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A NEW VIEW OF FUZZY VECTOR SPACE OVER FUZZY FIELD

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Abstract. In this paper, we determine a new view of fuzzy vector space over fuzzy field by using Yuan and Lee's definition of the fuzzy group, Aktaş and Çağman's definition of fuzzy ring and Yetkin and Olgun's definition of fuzzy module. Moreover the concepts of fuzzy vector subspace are studied and some of their basic properties are presented as analogous to ordinary vector space theory.

1. Introduction

The concept of fuzzy group was first introduced by Rosenfeld in [5]. Since then many researchers have studied fuzzy structures in many different types, as you can see in [1], [2], [3], [6], [4], [7]. In the definition of fuzzy subgroups, two types of fuzzy structures are observed in general. In the first type, the subset of a group G is fuzzy and the binary operation on G is nonfuzzy in the same classical sense as Rosenfeld's definition [5]. In the second one, the set is nonfuzzy or classical and the binary operation is fuzzy in the same fuzzy sense as in Yuan and Lee's [8] definition. By using of Yuan and Lee's definition of fuzzy group based on fuzzy binary operation, Aktaş and Çağman [1] defined a new type of fuzzy ring and the concepts of fuzzy subring, fuzzy ideal and fuzzy ring homomorphism are introduced. Yetkin and Olgun [7] presented a new type of fuzzy module by using Yuan and Lee's definition and

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discussed related results. Also, in 2010, Öztürk, Jun and Yazarli [4] introduced a new kind of fuzzy gamma rings and the concepts of fuzzy gamma ring, fuzzy ideal, fuzzy quotient gamma ring, and fuzzy gamma homomorphism are introduced.

2. Preliminaries

In this section we will formulate the preliminary definitions and results that are required in this paper. Let $\theta \in [0, 1)$. Malik and Mordeson [3] gave the following definition:

Definition 2.1. [8] Let R and S be nonempty sets and f be a fuzzy subset of $R \times S$. Then f is called a fuzzy function R into S if

(1) $\forall x \in R, \exists y \in S$ such that $f(x, y) > \theta$;

(2) $\forall x \in R, \forall y_1, y_2 \in S, f(x, y_1) > \theta$ and $f(x, y_2) > \theta$ implies $y_1 = y_2$.

By using of definition 2.1, Yuan and Lee gave the following definition:

Definition 2.2. [8] Let G be a nonempty set and R be a fuzzy subset of $G \times G \times G$. Then R is called a fuzzy binary operation on G if

(1) $\forall a, b \in G, \exists c \in G$ such that $R(a, b, c) > \theta$;

(2) $\forall a, b, c_1, c_2 \in G, R(a, b, c_1) > \theta$ and $R(a, b, c_2) > \theta$ implies $c_1 = c_2$.

Let R be a fuzzy binary operation on G. Then we have a mapping $R : F(G) \times F(G) \to F(G)$ defined by $(A, B) \to R(A, B)$ where $F(G) = \{A | A : G \to [0, 1]$ is a mapping $\}$ and $R(A, B)(c) = \bigvee_{a,b \in G}(A(a) \wedge B(b) \wedge R(a, b, c))$ for all $c \in G$.
Let $A = \{a\}$, $B = \{b\}$ and let $R(A, B)$ be denoted as $a \circ b$. Then $(a \circ b)(c) = R(a, b, c)$ for all $c \in G$. Also, $((a \circ b) \circ c)(z) = \bigvee_{d \in G}(R(a, b, d) \wedge R(d, c, z))$ and $(a \circ (b \circ c))(z) = \bigvee_{d \in G}(R(b, c, d) \wedge R(a, d, z))$ for all $z \in G$.

Definition 2.3. [8] Let G be a nonempty set and R be a fuzzy binary operation on G. Then (G, R) is called a fuzzy group if the following conditions are true:

(1) $\forall a, b, c, x_1, x_2 \in G, \Big((a \circ b) \circ c\Big)(x_1) > \theta$ and $\Big(a \circ (b \circ c\Big)(x_2) > \theta$ implies $x_1 = x_2$;

(2) $\exists e_R \in G$ such that $(e_R \circ a)(a) > \theta$ and $(a \circ e_R)(a) > \theta$ for any $a \in G(e_R$ is called an identity element of G);

(3) $\forall a \in G, \exists b \in G$ such that $(a \circ b)(e_R) > \theta$ and $(b \circ a)(e_R) > \theta(b$ is called an inverse element of a and denoted as a^{-1})where $\Big((a \circ b) \circ c\Big)(x_1) = \bigvee_{y \in G} R(a, b, y) \wedge R(y, c, x_1)$
$\Big((a \circ (b \circ c\Big)(x_2) = \bigvee_{y \in G} R(a, y, x_2) \wedge R(b, c, y)$.

Example 2.1. *Let $V = \{e, a\}$ be a nonempty set. Let us define E fuzzy binary operation on V, all with the same value of $\theta = 0.6$, as follows:*

$$E(e, e, e) = 0.9,\ E(e, a, e) = 0.2,\ E(a, e, e) = 0.4,\ E(a, a, e) = 0.7,$$

$$E(e, e, a) = 0.1,\ E(e, a, a) = 0.8,\ E(a, e, a) = 0.8,\ E(a, a, a) = 0.5.$$

Then (V, E) is a abelian fuzzy group, where $e = e_E$ is an identity element of V.

Proposition 2.1. [8] *Let (G, R) be a fuzzy group. Then*

$$\Big((a \circ b) \circ c\Big)(d) > \theta \Leftrightarrow \Big(a \circ (b \circ c)\Big)(d) > \theta,\ \forall a, b, c \in G.$$

Definition 2.4. [8] Let (H, R) be a fuzzy group. If

$$(a \circ b)(c) > \theta \Longleftrightarrow (b \circ a)(c) > \theta, \forall a, b, c \in G,$$

then (G, R) is called an abelian fuzzy group.

