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MULTIPLE WEIGHTED NORM INEQUALITIES FOR COMMUTATORS OF MULTILINEAR CALDERÓN-ZYGMUND AND POTENTIAL TYPE OPERATORS

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Abstract. In this paper, the authors study the multiple weighted boundedness of commutators generated by multilinear Calderón-Zygmund and potential type singular integral operators and BMO function. Furthermore, the two weighted norm inequalities of Calderón-Zygmund and potential type singular integral operators are obtained with A_∞ condition.

1. Introduction

The multilinear Calderón-Zygmund theory originated in the works of Coifaman Meyer in the 70s, see [5] and [6]. This topic has been attracting a lot of attention in the few decades. Recently, there are many studies on multilinear singular integrals under integral type regularity conditions so that they fall outside the standard Calderón-Zygmund classes (see [2], [7], [8] and so on).

In [2], Calderón-Zygmund and potential type multilinear operator was introduced. For $0 \le \alpha < nN$, assume that T_α is a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered

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distributions

$$T_\alpha : S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n) \to S^{'}(\mathbb{R}^n).$$

Assume also that for a certain kernel function K,

$$T_\alpha(\mathbf{f})(x) = \int_{(\mathbb{R}^n)^N} K(x, y_1, \cdots, y_N) f_1(y_1) \cdots f_N(y_N) d\mathbf{y},$$

for $f_i \in C_0^\infty(\mathbb{R}^n)$ (smooth functions with compact support), $i = 1, ..., N$ and $x \notin \cap_{j=i}^N \operatorname{supp} f_i$, where $\mathbf{f} = (f_1, \ldots, f_N)$ and $d\mathbf{y} = dy_1 \ldots dy_N$. The operator T_α satisfies the *multilinear $L^{r',\alpha}$-Hörmander contidion*, i.e.

$$\sup_Q \sup_{x,z \in \frac{1}{2}Q} \sum_{k=0}^{\infty} |2^k Q|^{\frac{N}{r} - \frac{\alpha}{n}} \left(\int_{(2^{k+1}Q)^N \setminus (2^k Q)^N} |K(x, \mathbf{y}) - K(z, \mathbf{y})|^{r'} d\mathbf{y} \right)^{\frac{1}{r'}} < \infty,$$

where $r > 1$ and r' is its dual exponent, Q is the cube in $\mathbb{R}^n$ with sides parallel to the axes, $Q^N = \underbrace{Q \times \cdots \times Q}_{N}$. When $r = 1$ the *multilinear $L^{\infty,\alpha}$-Hörmander contidion* is understood as

$$\sup_Q \sup_{x,z \in \frac{1}{2}Q} \sum_{k=0}^{\infty} |2^k Q|^{N - \frac{\alpha}{n}} \sup_{\mathbf{y} \in (2^{k+1}Q)^N \setminus (2^k Q)^N} |K(x, \mathbf{y}) - K(z, \mathbf{y})| < \infty.$$

It is easy to see that when $N = 1$ and $\alpha = 0$, the above condition reduces to the classical $L^{r'}$-Hörmander condition :

$$\sup_{x \in Q} \sum_{k=1}^{\infty} (2^k |Q|)^{\frac{1}{r}} \left(\int_{2^k Q^* \setminus 2^{k-1} Q^*} |K(x, y) - K(x_Q, y)|^{r'} dy \right)^{\frac{1}{r'}} < \infty,$$

where x_Q is the center of Q and Q^* is an appropriate dilation of Q. In [10], The authors defined the classical $L^{r'}$-Hörmander's condition, it was implicit in the work of D. Kurtz and R. Wheeden [11]. The multilinear version was introduced by Bui-Dong [8], i.e.

$$\left(\int_{S_{j_m}(Q^*)} \cdots \int_{S_{j_1}(Q^*)} |K(x, \mathbf{y}) - K(z, \mathbf{y})|^{r'} d\mathbf{y} \right)^{\frac{1}{r'}} \leq \frac{|x - z|^{N(\delta - \frac{n}{r})}}{|Q|^{\frac{N\delta}{n}}} 2^{-N\delta \max\{j_1, \ldots, j_m\}},$$

for all $x, z \in Q$ and $(j_1, \dots, j_N) \neq (0, \dots, 0)$, where $S_j(Q^*) = 2^j Q^* \setminus 2^{j-1} Q^*$ if $j \geq 1$ and $S_0(Q^*) = Q^*$.

Chaffee, Torres and Wu also [2] precisely identified the graded classes of multiple weights $\mathbf{A}(\mathbf{P}, q, r)$. They are the relevant multiple kind of weights for T_α.

Definition 1.1. [2] For $1 \leqslant r < p_1, \dots, p_N < \infty$, $\mathbf{P} = (p_1, \dots, p_N)$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_N}$, and $\frac{r}{N} < p \leq q < \infty$, we say that a vector of weight $\mathbf{w} = (\omega_1, \dots, \omega_m)$, is in the class $\mathbf{A}(\mathbf{P}, q, r)$, or it satisfies the $\mathbf{A}(\mathbf{P}, q, r)$ condition, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^N \left(\frac{1}{|Q|} \int_Q \omega_i(x)^{-\frac{p_i r}{(p_i - r)}} dx \right)^{\frac{(p_i - r)}{p_i r}} < \infty,$$

where $\omega = \prod_{i=1}^N \omega_i$.

The main result from [2] is the following.

