 6.10). As discussed above after Definition 6.1, we conclude that  $B_{\mathcal{J}} = \overline{co(ext(\mathcal{J}))}$ . That is,  $\mathcal{J}$  has the Bade-property.  $\square$

Let  $\mathfrak{S}$  be a unital  $JB^*$ -algebra. An element  $u$  of a  $JB^*$ -algebra  $\mathfrak{S}$  is called unitary if  $u^* = u^{-1}$ , the inverse of  $u$ . The set of all unitary elements of  $\mathfrak{S}$  will be denoted by  $U(\mathfrak{S})$ .

In the following we give the characterization of extreme points in  $JB^*$ -triples given by Kaup in ([12], Lemma 3.2 and Proposition 3.5).

**Definition 6.2.** *Let  $\mathfrak{S}$  be  $JB^*$ -triple. An element  $v \in \mathfrak{S}$  is an extreme point of the closed unit ball of  $B_{\mathfrak{S}}$  if and only if  $B(v, v) = 0$ . In  $JB^*$ -algebras, this translates into the form,*

$$a - 2\{vv^*a\} + \{v\{va^*v\}^*v\} = 0$$

for all  $a \in \mathfrak{S}$ .

Moreover, Russo-Dye Theorem holds for a big class in  $JB^*$ -algebras as proved by A. A. Siddiqui in [32]. He studied unitary elements in  $JB^*$ -algebras and established a sequence of results ending with the following lemma that leads to Russo-Dye Theorem in  $JB^*$ -algebras.

**Theorem 6.3** ([32], Theorem 2.2). *Let  $\mathfrak{S}$  be a unital  $JB^*$ -algebra,  $s \in B_{\mathfrak{S}}^{\circ}$  (the open unit ball) and  $v \in U(\mathfrak{S})$ . Then, for any positive integer  $n$ ,  $v + (n - 1)s = \sum_{i=1}^n u_i$  where the  $u_i$ 's are unitaries in  $\mathfrak{S}$*

$JB^*$ -algebras of topological stable rank 1 was studied by A. A. Siddiqui extending the notion of  $C^*$ -algebras of  $\text{tsr } 1$  and a  $JB^*$ -algebra  $\mathfrak{S}$  is said to be of topological stable rank 1 if the set  $\mathfrak{S}^{-1}$  of invertible elements in  $\mathfrak{S}$  is norm dense in  $\mathfrak{S}$ . He deduced the following interesting fact.

**Theorem 6.4** ([29], Corollary 6.10). *Let  $\mathfrak{S}$  be a unital  $JB^*$ -algebra of  $\text{tsr } 1$ . Then*

$$\text{ext}(B_{\mathfrak{S}}) = U(\mathfrak{S}).$$

*Hence, the distance from any  $x \in \mathfrak{S}$  to  $\text{ext}(B_{\mathfrak{S}})$ , is  $\text{dist}(x, U(\mathfrak{S}))$ .*

*Proof.* Since topological stable rank  $(\mathfrak{S}) = 1$ ,  $\mathfrak{S}^{-1}$  is norm dense in  $\mathfrak{S}$  such that  $\text{dist}(x, \mathfrak{S}^{-1}) = 0$  for all  $x \in \mathfrak{S}$ . So that  $\alpha(x) < 1$  for all  $x \in \text{ext}(B_{\mathfrak{S}})$ . Hence, from Theorem 6.7 [29] and the definition of the distance function  $\alpha(x)$ , there exists at least one  $y \in \mathfrak{S}^{-1}$  such that  $\|x - y\| < 1$ , thus  $\text{ext}(B_{\mathfrak{S}}) \subseteq U(\mathfrak{S})$ . The reverse inclusion is true for any  $JB^*$ -algebra since every unitary in  $\mathfrak{S}$  is an extreme point of the closed unit ball in  $\mathfrak{S}$ . □

The following result due to A. A. Siddiqui extends joint results of Kadison and Pedersen to general  $JB^*$ -algebras.

**Theorem 6.5.** *(Russo-Dye) ([32], Theorem 2.3)*

- (1) *Let  $x$  be an element of a  $JB^*$ -algebra  $\mathfrak{S}$  with unit  $e$  such that  $\|x\| < 1 - 2n^{-1}$  for some  $n \geq 3$ . Then there exist  $u_i \in U(\mathfrak{S}), i = 1, 2, 3, \dots, n$  such that  $x = \frac{1}{n} \sum_{i=1}^n u_i$ .*



- (2)  $B_{\mathfrak{S}}^{\circ} \subseteq co(U(\mathfrak{S}))$ .  
 (3)  $\overline{co(U(\mathfrak{S}))} = B_{\mathfrak{S}}$ .