Definition 2.5. [1] Let R be a nonempty set and G and let H be two fuzzy binary operations on R. Then (R, G, H) is called a fuzzy ring if the following conditions are met:

(1) (R, G) is an abelian fuzzy group;

(2) $\forall a, b, c, x_1, x_2 \in R, \Big((a * b) * c\Big)(x_1) > \theta$ and $\Big(a * (b * c)\Big)(x_2) > \theta$ implies $x_1 = x_2$;

(3) $\forall a, b, c, x_1, x_2 \in R$, $\Big((a \circ b) * c\Big)(x_1) > \theta$, $\Big((a * c) \circ (b * c)\Big)(x_2) > \theta$ implies $x_1 = x_2$ and
$\Big(a * (b \circ c)\Big)(x_1) > \theta$ and $\Big((a * b) \circ (a * c)\Big)(x_2) > \theta$ implies $x_1 = x_2$ where
$\Big(a * (b \circ c)\Big)(x_1) = \bigvee_{y \in R} H(a, y, x_1) \wedge G(b, c, y)$ and
$\Big((a * b) \circ (a * c)\Big)(x_2) = \bigvee_{z_1, z_2 \in R} H(a, b, z_1) \wedge H(a, c, z_2) \wedge G(z_1, z_2, x_2)$.

If $(a * b)(x) > \theta \Leftrightarrow (b * a)(x) > \theta$ for all $a, b, x \in R$, then (R, G, H) is said to be a commutative fuzzy ring. If (R, G, H) contains an element e_H such that $(a * e_H)(a) > \theta$ and $(e_H * a)(a) > \theta, \forall a \in R$, then (R, G, H) is said to be a fuzzy ring with identity. The identity element e_G is called the zero element of the fuzzy ring. If $\forall a \in R, \exists b \in R$ such that $(a * b)(e_H) > \theta$ and $(b * a)(e_H) > \theta$, then b is said to be an inverse element of a and is denoted by a^{-1}.

Definition 2.6. A commutative fuzzy ring with identity e_H, (R, G, H) is said to be fuzzy field if for each element $a \in R$ there exists some $b \in R$ such that $(a * b)(e_H) > \theta$ and $(b * a)(e_H) > \theta$.

Example 2.2. *Let $H = \{e, b, c\}$ be a nonempty set. Let us define R and S fuzzy binary operation on H, all with the same value of $\theta = 0.6$, as follows:*

$R(e, e, e) = 0.7$, $R(b, e, e) = 0.5$, $R(c, e, e) = 0.4$, $R(e, e, b) = 0.2$, $R(b, e, b) = 0.8$,

$R(c, e, b) = 0.5$, $R(e, e, c) = 0.3$, $R(b, e, c) = 0.4$, $R(c, e, c) = 0.9$, $R(e, b, e) = 0.5$,

$R(b, b, e) = 0.2$, $R(c, b, e) = 0.8$, $R(e, b, b) = 0.8$, $R(b, b, b) = 0.3$, $R(c, b, b) = 0.1$,

$R(e, b, c) = 0.4$, $R(b, b, c) = 0.9$, $R(c, b, c) = 0.4$, $R(e, c, e) = 0.2$, $R(b, c, e) = 0.8$,

$R(c, c, e) = 0.3$, $R(e, c, b) = 0.1$, $R(b, c, b) = 0.4$, $R(c, c, b) = 0.9$, $R(e, c, c) = 0.9$,

$R(b, c, c) = 0.4$, $R(c, c, c) = 0.4$, $S(e, e, e) = 0.8$, $S(b, e, e) = 0.9$, $S(c, e, e) = 0.8$,

$S(e, e, b) = 0.2$, $S(b, e, b) = 0.4$, $S(c, e, b) = 0.3$, $S(e, e, c) = 0.3$, $S(b, e, c) = 0.2$,

$S(c, e, c) = 0.2$, $S(e, b, e) = 0.9$, $S(b, b, e) = 0.3$, $S(c, b, e) = 0.5$, $S(e, b, b) = 0.2$,

$$S(b,b,b)=0.8,\ S(c,b,b)=0.4,\ S(e,b,c)=0.3,\ S(b,b,c)=0.3,\ S(c,b,c)=0.9,$$

$$S(e,c,e)=0.9,\ S(b,c,e)=0.1,\ S(c,c,e)=0.4,\ S(e,c,b)=0.1,\ S(b,c,b)=0.3,$$

$$S(c,c,b)=0.8,\ S(e,c,c)=0.2,\ S(b,c,c)=0.9,\ S(c,c,c)=0.3.$$

Then (H,R,S) *is a fuzzy field, where* $e_R=e$ *and* $e_S=a$ *are identity fuzzy field* (H,R,S) *respectively.*

3. Fuzzy vector over Fuzzy field

Definition 3.1. Let (R,G,H) be a fuzzy field and (V,E) be an abelian fuzzy group and let P be a fuzzy subset of $R\times V\times V$ such that

(1) $\forall a\in R,\forall u\in V,\exists v\in V$ such that $P(a,u,v)>\theta$;

(2) $\forall a\in R,\forall u,v_1,v_2\in V,R(a,u,v_1)>\theta$ and $R(a,u,v_2)>\theta$ implies $v_1=v_2$.