Theorem 1.1. [2] *Let* $1 \leqslant r < p_1, \dots, p_N < \infty$, $\mathbf{P} = (p_1, \dots, p_N)$, $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_N}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 \leq r < \min(p_1, \dots, p_N, \frac{nN}{\alpha}, Np)$ *and* $\frac{1}{r^*} = \frac{N}{r} - \frac{\alpha}{n}$. *If*

$$T_\alpha : L^r \times \dots \times L^r \to L^{r^*, \infty} \tag{1.1}$$

and satisfies the multilinear $L^{r', \alpha}$*-Hörmander contidion, then*

$$T_\alpha : L^{p_1}(\omega_1^{p_1}) \times \dots \times L^{p_N}(\omega_N^{p_N}) \to L^q(\omega^q)$$

for all $\mathbf{w} \in \mathbf{A}(P, q, r)$.

Given a collection of locally integral function $\vec{b} = (b_1, \dots, b_m)$, we define the m-linear commutator of $\vec{b}$ and the multilinear operator T_α as follows,

$$T_{\alpha, \vec{b}}(\mathbf{f}) = \sum_{i=1}^N T^i_{\alpha, \vec{b}}(\mathbf{f})$$

where each term is the commutator of b_i and T_α in the ith entry of T_α, that is

$$T^i_{\alpha, \vec{b}}(\mathbf{f}) = b_i T_\alpha(f_1, \dots, f_i, \dots, f_N) - T_\alpha(f_1, \dots, b_i f_i, \dots, f_N).$$

We say $\vec{b} \in (BMO)^m$, if $\|\vec{b}\|_{(BMO)^m} = \max\{\|b_i\|_{BMO} : i = 1, \dots, N\}$. On the base of [2], our main result is the following conclusion.

Theorem 1.2. *Let* $1 \leqslant r < p_1, \dots, p_N < \infty$, $\mathbf{P} = (p_1, \dots, p_N)$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 \leq r < \min(p_1, \dots, p_N, \frac{nN}{\alpha}, Np)$ *and* $\frac{1}{r^*} = \frac{N}{r} - \frac{\alpha}{n}$. *If*

$$T_\alpha : L^r \times \cdots \times L^r \to L^{r^*,\infty}$$

and satisfies the multilinear $L^{r',\alpha}$*-Hörmander contidion, then*

$$T_{\alpha,\vec{b}} : L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N}) \to L^q(\omega^q)$$

for all $\mathbf{w} \in \mathbf{A}(P, q, r)$, *where* $\vec{b} \in (BMO)^m$.

We also extend Theorem 1.1 to two weights result of T_α.

Theorem 1.3. *Let* $1 \leqslant r < p_1, \dots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, $p \leqslant q$, $1 \leq r < \min(p_1, \dots, p_N, \frac{nN}{\alpha}, Np)$ *and* $\frac{1}{r^*} = \frac{N}{r} - \frac{\alpha}{n}$. *If*

$$T_\alpha : L^r \times \cdots \times L^r \to L^{r^*,\infty}$$

and satisfies the multilinear $L^{r',\alpha}$*-Hörmander contidion,* $(u, \mathbf{w})$ *are weights that satisfy*

$$\sup_Q |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \left(\frac{1}{|Q|} \int_Q u(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^N \left(\frac{1}{|Q|} \int_Q \omega_i(x)^{-\frac{tp_i r}{p_i - r}} dx \right)^{\frac{p_i - r}{tp_i r}} < \infty$$

for some $t > 1$ *with* $u^q \in A_\infty$, *then*

$$T_\alpha : L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N}) \to L^q(u^q).$$

It is well known that A_∞ weights satisfy the reverse Hölder condition, the following corollary is a direct consequence of this fact.

Corollary 1.1. *Let* $1 \leqslant r < p_1, \dots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, $p \leqslant q$, $1 \leq r < \min(p_1, \dots, p_N, \frac{nN}{\alpha}, Np)$ *and* $\frac{1}{r^*} = \frac{N}{r} - \frac{\alpha}{n}$. *If*

$$T_\alpha : L^r \times \cdots \times L^r \to L^{r^*,\infty}$$

and satisfies the multilinear $L^{r',\alpha}$*-Hörmander contidion,* $(u, \mathbf{w})$ *are weights that satisfy*

$$\sup_Q |Q|^{\frac{1}{q}-\frac{1}{p}+\frac{\alpha}{n}} \left(\frac{1}{|Q|} \int_Q u(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^{N} \left(\frac{1}{|Q|} \int_Q \omega_i(x)^{\frac{-p_i r}{p_i - r}} dx \right)^{\frac{p_i - r}{p_i r}} < \infty$$

with $u^q, \omega_1, \dots, \omega_N \in A_\infty$, *then*

$$T_\alpha : L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N}) \to L^q(u^q).$$

In section 4 we give two generalized versions of Theorem 1.3 on Banach function spaces.

2. Preliminaries

We first recall the definition of other classes of multiple weights.