*Proof.* (1) Since  $\|x\| < 1 - 2n^{-1}$ , we have  $\|(n-1)^{-1}(nx - e)\| < 1$ . Hence, by taking  $v = e$  and  $s = (n-1)^{-1}(nx - e)$  in Theorem 6.3, we get  $nx = \sum_{j=1}^n u_i$ , for some unitaries  $u_i \in \mathfrak{S}$ .

(2) Suppose  $x \in B_{\mathfrak{S}}^{\circ}$ . Then there exists an integer  $n \geq 3$  such that  $\|x\| < 1 - 2n^{-1}$ . Therefore,  $x \in co(U(\mathfrak{S}))$  by (1).

(3) Clearly,  $\overline{co(U(\mathfrak{S}))} \subseteq B_{\mathfrak{S}}$ . On the other hand, we have  $B_{\mathfrak{S}}^{\circ} \subseteq co(U(\mathfrak{S}))$  by (1). Thus  $B_{\mathfrak{S}} \subseteq \overline{co(U(\mathfrak{S}))}$  because  $\overline{B_{\mathfrak{S}}^{\circ}} = B_{\mathfrak{S}}$ .  $\square$

**Corollary 6.1.** *Any  $JB^*$ -algebra of topological stable rank 1 has Bade-property.*

*Proof.* Immediate from definition of  $JB^*$ -algebras of topological stable rank 1 and Theorem 6.5  $\square$

**Remark 6.1** ([21], Proposition 1.7). *Recall that the  $C^*$ -algebra  $C(X)$  of all complex-valued continuous functions defined on a compact space  $X$  of covering dimension 1 or zero and any finite von Neumann algebra are  $tsr$  1. Also, any finite-dimensional  $JB^*$ -algebra is of  $tsr$  1 [29]. Therefore,  $C^*$ -algebras and  $JB^*$ -algebras of  $tsr$  1 are large and satisfy the Bade-property.*

For  $JB^*$ -triples, which are not  $JB^*$ -algebras, there is no formal generalization of topological stable rank 1  $JB^*$ -triple. Thus, we will consider another property, called extremal richness of  $JB^*$ -triples defined previously in section 4.

**Theorem 6.6.** *Let  $\mathfrak{S}$  be a  $JB^*$ -triple with  $ext(\mathfrak{S}_1) \neq 0$ . If  $\mathfrak{S}$  is extremally rich, then  $\mathfrak{S}$  has the Bade-property.*

*Proof.* First we shall prove that extremal richness of  $\mathfrak{S}$  implies that  $\alpha(ext(\mathfrak{S})) + (1 - \alpha)(ext(\mathfrak{S}))$  is norm dense in  $\mathfrak{S}_1$ . Let  $x \in \mathfrak{S}_1$ . By definition, there exists a sequence  $(x_n)$  in  $\mathfrak{S}_q^{-1}$  which converges uniformly to  $x$ . Putting  $a_n = (\max\{1, \|x_n\|\})^{-1}$  we see that

$$\|x_n - a_n x_n\| \leq \|x - x_n\| + \|x_n - a_n x_n\|.$$

Note that

$$\|x_n - a_n x_n\| = (1 - a_n)\|x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

since  $a_n \rightarrow 1$  as  $\|x_n\| \rightarrow \|x\| \leq 1$  when  $n \rightarrow \infty$ . Therefore,  $\|x - a_n x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Also, we note that for each  $n$ ,  $a_n x_n \in \mathfrak{S}_1 \cap \mathfrak{S}_q^{-1}$  because  $x_n \in \mathfrak{S}_q^{-1}$ , so

$$\|x_n a_n\| = \begin{cases} \|x_n\| & \text{if } a_n = 1; \\ \| \|x_n\|^{-1} x_n \| = 1 & \text{otherwise.} \end{cases}$$

From ([9], Corollary 3.5), it follows that

$$a_n x_n \in \alpha(\text{ext}(\mathfrak{S})) + (1 - \alpha)(\text{ext}(\mathfrak{S})).$$

Hence,  $\alpha(\text{ext}(\mathfrak{S})) + (1 - \alpha)(\text{ext}(\mathfrak{S}))$  is norm dense in  $\mathfrak{S}_1$ .

Next, recall that Russo-Dye theorem for  $JB^*$ -triples ([37], Theorem 16) gives the inclusion;  $\mathfrak{S}_1 \subseteq \overline{\text{co}(\text{ext}(\mathfrak{S}_1))}$ , hence  $\mathfrak{S}$  has the Bade-property. □

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