Then we have a mapping

$$P:F(R)\times F(V)\to F(V)$$
$$(A,S)\to P(A,S)$$

where $F(R)=\{A|A:R\to[0,1]$ is a mapping $\}$, $F(V)=\{S|S:V\to[0,1]$ is a mapping $\}$ and $P(A,S)(x)=\bigvee_{(r,u)\in R\times V}A(r)\wedge S(u)\wedge P(r,u,x)$. Let $A=\{r\}$ and $S=\{u\}$, and let $P(A,S)$ and $E(u,v)$ be a denoted as $r\odot u$ and $u\oplus v$, respectively. Then

$$(r\odot u)(x)=P(r,u,x),\ (u\oplus v)(x)=E(u,v,x),\ \forall x\in V$$
$$\Big(r\odot(u\oplus v)\Big)(x)=\bigvee_{y\in V}P(r,y,x)\wedge E(u,v,y)$$
$$\Big((r\odot u)\oplus(r\odot v)\Big)(x)=\bigvee_{y_1,y_2\in V}P(r,u,y_1)\wedge P(r,v,y_2)\wedge E(y_1,y_2,x)$$
$$\Big((r_1\circ r_2)\odot u\Big)(x)=\bigvee_{r\in R}G(r_1,r_2,r)\wedge P(r,u,x)$$
$$\Big(r_1\odot(r_2\odot u)\Big)(x)=\bigvee_{y\in V}P(r_1,y,x)\wedge P(r_2,u,y)$$

$$\Big((r_1 * r_2) \odot u\Big)(x) = \bigvee_{r \in R} H(r_1, r_2, r) \wedge P(r, u, x).$$

Definition 3.2. Let (R, G, H) be a fuzzy field and let (V, E) be an abelian group. Then V is called a fuzzy vector space over R if the following axioms hold:

(1) For any $u, v, x, y \in V$ and $r \in R$, $\Big(r \odot (u \oplus v)\Big)(x) > \theta$, $\Big((r \odot u) \oplus (r \odot v)\Big)(y) > \theta$ implies $x = y$;

(2) For any $u, x, y \in V$ and $r_1, r_2 \in R$, $\Big((r_1 \circ r_2) \odot u\Big)(x) > \theta$, $\Big((r_1 \odot u) \oplus (r_2 \odot u)\Big)(y) > \theta$ implies $x = y$;

(3) For any $u, x, y \in V$ and $r_1, r_2 \in R$, $\Big((r_1 * r_2) \odot u\Big)(x) > \theta, \Big(r_1 \odot (r_2 \odot u)\Big)(y) > \theta$ implies $x = y$;

(4) For any $u \in V$, $(e_H \odot u)(u) > \theta$.

Example 3.1. *Let (V, E) and (H, R, S) be abelian fuzzy group and fuzzy field which were mention in examples 2.4 and 2.8 respectivly. Let us define P fuzzy function of H into V, all with the same value of $\theta = 0.6$, as follows:*

$$P(e_R, e_E, e_E) = 0.8,\ P(e_R, a, e_E) = 0.7,\ P(b, e_E, e_E) = 0.9,\ P(e_R, e_E, a) = 0.2,$$

$$P(e_R, a, a) = 0.4,\ P(b, e_E, a) = 0.3,\ P(b, a, e_E) = 0.4,\ P(c, e_E, e_E) = 0.8,$$

$$P(c, a, e_E) = 0.1,\ P(b, a, a) = 0.8,\ P(c, e_E, a) = 0.3,\ P(c, a, a) = 0.9.$$

Then V is a fuzzy vector space over H together with a fuzzy function P, since the following conditions an true:

(1) *For any $u, v, x, y \in V$ and $r \in R$, $\big(r \odot (u \oplus v)\big)(x) > \theta$, $\big((r \odot u)\big) \oplus (r \odot v)\big)(y) > \theta$ implies $x = y$;*

(2) *For any $u, x, y \in V$ and $r_1, r_2 \in R$, $\big((r_1 \circ r_2) \odot u\big)(x) > \theta$, $\big((r_1 \odot u)\big) \oplus (r_2 \odot u)\big)(y) > \theta$ implies $x = y$;*

(3) *For any $u, x, y \in V$ and $r_1, r_2 \in R$, $\big((r_1 * r_2) \odot u\big)(x) > \theta$, $\big(r_1 \odot (r_2 \odot u)\big)(y) > \theta$ implies $x = y$;*

(4) *For any* $u \in V$, $(a \odot u)(u) > \theta$.

By proposition 18 of [7] we have:

Theorem 3.1. *Let V be a fuzzy vector space over a fuzzy field R. Then for any $u, v, x \in V$ and for any $r, r_1, r_2 \in R$,*

(i) $\Big(r \odot (u \oplus v)\Big)(x) > \theta$ *if and only if* $\Big((r \odot u) \oplus (r \odot v)\Big)(x) > \theta$;

(ii) $\Big((r_1 \circ r_2) \odot u\Big)(x) > \theta$ *if and only if* $\Big((r_1 \odot u) \oplus (r_2 \odot u)\Big)(x) > \theta$;

(iii) $\Big((r_1 * r_2) \odot u\Big)(x) > \theta$ *if and only if* $\Big(r_1 \odot (r_2 \odot u)\Big)(x) > \theta$.

Theorem 3.2. *Let V be a fuzzy vector space over a fuzzy filed R. Then for all $u \in V$ and $r \in R$,*

(i) $(r \odot e_E)(e_E) > \theta$;

(ii) $(e_G \odot u)(e_E) > \theta$;

(iii) *If* $(r \odot u)(e_E) > \theta$, *then* $r = e_G$ *or* $u = e_E$.

Proof. To prove (i) and (ii) see proposition 19 of [7].
(iii) Let $(r \odot u)(e_E) > \theta$ and $r \neq e_G$. Then there exists $r^{-1} \in R$ such that $(r^{-1} * r)(e_H) > \theta$, since $\Big(r^{-1} \odot (r \odot u)\Big)(e_E) > P(r^{-1}, e_E, e_E) \wedge P(r, u, e_E) > \theta$. By theorem 3.1, we have $\Big((r^{-1} * r) \odot u\Big)(e_E) = \bigvee_{a \in R} H(r^{-1}, r, a) \wedge P(a, u, e_E) > \theta$. Then there exists $a_1 \in R$ such that $H(r^{-1}, r, a_1) > \theta$ and $P(a_1, u, e_E) > \theta$. Thus $a_1 = e_H$. Since $(e_H \odot u)(u) > \theta$, hence $u = e_E$. □

Definition 3.3. Let V be a fuzzy vector space over a fuzzy filed R. A nonempty subset W of V is called a fuzzy subspace of V if W is itself a fuzzy space over R with the fuzzy operation on V, i.e.