Let $1 < p_1, \dots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$. Given $\mathbf{w} = (\omega_1, \dots, \omega_N)$, set $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}$. As it is defined in [3], we say that $\mathbf{w} \in A_{\mathbf{P}}$ if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v_{\vec{\omega}}(x) dx \right)^{\frac{1}{p}} \prod_{j=1}^{N} \left(\frac{1}{|Q|} \int_Q \omega_j^{1-p_j'}(x) dx \right)^{\frac{1}{p_j'}} < \infty,$$

where p' is the conjugate index of p. On the other hand Moen [4] considered for $1 < p_1, \dots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, and q be a number $\frac{1}{N} < p \leqslant q < \infty$, the classes $\mathbf{A}(\mathbf{P}, q)$ of vector weights $\mathbf{w} = (\omega_1, \dots, \omega_m)$ such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \prod_{i=1}^{N} \omega_i^q(x) dx \right)^{\frac{1}{q}} \prod_{i=1}^{m} \left(\frac{1}{|Q|} \int_Q \omega_i^{-p_i'}(x) dx \right)^{\frac{1}{p_i'}} < \infty.$$

Next we give the following properties, the detail proofs to see [2].

Lemma 2.1. *Let* $1 < s < r < p_1, \ldots, p_N, Np < \infty$ *and* $1 \leq q < \infty$*. The following properties hold.*

(i) $\mathbf{A}(\mathbf{P}, q, 1) = \mathbf{A}(\mathbf{P}, q)$ *and* $[\omega]_{\mathbf{A}(\mathbf{P},p,1)} = [\omega^p]_{A_p}^{\frac{1}{p}}$ *when* $N = 1$*.* A_p *denote Muckenhoupt classes.*

(ii) $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$ *if and only if* $\mathbf{w}^r = (\omega_1^r, \ldots, \omega_N^r) \in \mathbf{A}(\frac{\mathbf{P}}{r}, \frac{q}{r})$*.*

(iii) $\mathbf{A}(\mathbf{P}, q, s) \subset \mathbf{A}(\mathbf{P}, q, r) \subset \mathbf{A}(\mathbf{P}, q)$*.*

Lemma 2.2. *The weight* $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$ *if and only if*

$$\omega^q \in A_{Nq/r} \text{ and } \omega_i^{-\frac{p_i r}{p_i - r}} \in \mathbf{A}_{N(p_i/r)'},\ i = 1, \ldots N.$$

Lemma 2.3. *Assume that* $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$*. Then there exists* $t > r$ *such that* $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, t)$*.*

We will use several maximal functions. For $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function $M(f)$ is defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\varepsilon > 0$, $M^\varepsilon(f)(x) = (M(|f|^\varepsilon)(x))^{\frac{1}{\varepsilon}}$. For $0 < \alpha < n$ and $f \in L^1_{loc}(\mathbb{R}^n)$, the fractional function defined by

$$M_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy.$$

We also recall the Sharp maximal function defined by

$$M^\sharp(f)(x) = \sup_{Q \ni x} \inf_{c \in \mathbf{R}} \frac{1}{|Q|} \int_Q |f(y) - c| dy,$$

and for $\delta > 0$, define $M^\sharp_\delta(f) = M^\sharp(|f|^\delta)^{\frac{1}{\delta}}$.

For $0 \leqslant \alpha < nN$ and $\mathbf{f} = (f_1, \dots, f_N) \in (L^1_{loc}(\mathbf{R}^n))^N$, the multilinear fractional maximal functions $\mathcal{M}_\alpha(\mathbf{f})$ and $\mathcal{M}_{L(\log L),\alpha}(\mathbf{f})$ are defined by

$$\mathcal{M}_\alpha(\mathbf{f}) = \sup_{Q \ni x} \prod_{i=1}^{N} \frac{1}{|Q|^{1-\frac{\alpha}{nN}}} \int_Q |f_i(y_i)| dy_i,$$

$$\mathcal{M}_{L(\log L),\alpha}(\mathbf{f})(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \prod_{i=1}^{N} \| f_i \|_{L \log L, Q},$$

where $\| f \|_{L\log L, Q} = \inf \left\{ \lambda > 0 : \int_Q \frac{|f(x)|}{\lambda} \log^+ \frac{|f(x)|}{\lambda} dx \leqslant 1 \right\}$, $\log^+ t = \max(\log t, 0)$. Finally, for $1 \leq r < \frac{nN}{\alpha}$, we also define

$$\mathcal{M}_{\alpha,r}(\mathbf{f}) = \sup_{Q \ni x} \left(\prod_{i=1}^{N} \frac{1}{|Q|^{1-\frac{r\alpha}{nN}}} \int_Q |f_i(y_i)|^r dy_i \right)^{\frac{1}{r}},$$

$$\mathcal{M}_{L(\log L),\alpha,r}(\mathbf{f})(x) = \sup_{Q \ni x} |Q|^{\frac{\alpha}{n}} \prod_{i=1}^{N} \| f_i^r \|^{\frac{1}{r}}_{L(\log L),Q} .$$

By the generalized Hölder inequality (see section 4), we get

$$\frac{1}{|Q|} \int_Q |f(x)| dx \leq C \| f \|_{L(\log L),Q} .$$

We have the following weighted estimate for $\mathcal{M}_{L\log L,\alpha,r}(f)$.

Lemma 2.4. *Let* $1 \leqslant r < p_1, \dots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, $0 < \alpha < \frac{N}{p}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then* $\mathcal{M}_{L\log L,\alpha,r}(f)$ *is bounded from* $L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N})$ *to* $L^q(\omega^q)$ *if* $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$.

