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2-ABSORBING PRIMARY IDEALS OF SO-RINGS

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Abstract. A partial semiring is a structure possessing an infinitary partial addition and a binary multiplication, subject to a set of axioms. The partial functions under disjoint-domain sums and functional composition is a partial semiring. In this paper we obtain equivalent conditions and some characteristics of 2-absorbing primary ideals in so-rings.

1. Introduction

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming language semantics. The general cardinal algebras studied by Tarski in 1949, Housdorff topological commutative groups studied by Bourbaki in 1966, Σ-structures studied by Higgs in 1980, sum ordered partial monoids and sum ordered partial semirings (so-rings) studied by Arbib, Manes and Benson[2], [4], and Streenstrup[13] are some of the algebraic structures of the above type.

In [7], we studied some characteristics of 2-absorbing ideals in so-rings. In this paper, we consider the 2-absorbing primary ideals of so-rings and obtain various equivalent conditions of it. Also we obtain some characterizations of $\sqrt{I}$ in the 2-absorbing primary ideals of so-rings.

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2. Preliminaries

In this section we collect some important definitions, results and examples for our use in this paper.

Let M be a nonempty set, and let I be a set. An $I-indexed\ family$ in M is a function $x : I \to M$. Such a family is denoted by $(x_i : i \in I)$, where $x_i = ix$ for each i in I. The $empty\ family$ is the unique such family indexed by $\emptyset$.

Now let us consider an infinitary operation Σ which takes families in M to elements of M, but which may not be defined for all families in M. By "infinitary", we mean that Σ may be applied to a family $(x_i : i \in I)$ in M, for which the cardinality of the index set I is infinite. Since $\Sigma(x_i : i \in I)$ need not be defined for an arbitrary family $(x_i : i \in I)$ in M, Σ is said to be *partially-defined.* A family $(x_i : i \in I)$ in M is said to be *summable* if $\Sigma(x_i : i \in I)$ is defined and is in M.

Definition 2.1. [4] *A positive partial monoid or partial monoid, for short, is a pair (M, Σ) where M is a nonempty set and Σ is a partial addition defined on some, but not necessarily all, families $(x_i : i \in I)$ in M subject to the following axioms:*
(1) Unary Sum Axiom. If $(x_i : i \in I)$ is a one element family in M and $I = \{j\}$, then $\Sigma(x_i : i \in I)$ is defined and equals x_j.
(2) Partition-Associativity Axiom. If $(x_i : i \in I)$ is a family in M and $(I_j : j \in J)$ is a partition of I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J and $(\Sigma(x_i : i \in I_j) : j \in J)$ is summable. We write
$\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j) : j \in J).$

Definition 2.2. [4] *A partial semiring is a quadruple $(R, \Sigma, \cdot, 1)$, where (R, Σ) is a partial monoid, $(R, \cdot, 1)$ is a monoid with multiplicative operation '$\cdot$' and unit 1, and the additive and multiplicative structures obey the following distributive laws: If $\Sigma(x_i : i \in I)$ is defined in R, then for all y in R, $\Sigma(y \cdot x_i : i \in I)$ and $\Sigma(x_i \cdot y : i \in I)$*

are defined and

$y \cdot \Sigma(x_i : i \in I) = \Sigma(y \cdot x_i : i \in I)$, $\Sigma(x_i : i \in I) \cdot y = \Sigma(x_i \cdot y : i \in I)$.

Definition 2.3. [4] *A partial semiring* $(R, \Sigma, \cdot, 1)$ *is said to be commutative, if* $xy = yx \ \forall \ x, \ y \in R$.

Definition 2.4. [4] *The sum ordering on a partial monoid* (M, Σ) *is the binary relation* $\leq$ *such that* $x \leq y$ *if and only if there exists a* h *in* M *such that* $y = x + h$, *for* $x, \ y \in M$.

The binary relation sum ordering on a partial monoid (M, Σ) is always reflexive and transitive.

Definition 2.5. [4] *A sum ordered partial semiring or so-ring, for short, is a partial semiring in which the sum ordering is a partial order.*

Let X and Y be two sets. A *partial function* f from X to Y is a function f from X' to Y, for some subset X' of X.