(i) $\forall w_1, w_2 \in W, \forall u \in V, (w_1 \oplus w_2)(u) > \theta$ implies $u \in W$;

(ii) $w \in W$ implies $w^{-1} \in W$

(iii) $\forall r \in R, \forall w \in W, \forall u \in V, (r \odot w)(u) > \theta$ implies $u \in W$.

Theorem 3.3. *Let V be a fuzzy vector space over a fuzzy filed R. A nonempty subset W of V is a fuzzy subspace of V if and only if*

$(*)$ $\forall r_1, r_2 \in R, \forall w_1, w_2 \in W, \forall u \in V,$ $\Big((r_1 \odot w_1) \oplus (r_2 \odot w_2)\Big)(u) > \theta$ *implies* $u \in W$.

Proof. Let W be a fuzzy subspace of V and $\Big((r_1 \odot w_1) \oplus (r_2 \odot w_2)\Big)(u) > \theta$. Then there exists $y_1, y_2 \in V$, such that

$$P(r_1, w_1, y_1) > \theta, P(r_2, w_2, y_2) > \theta, E(y_1, y_2, u) > \theta.$$

Since $w_1, w_2 \in W$, so we obtain $y_1, y_2 \in W$ and consequently $u \in W$.
Conversely, assume that (*) hold and let $(w_1 \oplus w_2)(u) > \theta$, $\forall w_1, w_2 \in W, \forall u \in V$. Since

$$\Big((e_H \odot w_1) \oplus (e_H \odot w_2)\Big)(u) > P(e_H, w_1, w_1) \wedge P(e_H, w_2, w_2) \wedge E(w_1, w_2, u) > \theta,$$

thus $u \in W$. Now let w be an element of W. Since

$$\Big((e_H \odot e_E) \oplus (e_H \odot w^{-1})\Big)(w^{-1}) > P(e_H, e_E, e_E) \wedge P(e_H, w^{-1}, w^{-1}) \wedge E(e_E, w^{-1}, w^{-1}) > \theta.$$

Thus $w^{-1} \in W$.
If $\forall r \in R, \forall w \in W, \forall u \in V, (r \odot w)(u) > \theta$, then

$$\Big((e_H \odot e_E) \oplus (r \odot w)\Big)(u) > P(e_H, e_E, e_E) \wedge P(r, w, u) \wedge E(e_E, u, u) > \theta.$$

Thus $u \in W$. Hence by definition 3.3, W is a fuzzy subspace of V. □

Remark 1. If $\{W_i | i \in I\}$ is a family of fuzzy subspace of a fuzzy vector space of V, then $\bigcap_{i \in I} W_i$ is a fuzzy subspace of V.

Proof. Let $u \in V$ and $r \in R$. (1) If $w_1, w_2 \in \bigcap_{i \in I} W_i$ and $(w_1 \oplus w_2)(u) > 0$, since W_i is fuzzy subspace of V, then for all $i \in I$, $u \in W_i$. Therefore $u \in \bigcap_{i \in I} W_i$. (2) If $w \in \bigcap_{i \in I} W_i$, then for all $i \in I$, $w \in W_i$. So for all $i \in I$, $w^{-1} \in W_i$. Therefore $w^{-1} \in \bigcap_{i \in I} W_i$. (3) If $w \in \bigcap_{i \in I} W_i$ and $(r \odot w)(u) > 0$, then for all $i \in I$, $w \in W_i$, since W_i is fuzzy subspace of V, thus for all $i \in I$, $u \in W_i$. Therefore $u \in \bigcap_{i \in I} W_i$. □

Theorem 3.4. *Let V be a fuzzy vector space over fuzzy filed R. Then $FL(v) = \{u \in V | \exists r \in R, (r \odot v)(u) > \theta\}$ where $v \in V$, is the smallest fuzzy subspace of V containing v; hence, it is called the fuzzy subspace spanned or generated by v.*

Proof. Since $(e_H \odot v)(v) > \theta$, thus $v \in FL(v)$. Now suppose $u, u' \in FL(v)$, $x \in V$ and $(u \oplus u')(x) > \theta$. Then there exist $r, r' \in R$ such that $(r \odot v)(u) > \theta$ and $(r' \odot v)(u') > \theta$. Since $\Big((r \odot v) \oplus (r' \odot v)\Big)(x) > P(r, v, u) \wedge P(r', v, u') \wedge E(u, u', x) > \theta$. By the theorem 3.1, $\Big((r \circ r') \odot v\Big)(x) = \bigvee_{a \in R} G(r, r', a) \wedge P(a, v, x) > \theta$, hence there exist $a \in R$, such that $G(r, r', a) \wedge P(a, v, x) > \theta$. Then $P(a, v, x) = (a \odot v)(x) > \theta$. Thus $x \in FL(v)$. Let $u \in FL(v)$ and let $r \in R, x \in V$ such that $(r \odot v)(u) > \theta$ and $P(r^{-1}, v, x) > \theta$. Since $\Big((r \circ r^{-1}) \odot v\Big)(e_E) > G(r, r^{-1}, e_G) \wedge P(e_G, v, e_E) > \theta$. Thus $\Big((r \odot v) \oplus (r^{-1} \odot v)\Big)(e_E) > \theta$. Therefore

$$\exists y_1, y_2 \in V : P(r, v, y_1) > \theta, P(r^{-1}, v, y_2) > \theta, E(y_1, y_2, e_E) > \theta.$$