Proof : It is easy to check that

$$\| \mathcal{M}_{L(\log L),\alpha,r}(\mathbf{f}) \|_{L^q(\omega^q)} = \| \mathcal{M}_{L(\log L),r\alpha}(\mathbf{f}^r) \|^{\frac{1}{r}}_{L^{\frac{q}{r}}(\omega^q)},$$

where $\mathbf{f}^r = (|f_1|^r, \dots, |f_N|^r)$. The result is a consequence of [1, Proposition 7.2]. $\square$

3. Proof of the Main Results

We will use the following form of the classical result of Fefferman and Stein [9]. Let $0 < p, \delta < \infty$ and let ω be a weight in A_∞. Then, there exists $C > 0$, such that

$$\int_{\mathbb{R}^n} (M^\delta(f)(x))^p \omega(x)dx \leq C \int_{\mathbb{R}^n} (M_\delta^\sharp(f)(x))^p \omega(x)dx$$

for all function f if the left-hand is finite.

Lemma 3.1. *Assume that T_α satisfy the hypothesis of Theorem 1.2. If $0 < \delta < \varepsilon < \min(1, rn/(nN - r\alpha)$, then for all $\mathbf{f} \in L^{l_1} \times L^{l_2} \cdots \times L^{l_N}$ with $r < s < l_1, \ldots, l_N < \infty$,*

$$M_\delta^\sharp(T_{\alpha,\vec{b}}(\mathbf{f}))(x) \leq C \parallel \vec{b} \parallel_{(BMO)^m} \left(\mathcal{M}_{L(\log L),\alpha,s}(\mathbf{f})(x) + M^\varepsilon(T_\alpha(\mathbf{f})(x)\right).$$

Proof: It suffices to prove the $T^i_{\alpha,\vec{b}}$. Note that for any constant λ we have

$$\begin{aligned} T^i_{\alpha,\vec{b}}(\mathbf{f})(x) =& (b_i(x) - \lambda)T_\alpha(f_1, \ldots, f_i, \ldots, f_N)(x) \\ & - T_\alpha(f_1, \ldots, (b_i - \lambda)f_i, \ldots, f_N)(x). \end{aligned}$$

Fix $Q \subset \mathbb{R}^n$ and $x \in Q$. For $0 < \delta < 1$, $0 < b < a$, we know that $a^\delta - b^\delta < |a - b|^\delta$ and $(a + b)^{1/\delta} < 2^{1/\delta}(a^{1/\delta} + b^{1/\delta})$. We can estimate

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \left||T^i_{\alpha,\vec{b}}(\mathbf{f})(z)|^\delta - |c|^\delta\right| dz\right)^{\frac{1}{\delta}} \\ &\leq \left(\frac{1}{|Q|}\int_Q \left|T^i_{\alpha,\vec{b}}(\mathbf{f})(z) - c\right|^\delta dz\right)^{\frac{1}{\delta}} \\ &\leq \left(\frac{C}{|Q|}\int_Q |(b_i(z) - \lambda)T_\alpha(\mathbf{f})(z)|^\delta dz\right)^{\frac{1}{\delta}} \\ &\quad + \left(\frac{C}{|Q|}\int_Q |T_\alpha(f_1, \ldots, (b_i - \lambda)f_i, \ldots, f_N) - c|^\delta dz\right)^{\frac{1}{\delta}} \\ &\quad = I + II. \end{aligned}$$

Let $\lambda = (b_i)_{Q^*}$ be the average of b_i on Q^*. For any $1 < u < \frac{\varepsilon}{\delta}$, by the Hölder inequality and Jensen inequality, we obtain

$$
\begin{aligned}
I &\le C\Big(\frac{1}{|Q|}\int_Q |b_i(z)-\lambda|^{\delta u'}dz\Big)^{\frac{1}{u'\delta}}\Big(\frac{1}{|Q|}\int_Q |T_\alpha(\mathbf{f})(z)|^{\delta u}dz\Big)^{\frac{1}{\delta u}}\\
&\le C \parallel b_i \parallel_{BMO} M^{\delta u}(T_\alpha(\mathbf{f}))(x)\\
&\le C \parallel \vec{b} \parallel_{(BMO)^m} M^{\varepsilon}(T_\alpha(\mathbf{f}))(x).
\end{aligned}
$$

To estimate II for $i = 1, \ldots, N$, let $f_i^0 = f_i\chi_{Q^*}$ and $f_i^\infty = f_i(1-\chi_{Q^*})$. Then $f_i = f_i^0 + f_i^\infty$ and

$$
\begin{aligned}
\prod_{i=1}^N f_i(y_i) &= \prod_{i=1}^N (f_i^0(y_i) + f_i^\infty(y_i))\\
&= \prod_{i=1}^N f_i^0(y_i) + \sum_{(\beta_1,\ldots,\beta_N)\in I}\prod_{i=1}^N f_1^{\beta_1}(y_1)\cdots f_N^{\beta_N}(y_N),
\end{aligned}
$$

where I is the collection of all N-tuples $\beta = (\beta_1, \ldots, \beta_N)$ with each $\beta_i = 0$ or ∞ and at least one $\beta_j \neq 0$.

We select $c = \sum_{\beta\in I} c_{\beta_1,\ldots,\beta_N}$ with $c_{\beta_1,\ldots,\beta_N} = T_\alpha(f_1^{\beta_1}, \ldots, (b_i-\lambda)f_i^{\beta_i}, \ldots, f_N^{\beta_N})(x)$ and get

$$
\begin{aligned}
II &\le C\Big(\frac{1}{|Q|}\int_Q |T_\alpha(f_1^0, \ldots, (b_i-\lambda)f_i^0, \ldots, f_N^0)(z)|^{\delta}dz\Big)^{\frac{1}{\delta}}\\
&\quad + C\sum_{\beta\in I}\Big(\frac{1}{|Q|}\int_Q |T_\alpha(f_1^{\beta_1}, \ldots, (b_i-\lambda)f_i^{\beta_i}, \ldots, f_N^{\beta_N})(z) - c_{\beta_1,\ldots,\beta_N}|^{\delta}dz\Big)^{\frac{1}{\delta}}\\
&\le II_0 + \sum_{\beta\in I} II_{\beta_1,\ldots,\beta_N}.
\end{aligned}
$$

Using the weak-type estimate on T_α and the Kolmogorov inequality we have

$$\begin{aligned}
II_0 &= C\bigg(\frac{1}{|Q|}\int_Q |T_\alpha(f_1^0,\dots,(b_i-\lambda)f_i^0,\dots,f_N^0)(z)|^\delta dz\bigg)^{\frac{1}{\delta}} \\
&\le C \parallel T_\alpha(f_1^0,\dots,(b_i-\lambda)f_i^0,\dots,f_N^0) \parallel_{L^{r^*,\infty}(Q,\frac{dz}{|Q|})} \\
&\le C|Q^*|^{\frac{\alpha}{n}}\bigg(\frac{1}{|Q^*|}\int_{Q^*}|(b_i-\lambda)f_i^0|^r dz\bigg)^{1/r}\bigg(\frac{1}{|Q^*|}\int_{Q^*}\prod_{\substack{j=1,\\ j\neq i}}^N |f_j(y_j)|^r dy_i\bigg)^{\frac{1}{r}} \\
&\le C \parallel b_i \parallel_{BMO} |Q^*|^{\frac{\alpha}{n}}\left(\prod_{j=1}^N \frac{1}{|Q^*|}\int_{Q^*}|f_j(y_j)|^s dy_i\right)^{\frac{1}{s}} \\
&\le C \parallel \vec{b} \parallel_{(BMO)^m} \mathcal{M}_{L(\log L),\alpha,s}(\mathbf{f})(x)
\end{aligned}$$

We consider the term $II_{\infty,\dots,\infty}$ and the terms $II_{\beta_1,\dots,\beta_N}$ such that $\beta_{j_1}=\cdots=\beta_{j_l}=0$ for some $\{j_1,\dots,j_l\}\subset\{1,\dots,N\}$, where $1\le l<N$. However, we can estimate all terms $II_{\beta_1,\dots,\beta_N}$ together, since for any $J=1,\dots,N-1$,

$$(Q^*)^{N-J}\times(\mathbb{R}^n\setminus Q^*)^J\subset\mathbb{R}^{nN}\setminus(Q^*)^N=\cup_{k=0}^\infty(2^{k+1}Q^*)^N\setminus(2^kQ^*)^N,$$

and similarly $(\mathbb{R}^n\setminus Q^*)^N\subset\cup_{k=0}^\infty(2^{k+1}Q^*)^N\setminus(2^kQ^*)^N$. Using Hölder's inequality and $L^{r',\alpha}$-Hörmander's condition, we have

$$\begin{aligned}
&\bigg(\frac{1}{|Q|}\int_Q |T_\alpha(f_1^{\beta_1},\dots,(b_i-\lambda)f_i^{\beta_i},\dots,f_N^{\beta_N})(z)-c_{\beta_1,\dots,\beta_N}|^\delta dz\bigg)^{\frac{1}{\delta}} \\
\le&\frac{C}{|Q|}\int_Q |T_\alpha(f_1^{\beta_1},\dots,(b_i-\lambda)f_i^{\beta_i},\dots,f_N^{\beta_N})(z)-c_{\beta_1,\dots,\beta_N}|dz \\
\le&\frac{C}{|Q|}\int_Q\int_{\mathbb{R}^{nN}\setminus(Q^*)^N}|K(z,\mathbf{y})-K(x,\mathbf{y})||(b(y_i)-\lambda)f_i(y_i)|\prod_{\substack{j=1,\\ j\neq i}}^N|f_j(y_j)|d\mathbf{y}dz
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{|Q|}\sum_{k=0}^{\infty}\int_Q\int_{(2^{k+1}Q^*)^N\setminus(2^kQ^*)^N}|K(z,\mathbf{y})-K(x,\mathbf{y})||(b(y_i)-\lambda)f_i(y_i)|\prod_{\substack{j=1,\\ j\neq i}}^{N}|f_j(y_j)|d\mathbf{y}dz\\
&\leq C\sup_{Q^*}\sup_{x,z\in\frac{1}{2}Q^*}\sum_{k=0}^{\infty}|2^kQ^*|^{N/r-\alpha/n}\left(\int_{(2^{k+1}Q^*)^N\setminus(2^kQ^*)^N}|K(z,\mathbf{y})-K(x,\mathbf{y})|^{r'}d\mathbf{y}\right)^{\frac{1}{r'}}\\
&\quad\times|2^{k+1}Q^*|^{\alpha/n}\left(\frac{1}{|2^{k+1}Q^*|^N}\int_{2^{k+1}Q^*}|(b(y_i)-\lambda)f_i(y_i)|^r\prod_{\substack{j=1,\\ j\neq i}}^{N}|f_j(y_j)|^r d\mathbf{y}\right)^{\frac{1}{r}}\\
&\leq C\parallel b_i\parallel_{BMO}\mathcal{M}_{L(\log L),\alpha,s}(\mathbf{f})(x)\\
&\leq C\parallel \vec{b}\parallel_{(BMO)^m}\mathcal{M}_{L(\log L),\alpha,s}(\mathbf{f})(x)
\end{aligned}
$$

$\square$

Taking the similar method, we get the following lemma.