Consequently $y_1 = u$ and $y_2 = x$. Furthermore, since $E(u, x, e_E) > \theta$ thus $x = u^{-1}$. Hence $P(r^{-1}, v, x) = P(r^{-1}, v, u^{-1}) > \theta$ implies $u^{-1} \in FL(v)$. We next show that if $u \in FL(v), r \in R$ and $(r \odot u)(x) > \theta$, then $x \in FL(v)$. From $u \in FL(v)$ implies $\exists a \in R, (a \odot v)(u) > \theta$. Since $\Big(r \odot (a \odot v)\Big)(x) > P(r, u, x) \wedge P(a, v, u) > \theta$. Consequently by theorem 3.1, $\Big((r * a) \odot v\Big)(x) > \theta$. Hence $\exists b \in R : P(b, v, x) > \theta$ implies $x \in FL(v)$. Accordingly, $FL(v)$ is a fuzzy subspace of V containing v. Finally, if W is a fuzzy subspace of V containing v and $u \in FL(v)$, then $u \in W$. Thus $FL(v)$ is the smallest fuzzy subspace of V containing v. □

Lemma 3.1. *Let V be a fuzzy vector space. Then*

$$\Big((u_1 \oplus u_2) \oplus (u_3 \oplus u_4)\Big)(u) > \theta \Leftrightarrow \Big[\Big((u_1 \oplus u_2) \oplus u_3\Big) \oplus u_4\Big](u) > \theta,$$

where $u_1, u_2, u_3, u_4, u \in V$.

Proof. Let $\Big((u_1 \oplus u_2) \oplus (u_3 \oplus u_4)\Big)(u) > \theta$. Then there exist $y_1, y_2 \in V$ such that, $E(u_1, u_2, y_1) > \theta, E(u_3, u_4, y_2) > \theta$ and $E(y_1, y_2, u) > \theta$. Now, suppose $v, w \in V$ such that, $E(y_1, u_3, v) > \theta$ and $E(v, u_4, w) > \theta$. Which implies that $E(u_3, y_1, v) > \theta$ and $E(u_4, v, w) > \theta$. Then

$$\Big[(u_1 \oplus u_2) \oplus u_3\Big](v) > E(u_1, u_2, y_1) \wedge E(y_1, u_3, v) > \theta.$$

Consequently

$$\Big[\Big((u_1 \oplus u_2) \oplus u_3\Big) \oplus u_4\Big](w) = \bigvee_{y \in V} \Big[(u_1 \oplus u_2) \oplus u_3\Big](y) \wedge E(y, u_4, w) > \theta.$$

Since $\Big[(u_4 \oplus u_3) \oplus y_1\Big](u) > E(u_4, u_3, y_2) \wedge E(y_2, y_1, u) > \theta$ and

$$\begin{aligned}\Big[(u_4 \oplus u_3) \oplus y_1\Big](w) &= \bigvee_{x \in V} E(u_4, x, w) \wedge E(u_3, y_1, x) \\ &> E(u_4, v, w) \wedge E(u_3, y_1, v) > \theta.\end{aligned}$$

Hence by theorem 3.1, $w = u$.
Similarly, it is proved that if $\Big[\Big((u_1 \oplus u_2) \oplus u_3\Big) \oplus u_4\Big](u) > \theta$, then $\Big((u_1 \oplus u_2) \oplus (u_3 \oplus u_4)\Big)(u) > \theta$. □

Corollary 3.1. *If V is a fuzzy vector space, then*

$$\Big[\Big(u_1 \oplus (u_2 \oplus u_3)\Big) \oplus u_4\Big](u) > \theta \Leftrightarrow \Big[\Big((u_1 \oplus u_2) \oplus u_3\Big) \oplus u_4\Big](u) > \theta,$$

where $u_1, u_2, u_3, u_4, u \in V$.

Proof. The proof is similar to the proof of lemma 3.1. □

Lemma 3.2. *If V is a fuzzy vector space, then for all $n \geq 2$,*

$$E(u_1, \cdots, u_n, u) \geq \theta \Leftrightarrow E(u_1^{-1}, \cdots, u_n^{-1}, u^{-1}) > \theta,$$

where $u_1, ..., u_n, u \in V$.

Proof. By induction on n. Let $n = 2$ and $E(u_1, u_2, u) > \theta$. On the other hand we have $\Big(u_1 \oplus (u_2 \oplus u_2^{-1})\Big)(u_1) > \theta$. Consequently,

$$\Big[\Big(u_1 \oplus (u_2 \oplus u_2^{-1})\Big) \oplus u_1^{-1}\Big](e_E) = \bigvee_{y \in V} E(y, u_1^{-1}, e_E) \wedge \Big(u_1 \oplus (u_2 \oplus u_2^{-1})\Big)(y)$$

$$> E(u_1, u_1^{-1}, e_E) \wedge \Big(u_1 \oplus (u_2 \oplus u_2^{-1})\Big)(u_1) > \theta.$$

By lemma 3.1, we have $\Big((u_1 \oplus u_2) \oplus (u_2^{-1} \oplus u_1^{-1})\Big)(e_E) > \theta$. Hence, there exist $y_1, y_2 \in V$ such that:

$$E(u_1, u_2, y_1) > \theta, E(u_2^{-1}, u_1^{-1}, y_2) > \theta, E(y_1, y_2, e_E) > \theta.$$

Since $E(u_1, u_2, u) > \theta$, thus $y_1 = u$. Furthermore, since $E(y_1, y_2, e_E) > \theta$, thus $y_2 = u^{-1}$. Consequently,

$$E(u_1^{-1}, u_2^{-1}, u^{-1}) = E(u_2^{-1}, u_1^{-1}, u^{-1}) = E(u_2^{-1}, u_1^{-1}, y_2) > \theta.$$

Now, suppose that the lemma is true for $n-1$ and $E(u_1, \cdots, u_n, u) > \theta$. Then there exists $y \in V$ such that $E(u_1, \cdots, u_{n-1}, y) > \theta$ and $E(y, u_n, u) > \theta$, which implies $E(u_1^{-1}, \cdots, u_{n-1}^{-1}, y^{-1}) > \theta$ and $E(y^{-1}, u_n^{-1}, u^{-1}) > \theta$. Thus $E(u_1^{-1}, \cdots, u_n^{-1}, u^{-1}) > \theta$. □

Theorem 3.5. *Let V be a fuzzy vector space over a fuzzy field R and S be nonempty subsets of V. Then*

$$FL(S) = \{u \in V | \exists a_i \in R, u_i \in S, (\sum_{i=1}^{n} a_i \odot u_i)(u) > \theta, n \in N\},$$

is the smallest fuzzy subspace of V containing S.

Proof. Let $u, v \in FL(S)$, $w \in V$ and $(u \oplus v)(w) > \theta$. Then there exist $a_i, b_j \in R$; $u_i, v_j \in S$ such that

$$(\sum_{i=1}^{n} a_i \odot u_i)(u) > \theta, (\sum_{j=1}^{m} b_j \odot v_j)(v) > \theta.$$

Hence there exists $x_i, y_j \in V$ such that

$$\bigwedge_{i=1}^{n} P(a_i, u_i, x_i) \wedge E(x_1, \cdots, x_n, u) > \theta, \bigwedge_{j=1}^{m} P(b_j, v_j, y_j) \wedge E(y_1, \cdots, y_m, v) > \theta$$

Now suppose,

$$w_r = \begin{cases} u_r & \text{if } 1 \leq r \leq n \\ v_{r-n} & \text{if } n < r \leq n+m \end{cases}, c_r = \begin{cases} a_r & \text{if } 1 \leq r \leq n \\ b_{r-n} & \text{if } n < r \leq n+m \end{cases}$$

and

$$z_r = \begin{cases} x_r & \text{if } 1 \leq r \leq n \\ y_{r-n} & \text{if } n < r \leq n+m \end{cases}.$$

Since

$$E(x_1, \cdots, x_n, y_1, \cdots, y_m, w) = \bigvee_{c,d \in V} E(x_1, \cdots, x_n, c) \wedge E(y_1, \cdots, y_m, d) \wedge E(c, d, w)$$

$$> E(x_1, \cdots, x_n, u) \wedge E(y_1, \cdots, y_m, v) \wedge E(u, v, w) > \theta.$$

Thus

$$\Big(\bigwedge_{i=1}^{n} P(a_i, u_i, x_i)\Big) \wedge \Big(\bigwedge_{j=1}^{m} P(b_j, v_j, y_j)\Big) \wedge E(x_1, \cdots, x_n, y_1, \cdots, y_m, w)$$
$$= \bigwedge_{r=1}^{n+m} P(c_r, w_r, z_r) \wedge E(z_1, \cdots, z_{n+m}, w) > \theta.$$

And so, $\bigvee_{z_r \in V} \Big(\bigwedge_{r=1}^{n+m} P(c_r, w_r, z_r) \wedge E(z_1, \cdots, z_{n+m}, w)\Big) > \theta$, which implies $w \in FL(S)$. Now suppose $u \in FL(S)$ and let there exist $u_i \in S$, $x_i \in V$, $a_i \in R$ such that for all $i, P(a_i, u_i, x_i) > \theta$ and $E(x_1, \cdots, x_n, u) > \theta$. And so for all i $P(a_i^{-1}, u_i^{-1}, x_i^{-1}) > \theta$ and $E(x_1^{-1}, \cdots, x_n^{-1}, u^{-1}) > \theta$. Thus

$$\bigwedge_{i=1}^{n} P(a_i^{-1}, u_i^{-1}, x_i^{-1}) \wedge E(x_1^{-1}, \cdots, x_n^{-1}, u^{-1}) > \theta.$$

Hence $(\sum_{i=1}^{n} a_i^{-1} \odot u_i^{-1})(u^{-1}) > \theta$, which implies $u^{-1} \in FL(S)$.

On the other hand, if $u \in FL(S), r \in R, v \in V$ and $(r \odot u)(v) > \theta$, then for some $u_i \in S$, $x_i \in V$, $a_i \in R$

$$\Big(r \odot (\sum_{i=1}^{n} a_i \odot u_i)\Big)(v) = \bigvee_{y \in V} P(r, y, v) \wedge (\sum_{i=1}^{n} a_i \odot u_i)(y)$$

$$= \bigvee_{y \in V} \Big[P(r, y, v) \wedge \Big(\bigvee_{x_i \in V} P(a_1, u_1, x_1) \wedge \cdots \wedge P(a_n, u_n, x_n) \wedge E(x_1, \ldots, x_n, u)\Big)\Big]$$

$$> P(r, y, v) \wedge P(a_1, u_1, x_1) \wedge \cdots \wedge P(a_n, u_n, x_n) \wedge E(x_1, \ldots, x_n, u) > \theta,$$

and so

$$\Big[\Big((r * a_1) \odot u_1\Big) \oplus \cdots \oplus \Big((r * a_n) \odot u_n\Big)\Big](v) =$$

$$\bigvee_{z_i \in V} \Big(\bigvee_{b_1 \in R} H(r, a_1, b_1) \wedge P(b_1, u_1, z_1)\Big) \wedge \cdots$$

$$\wedge \Big(\bigvee_{b_n \in R} H(r, a_n, b_n) \wedge P(b_n, u_n, z_n)\Big) \wedge E(z_1, \cdots, z_n, v) > \theta.$$

Hence there exists $b_i \in R$ such that $\bigwedge_{i=1}^{n} P(b_i, u_i, z_i) > \theta$ and $E(z_1, \cdots, z_n, v) > \theta$. And so $\Big(\sum_{i=1}^{n} b_i \odot u_i\Big)(v) > \theta$, which implies $v \in FL(S)$. Thus $FL(S)$ is a fuzzy subspace of V. Now suppose W is a fuzzy subspace of V containing S and suppose $u \in FL(S)$. Then

$$\exists a_i \in R, u_i \in S,\ x_i \in V : (\sum_{i=1}^{n} a_i \odot u_i)(u) > \theta,\ \bigwedge_{i=1}^{n} P(a_i, u_i, x_i) \wedge E(x_1, \cdots, x_n, u) > \theta.$$

Since $u_i \in W$ and $P(a_i, u_i, x_i) > \theta$ for all i, thus $x_i \in W$. Also, since $x_i \in W$ and $E(x_1, \cdots, x_n, u) > \theta$, we have $u \in W$. Thus $FL(S) \subseteq W$. Hence, $FL(S)$ is the smallest fuzzy subspace of V containing S. □

Theorem 3.6. *Let W_1 and W_2 be fuzzy subspaces of a fuzzy vector space V. Then*

$$W_1 \oplus W_2 = \{u \in V | \exists w_1 \in W_1, w_2 \in W_2, (w_1 \oplus w_2)(u) > \theta\},$$

is a fuzzy subspace of V containing W_1 and W_2.