Lemma 3.2. *Let T_α satisfy the hypothesis of Theorem 1.2. If $0<\delta<\min(1,\frac{rn}{nN-r\alpha})$, then for all $\mathbf{f}\in L^{l_1}\times L^{l_2}\cdots\times L^{l_N}$ with $r<l_1,\ldots,l_N<\infty$,*

$$M_\delta^\sharp(T_\alpha(\mathbf{f}))(x)\leq\mathcal{M}_{L(\log L),\alpha,r}(\mathbf{f})(x).$$

Proof of theorem 1.2 : Note that we always have $\omega^q\in A_\infty$ by Lemma 2.2, so for exponents $0<\delta<\varepsilon<\min(1,\frac{rn}{nN-r\alpha})$, applying the Fefferman-Stein inequality ,Lemma 3.1, Lemma 2.3, Lemma 2.4, Lemma 3.2 and Lemma 2.4 we have,

$$
\begin{aligned}
\parallel T_{\alpha,\vec{b}}(\mathbf{f})\parallel_{L^q(\omega^q)} &\leq \parallel M_\delta(T_{\alpha,\vec{b}}(\mathbf{f}))\parallel_{L^q(\omega^q)}\\
&\leq C\parallel M_\delta^\sharp(T_{\alpha,\vec{b}}(\mathbf{f}))\parallel_{L^q(\omega^q)}\\
&\leq C\parallel\vec{b}\parallel_{(BMO)^m}\left(\parallel\mathcal{M}_{L(\log L),\alpha,s}(\mathbf{f})\parallel_{L^q(\omega^q)}+\parallel M^\varepsilon(T_\alpha(\mathbf{f})\parallel_{L^q(\omega^q)}\right)\\
&\leq C\parallel\vec{b}\parallel_{(BMO)^m}\left(\parallel\mathcal{M}_{L(\log L),\alpha,s}(\mathbf{f})\parallel_{L^q(\omega^q)}+\parallel M_\varepsilon^\sharp(T_\alpha(\mathbf{f})\parallel_{L^q(\omega^q)}\right)\\
&\leq C\parallel\vec{b}\parallel_{(BMO)^m}\parallel\mathcal{M}_{L(\log L),\alpha,s}(\mathbf{f})\parallel_{L^q(\omega^q)}\\
&\leq C\parallel\vec{b}\parallel_{(BMO)^m}\parallel\prod_{i=1}^{N}\parallel f_i\parallel_{L^{p_i}(\omega_i^{p_i})}.
\end{aligned}
$$

□

Remark 1. *The Fourier multiplier T_m is given by*

$$T_m(f)(x) = \int_{\mathbb{R}^{nN}} e^{i\cdot(\xi_1+\cdots+\xi_N)} m(\xi_1+\cdots+\xi_N)\widehat{f}_1(\xi_1)\ldots\widehat{f}_N(\xi_N)d\xi_1\ldots d\xi_N$$

for $f_1,\ldots,f_N \in \mathcal{S}(\mathbb{R}^n)$, and where the function m satisfies some regularity property defined in terms of Sobolev spaces estimates.

For $s\in\mathbb{R}$, we say $F\in\mathcal{S}'$ in the Sobolev space $W^s(\mathbb{R}^{nN})$ if

$$\| F \|_{W^s(\mathbb{R}^{nN})} = \left(\int_{\mathbb{R}^{nN}} (1+|\xi|^2)^s |\widehat{F}(\xi)|^2 d\xi_1\ldots d\xi_N\right)^{\frac{1}{2}}$$

Let $\Psi\in\mathcal{S}(\mathbb{R}^{nN})$ be such that supp $\Psi\subset\{\xi\in\mathbb{R}^{nN} : \frac{1}{2}\le|\xi|\le 2\}$ and $\sum_{k\in\mathbb{Z}}\Psi(\xi/2^k) = 1$ for all $\xi\in\mathbb{R}^{nN}\setminus\{0\}$. We set

$$m_\alpha^k(\xi) = 2^{k\alpha} m(2^k\xi)\Psi(\xi)$$

for a function m, $\alpha>0$ and $k\in\mathbb{Z}$.

In [2] they proved T_m is bounded from $L^{p_1}(\omega_1^{p_1})\times\cdots\times L^{p_N}(\omega_N^{p_N})$ to $L^q(\omega^p)$ with norm $\| T_m \|$ at most multiple $\sup_{k\in\mathbb{Z}} \| m_\alpha^k \|_{W(\mathbb{R}^n)}$ by Theorem 1.1, i.e the Theorem 5.1 in [2]. Naturally, the conclusion of the Theorem 5.1 in [2] is valid to the commutator $T_{m,\vec{b}}$ of Fourier multiplier operator T_m with BMO function.

To prove Theorem **1.3**, we need the following lemmas.

Lemma 3.3. [2] *Let T_α satisfy the hypothesis of Theorem 1.2. If $0<\delta<\min\{1, \frac{rn}{nN-r\alpha,}\}$, then for all $\mathbf{f}\in L^{l_1}\times L^{l_2}\cdots\times L^{l_N}$ with $r<l_1,\ldots,l_N<\infty$,*

$$M_\delta^\sharp(T_\alpha(\mathbf{f}))(x) \le \mathcal{M}_{\alpha,r}(\mathbf{f})(x).$$

Lemma 3.2 also can be obtained by Lemma 3.3 since $\mathcal{M}_{\alpha,r}(\mathbf{f}) \leqslant \mathcal{M}_{L(\log L),\alpha,r}(\mathbf{f})$.