Proof. Let $u \in W_1$. By hypothesis W_2 is a fuzzy subspace of V and so $e_E \in W_2$. Since $(u \oplus e_E)(u) > \theta$, thus $u \in W_1 \oplus W_2$. Therefore, W_1 is contained in $W_1 \oplus W_2$. Similarly, W_2 is contained in $W_1 \oplus W_2$. Now, suppose $u, u' \in W_1 \oplus W_2$, $v \in V$ and $(u \oplus u')(v) > \theta$. Then, there exist $w_1, w_2 \in W_1$ and $w_2, w_2' \in W_2$ such that $(w_1 \oplus w_2)(u) > \theta, (w_1' \oplus w_2')(u') > \theta$. Hence, by lemmas 3.1 and 3.1, we have

$$\Big((w_1 \oplus w_2) \oplus (w_1' \oplus w_2')\Big)(v) = \Big((w_1 \oplus w_1') \oplus (w_2 \oplus w_2')\Big)(v)$$
$$= \bigvee_{y_1, y_2 \in V} E(w_1, w_1', y_1) \wedge E(w_2, w_2', y_2) \wedge E(y_1, y_2, v) > \theta.$$

So, there exist $y_1, y_2 \in V$ such that

$$E(w_1, w_1', y_1) > \theta, E(w_2, w_2', y_2) > \theta, E(y_1, y_2, v) > \theta.$$

Therefore $y_1 \in W_1, y_2 \in W_2$ and so, $v \in W_1 \oplus W_2$.

On the other hand, if $u \in W_1 \oplus W_2$ and $(w_1 \oplus w_2)(u) > \theta$ where $w_1 \in W_1$, $w_2 \in W_2$; then $E(w_1^{-1}, w_2^{-1}, u^{-1}) > \theta$. Therefore, $(w_1^{-1} \oplus w_2^{-1})(u^{-1}) > \theta$. Thus, $u^{-1} \in W_1 \oplus W_2$.

If $u \in W_1 \oplus W_2, a \in R, v \in V$ and $(a \odot u)(v) > \theta$, then $(w_1 \oplus w_2)(u) > \theta$ for some $w_1 \in W_1$, $w_2 \in W_2$. Since

$$\Big(a \odot (w_1 \oplus w_2)\Big)(v) = \bigvee_{y \in V} P(a, y, v) \wedge E(w_1, w_2, y)$$
$$> P(a, u, v) \wedge E(w_1, w_2, u) > \theta,$$

and so, $\Big((a \odot w_1) \oplus (a \odot w_2)\Big)(v) > \theta$. Thus there exist $y_1, y_2 \in V$ such that

$$P(a, w_1, y_1) \wedge P(a, w_2, y_2) \wedge E(y_1, y_2, v) > \theta.$$

Hence $y_1 \in W_1, y_2 \in W_2$ and so $v \in W_1 \oplus W_2$. Consequently, $W_1 \oplus W_2$ is a fuzzy subspace of V containing W_1 and W_2. □

Corollary 3.2. *If W_1 and W_2 are fuzzy subspaces of V, then*

$$W_1 \oplus W_2 = FL(W_1, W_2).$$

Proof. Obvious. □

Theorem 3.7. *Suppose U and W are fuzzy subspaces of a fuzzy vector space V. If $U = FL(u_i)_{i=1}^n$ and $W = FL(w_j)_{j=1}^n$, then $U \oplus W = FL(u_i, w_j)$.*

Proof. Let $x \in U \oplus W$. Then $(u \oplus w)(x) > \theta$, where $u \in U$ and $w \in W$. Since $\{u_i\}$ and $\{w_j\}$ generated U and W, respectively. Thus there exist $a_i, b_j \in R$ such that

$$(a_1 \odot u_1 \oplus \cdots \oplus a_n \odot u_n)(u) > \theta, (b_1 \odot w_1 \oplus \cdots \oplus b_m \odot w_m)(w) > \theta.$$

Hence, there exist $x_i, y_j \in V$ such that for all i, $P(a_i, u_i, x_i) > \theta, E(x_1, \cdots, x_n, u) > \theta$ and for all j, $P(b_j, w_j, y_j) > \theta, E(y_1, \cdots, y_m, w) > \theta$. Since $E(w, u, x) > \theta$, thus $E(y_1, \cdots, y_m, w, x) > \theta$, then $E(u, y_1, \cdots, y_m, x) > \theta$, and so $E(x_1, \cdots, x_n, y_1, \cdots, y_m, x) > \theta$. Therefore

$$(a_1 \odot u_1 \oplus \cdots \oplus a_n \odot u_n \oplus b_1 \odot w_1 \oplus \cdots \oplus b_m \odot w_m)(x) > \theta,$$

which implies that $x \in FL(u_i, w_j)$. Consequently, $U \oplus W \subseteq FL(u_i, w_j)$. Clearly, $FL(u_i, w_j) \subseteq U \oplus W$. This completes the proof. □

4. Basis of fuzzy vector space

Definition 4.1. Let V be a fuzzy vector space over a fuzzy field R. Then a subset S of V is called fuzzy linearly independent if for every vector $v_1, \cdots, v_n$ in S, and $a_1, \cdots, a_n \in R$.

$$(a_1 \odot v_1 \oplus a_2 \odot v_2 \oplus \cdots \oplus a_n \odot v_n)(e_E) > \theta,$$

which implies that $a_1 = a_2 = \cdots = a_n = e_0$. A subset S of V is called fuzzy linearly dependent if it is not fuzzy linearly independent.

Definition 4.2. A basis for V is a fuzzy linearly independent subset of V such that span V. We say that V has finite dimension if it has a finite basis.

Theorem 4.1. *Let v_1, v_2 be fuzzy linearly independent vectors, and suppose $v \in FL(v_1, v_2)$, say $(a_1 \odot v_1 \oplus a_2 \odot v_2)(v) > \theta$, where $a_1, a_2 \in R$. Show that the above representation of v is unique.*

Proof. Suppose $(b_1 \odot v_1 \oplus b_2 \odot v_2)(v) > \theta$, where $b_1, b_2 \in R$. We show that

$$a_1 = b_1, a_2 = b_2 \Leftrightarrow (a_1 \circ b_1^{-1})(e_0) > \theta, (a_2 \circ b_2^{-1})(e_0) > \theta$$

$$\Leftrightarrow \Big((a_1 \circ b_1^{-1}) \odot v_1 \oplus (a_2 \circ b_2^{-1}) \odot v_2\Big)(e_E) > \theta$$

$$\Leftrightarrow ((a_1 \odot v_1 \oplus b_1^{-1} \odot v_1) \oplus (a_2 \odot v_2 \oplus b_2^{-1} \odot v_2))(e_E) > \theta$$

By hypothesis,

$$\exists y_1, y_2 \in V : P(a_1, v_1, y_1) > \theta, P(a_2, v_2, y_2) > \theta, E(y_1, y_2, v) > \theta,$$

$$\exists x_1, x_2 \in V : P(b_1^{-1}, v_1, x_1^{-1}) > \theta, P(b_2^{-1}, v_2, x_2^{-1}) > \theta, E(x_1^{-1}, x_2^{-1}, v^{-1}) > \theta.$$

Let $z, u \in V$ such that $E(y_1, x_1^{-1}, z) > \theta$ and $E(y_2, x_2^{-1}, u) > \theta$. Then

$$(a_1 \odot v_1 \oplus b_1^{-1} \odot v_2)(z) > P(a_1, v_1, y_1) \wedge P(b_1^{-1}, v_2, x_1^{-1}) \wedge E(y_1, x_1^{-1}, z) > \theta$$

and

$$(a_2 \odot v_2 \oplus b_2^{-1} \odot v_2)(u) > P(a_2, v_2, y_2) \wedge P(b_2^{-1}, v_2, x_2^{-1}) \wedge E(y_2, x_2^{-1}, u) > \theta.$$

Hence by lemma 3.1, we have:

$$((y_1 \oplus x_1^{-1}) \oplus (y_2 \oplus x_2^{-1}))(e_E) > \theta \Leftrightarrow ((y_1 \oplus y_2) \oplus (x_1^{-1} \oplus x_2^{-1}))(e_E) > \theta.$$

Since

$$((y_1 \oplus y_2) \oplus (x_1^{-1} \oplus x_2^{-1}))(e_E) > E(y_1, y_2, v) \wedge E(x_1^{-1}, x_2^{-1}, v^{-1}) \wedge E(v, v^{-1}, e_E) > \theta.$$

Thus

$$((y_1 \oplus x_1^{-1}) \oplus (y_2 \oplus x_2^{-1}))(e_E) = \bigvee_{w_1, w_2 \in V} E(y_1, x_1^{-1}, w_1) \wedge E(y_2, x_2^{-1}, w_2) \wedge E(w_1, w_2, e_E) > \theta.$$

And so,

$$\exists w_1, w_2 \in V : E(y_1, x_1^{-1}, w_1) > \theta, E(y_2, x_2^{-1}, w_2) > \theta, E(w_1, w_2, e_E) > \theta.$$

Since $E(y_1, x_1^{-1}, z) > \theta, E(y_2, x_2^{-1}, u) > \theta$. Thus $w_1 = z, w_2 = u$. Hence $E(w_1, w_2, e_E) = E(z, u, e_E) > \theta$. Accordingly

$$((a_1 \odot v_1 \oplus b_1^{-1} \odot v_1) \oplus (a_2 \odot v_2 \oplus b_2^{-1} \odot v_2))(e_E) >$$

$$(a_1 \odot v_1 \oplus b_1^{-1} \odot v_1)(z) \wedge (a_2 \odot v_2 \oplus b_2^{-1} \odot v_2)(u) \wedge E(z, u, e_E) > \theta,$$

and the theorem is proved. □

By Theorem 4.1 and by induction on n, we have the following result:

Corollary 4.1. *Let $v_1, v_2, \cdots, v_n$ be independent vectors, and suppose $v \in FL(v_1, v_2, \cdots, v_n)$. If $(a_1 \odot v_1 \oplus a_2 \odot v_2 \oplus \cdots a_n \odot v_n)(v) > \theta$, where $a_1, a_2, \cdots, a_n \in R$, then the above representation of v is unique.*

5. Conclusions

Yuan and Lee's [8] definition of fuzzy group based on fuzzy binary operation. Also, Aktaş and Çağman in [1] defined a new type of fuzzy ring. In [7] Yetkin and Olgun presented a new type of fuzzy module by using Yuan and Lee's definition. In this paper, a new kind of fuzzy vector space over fuzzy field was introduced and their related properties were investigated. Moreover the concepts of fuzzy vector subspace are studied and some of their basic properties are presented as analogous to ordinary vector space theory. The questions of other types and their applications in fuzzy algebra remain. An investigation into this aspect of the work will be the subject of future research.

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