The next lemma is similar Lemma 2.4

Lemma 3.4. *Suppose* $0 \leqslant \alpha < nN$, $1 \leqslant r < p_1, \dots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, $\frac{r}{N} < p \leqslant q$. *And* $(u, \mathbf{w})$ *are weights that satisfy*

$$\sup_Q |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \left(\frac{1}{|Q|} \int_Q u(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^{N} \left(\frac{1}{|Q|} \int_Q \omega_i(x)^{-\frac{tp_i r}{p_i - r}} dx \right)^{\frac{p_i - r}{tp_i r}} < \infty$$

for some $t > 1$, *then* $\mathcal{M}_{\alpha,r}(\mathbf{f})$ *is bounded from* $L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N})$ *to* $L^q(u^q)$.

Proof : It is easy to check that

$$\| \mathcal{M}_{\alpha,r}(\mathbf{f}) \|_{L^q(u^q)} = \| \mathcal{M}_{r\alpha}(\mathbf{f}^r) \|_{L^{\frac{q}{r}}(u^q)}^{\frac{1}{r}} .$$

We just prove the boundedness for the dyadic version,

$$\mathcal{M}_{r\alpha}^d(\mathbf{f}^r)(x) = \sup_{Q \in \mathcal{D}: Q \ni x} \prod_{i=1}^{N} \frac{1}{|Q|^{1 - \frac{r\alpha}{nN}}} \int_Q |f_i(y_i)|^r dy_i,$$

where $\mathcal{D}$ is the collection of all the dyadic cubes. Let a be a constant satisfying $a > 2^{mn}$ and let

$$D_k = \{x \in \mathbb{R}^n : \mathcal{M}_{r\alpha}^d(\mathbf{f}^r)(x) > a^{kr}\}$$

If D_k is non-empty then we can write $D_k = \bigcup_j Q_{k,j}$ where $Q_{k,j}$ is the maximal dyadic cube satisfying

$$a^{kr} < \prod_{i=1}^{N} \frac{1}{|Q_{k,j}|^{1 - \frac{r\alpha}{nN}}} \int_{Q_{k,j}} |f_i(y_i)|^r dy_i \leqslant 2^{Nn} a^{kr}.$$

The sets $E_{k,j} = Q_{k,j} \setminus (Q_{k,j} \bigcup D_{k+1})$ are disjoint and satify

$$|Q_{k,j}| < \beta |E_{k,j}|$$

for some $\beta > 1$.

$$
\begin{aligned}
&\| \mathcal{M}^d_{r\alpha}(\mathbf{f}^r) \|^{\frac{1}{r}}_{L^{\frac{q}{r}}(u^q)} \\
=&\Big(\int_{\mathbb{R}^n} (\mathcal{M}^d_{r\alpha}(\mathbf{f}^r)(x))^{q/r} u(x)^q dx \Big)^{1/q} \\
=&\Big(\sum_k \int_{D_k \setminus D_{k+1}} (\mathcal{M}^d_{r\alpha}(\mathbf{f}^r)(x))^{q/r} u(x)^q dx \Big)^{1/q} \\
\leqslant& a\Big(\sum_{k,j} a^{kq} \int_{Q_{k,j}} u(x)^q dx \Big)^{1/q} \\
\leqslant& a\Big(\sum_{k,j} \Big(\prod_{i=1}^N \frac{1}{|Q|^{1-\frac{r\alpha}{nN}}} \int_{Q_{k,j}} |f_i(y_i)|^r \omega_i(y_i)^r \omega_i(y_i)^{-r} dy_i, \Big)^{q/r} \int_{Q_{k,j}} u(x)^q dx \Big)^{1/q}.
\end{aligned}
$$

The next is similar to [4, Theorem 2.8], we omit it. □

Proof of Theorem 1.3 : For $u^q \in A_\infty$ and exponents $0 < \delta < \varepsilon < \min\{1, \frac{rn}{nN - r\alpha}\}$, applying Fefferman-Stein's inequality, Lemma 3.3, and Lemma 3.4 we have,

$$
\begin{aligned}
\| T_\alpha(\mathbf{f}) \|_{L^q(u^q)} &\leq \| M_\delta(T_\alpha(\mathbf{f})) \|_{L^q(u^q)} \\
&\leq C \| M^\sharp_\delta(T_\alpha(\mathbf{f})) \|_{L^q(u^q)} \\
&\leq C \| \mathcal{M}_{\alpha,r}(\mathbf{f}) \|_{L^q(u^q)} \\
&\leq C \prod_{i=1}^N \| f_i \|_{L^{p_i}(\omega_i^{p_i})} .
\end{aligned}
$$

□

4. Two Weighted Inequalities on Banach function spaces

Let X be a Banach function space on $\mathbb{R}^n$ with respect to Lebesgue measure. We refer the readers to [12] for more details about Banach function spaces. Given a Banach function space X there is an associate Banach function space X' for which the generalized Hölder inequality

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \| f \|_X \| g \|_{X'}$$

holds. Lebesgue spaces, Lorentz spaces and Orlicz spaces are some examples of Banach function spaces. The Orlicz $L^B(\mathbb{R}^n)$ is defined by a Young function B (the detail to see [13]).

In [4], if $Y_1, \ldots, Y_N$ are Banach function spaces the author defines the multi(sub)-linear maximal function to be

$$\mathcal{M}_{\vec{Y}} \vec{f}(x) = \sup_{Q \ni x} \prod_{i=1}^{N} \| f_i \|_{Y_i,Q} .$$

The X average of f over Q as in [12] is

$$\| f \|_{X,Q} = \| \delta_{\ell(Q)}(f\chi_Q) \|_X,$$

where for $a > 0$, $\delta_a f(x) = f(ax)$. The definition of maximal operator associated to the Banach function space X is

$$M_X f(x) = \sup_{Q \ni x} \| f \|_{X,Q} .$$

We denote the M_X by M_B when X is the Orlicz space L^B. The following lemmas are also similar to Lemma 2.4, they can be easily gotten from corresponding [4, Theorem 6.4 and Theorem 6.3].

Lemma 4.1. *Suppose* $0 < \alpha < nN$, $1 \leqslant r < p_1, \ldots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, $\frac{r}{N} < p \leqslant q < \infty$, *and* $Y_1, \ldots, Y_N$ *are translation invariant Banach function spaces with*

$$\mathcal{M}_{\vec{Y}'} : L^{\frac{p_1}{r}}(\mathbb{R}^n) \times \cdots \times L^{\frac{p_N}{r}}(\mathbb{R}^n) \to L^{\frac{p}{r}}(\mathbb{R}^n).$$

If $(u, \mathbf{w})$ *are weights that satisfy*

$$\sup_Q |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \left(\frac{1}{|Q|} \int_Q u(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^{N} \| v_i^{-r} \|_{Y_i,Q}^{\frac{1}{r}} < \infty,$$

then $\mathcal{M}_{\alpha,r}(\mathbf{f})$ *is bounded from* $L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N})$ *to* $L^q(u^q)$.

Lemma 4.2. *Suppose* $0 < \alpha < nN$, $1 \leqslant r < p_1, \ldots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, $\frac{r}{N} < p \leqslant q < \infty$, *and* $\Phi_1, \ldots, \Phi_N$ *are Young functions that satisfy*

$$\int_c^\infty \frac{\Psi_i(t)}{t^{\frac{p_i}{p_i - r}}} \frac{dt}{t} < \infty, \ \ i = 1, \ldots, N$$

for some $c > 0$. *If* $(u, \mathbf{w})$ *are weights that satisfy*

$$\sup_Q |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \left(\frac{1}{|Q|} \int_Q u(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^N \| v_i^{-r} \|_{\Psi_i, Q}^{\frac{1}{r}} < \infty$$

then $\mathcal{M}_{\alpha,r}(\mathbf{f})$ *is bounded from* $L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N})$ *to* $L^q(u^q)$.

We can get two generalized versions of Theorem 1.3 from Lemma 4.1 and Lemma 4.2 additive $u^q \in A_\infty$.

Theorem 4.1. *Let* $1 \leqslant r < p_1, \ldots, p_N < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_N}$, $p \leqslant q$, $1 \leq r < \min(p_1, \ldots, p_N, \frac{nN}{\alpha}, Np)$ *and* $\frac{1}{r^*} = \frac{N}{r} - \frac{\alpha}{n}$. *If*

$$T_\alpha : L^r \times \cdots \times L^r \to L^{r^*, \infty}$$

and satisfies the multilinear $L^{r', \alpha}$*-Hörmander contidion, and further suppose that* $Y_1, \ldots, Y_N$ *are translation invariant Banach function spaces with*

$$\mathcal{M}_{\vec{Y}'} : L^{\frac{p_1}{r}}(\mathbb{R}^n) \times \cdots \times L^{\frac{p_N}{r}}(\mathbb{R}^n) \to L^{\frac{p}{r}}(\mathbb{R}^n).$$

And $(u, \mathbf{w})$ *are weights that satisfy*

$$\sup_Q |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \left(\frac{1}{|Q|} \int_Q u(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^N \| v_i^{-r} \|_{Y_i, Q}^{\frac{1}{r}} < \infty.$$

Then

$$T_\alpha : L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N}) \to L^q(u^q)$$

for $u^q \in A_\infty$.

Theorem 4.2. *Let* $1 \leqslant r < p_1, \ldots, p_N < \infty$, $1/p = 1/p_1 + \cdots + 1/p_N$, $p \leqslant q$, $1 \leq r < \min(p_1, \ldots, p_N, nN/\alpha, Np)$ *and* $1/r^* = N/r - \alpha/n$. *If*

$$T_\alpha : L^r \times \cdots \times L^r \to L^{r^*,\infty}$$

and satisfies the multilinear $L^{r',\alpha}$*-Hörmander contidion, and further suppose that* $\Phi_1, \ldots, \Phi_N$ *are Young functions that satisfy*

$$\int_c^\infty \frac{\Psi_i(t)}{t^{\frac{p_i}{p_i - r}}} \frac{dt}{t} < \infty, \ \ i = 1, \ldots, N.$$

for some $c > 0$. *And* $(u, \mathbf{w})$ *are weights that satisfy*

$$\sup_Q |Q|^{\frac{1}{q} - \frac{1}{p} + \frac{\alpha}{n}} \left(\frac{1}{|Q|} \int_Q u(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^N \| v_i^{-r} \|_{Y_i,Q}^{\frac{1}{r}} < \infty.$$

Then

$$T_\alpha : L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_N}(\omega_N^{p_N}) \to L^q(u^q)$$

for $u^q \in A_\infty$.

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