Example 2.6. [4] *Let* D *be a set and let the set of all partial functions from* D *to* D *be denoted by* $Pfn(D, D)$. *A family* $(x_i : i \in I)$ *is summable if and only if for* i, j *in* I, *and* $i \neq j$, $dom(x_i) \cap dom(x_j) = \emptyset$. *If* $(x_i : i \in I)$ *is summable then for any* d *in* D

$$d(\Sigma_i x_i) = \begin{cases} dx_i, \ if \ d \in dom(x_i) \ for \ some \ (necessarily \ unique) \ i \in I; \\ undefined, \ otherwise \end{cases}$$

and '$\cdot$' is defined as the usual functional composition. That is, for any $x, \ y \in Pfn(D, D)$ *and* d *in* D,

$$d(x \cdot y) = \begin{cases} (dx)y, \ if \ d \in dom(x) \ and \ dx \in dom(y); \\ undefined, \ otherwise \end{cases}$$

and the ordering as the extension of functions. Then $(Pfn(D, D),\ \Sigma,\ \cdot,\ 1)$*, where* 1 *is the identity function on* D*, is a so-ring.*

Example 2.7. [4] *Let* D *be a set. A multi-function* $x:\ D \rightarrow D$ *maps each element in* D *to an arbitrary subset of* D*. Such multi-functions correspond bijectively to relations* $r \subseteq D \times D$*, where* $(d,\ e) \in r$ *if and only if* $e \in dx$*. The set of all multi-functions from* D *to* D*, denoted by* $Mfn(D,\ D)$*, together with* Σ *defined such that* d *in* D*,* $d(\Sigma_i x_i) = \bigcup_i (dx_i)$*, and '$\cdot$' defined as the usual relational composition. That is, for each* d *in* D *and for* x, y *in* $Mfn(D, D)$*,* $d(x \cdot y) = \bigcup(ey : e \in dx)$*, and* $d1 = \{d\}$*. Then* $(Mfn(D, D),\ \Sigma,\ \cdot,\ 1)$ *is a so-ring.*

Definition 2.8. [1] *Let* R *be a so-ring. A subset* N *of* R *is said to be an ideal of* R *if the following are satisified*

(I1). If $(x_i : i \in I)$ *is a summable family in* R *and* $x_i \in N\ \forall\ i \in I$ *then* $\Sigma(x_i : i \in I) \in N$*,*

(I2). If $x \leq y$ *and* $y \in N$ *then* $x \in N$*,*

(I3). If $x \in N$ *and* $r \in R$ *then* $xr, rx \in N$*.*

A So-ring R is said to be *complete* if every family in R is summable.

Definition 2.9. [8] *Let* R *be a complete so-ring and* $a \in R$*. Then the smallest ideal generated by 'a' is* $< a > = \{x \in R \mid x \leq \Sigma_n a + \Sigma_i r_i a s_i,$ *where* $r_i,\ s_i \in R$ *and* n *is a positive integer*$\}$*. We call* $< a >$ *as the principal ideal generated by* a*.*

Definition 2.10. [8] *Let* I *and* J *be ideals of a so-ring* R*. Then* $IJ = \{x \in R \mid x \leq \Sigma_i a_i b_i\ for\ some\ a_i \in I,\ b_i \in J\}$*.*

Definition 2.11. [8] *Let* A *and* B *be any subsets of a so-ring* R*. Then we define* $(A : B) = \{r \in R \mid rB \subseteq A\}$ *where* $rB = \{x \in R \mid x \leq rb\ for\ some\ b \in B\}$*.*

Definition 2.12. [8] *Let* $(R, \Sigma, \cdot, 1)$ *and* $(R', \Sigma', *, 1)$ *be two so-rings. Then a mapping* $f : R \rightarrow R'$ *is said to be homomorphism if it satisfies the following:*
(i). Whenever $(x_i : i \in I)$ *is summable in* R*, then* $(f(x_i) : i \in I)$ *is summable in* R' *and* $f(\Sigma x_i) = \Sigma' f(x_i)$*. (ii).* $f(x \cdot y) = f(x) * f(y)$ *for every* x, y *in* R*.*

Definition 2.13. [13] *A proper ideal* P *of a so-ring* R *is said to be prime if and only if for any ideals* A, B *of* R, $AB \subset P$ *implies* $A \subset P$ *or* $B \subset P$*.*

Definition 2.14. [12] *A proper ideal* I *of a so-ring* R *is called 2-absorbing if for any* $a, b, c \in R$, $abc \in I$ *implies* $ab \in I$ *or* $bc \in I$ *or* $ac \in I$*.*

The set of all prime ideals of a so-ring R is denoted by $spec(R)$. Let I be a proper ideal of R. Then denote the set $\{H \in spec(R) \mid I \subset H\}$ by $V(I)$ and $\bigcap V(I)$ by $\sqrt{I}$.

Theorem 2.15. [8] *Let* I *be an ideal of a commutative so-ring* R*. Then* $\sqrt{I} = \{x \in R \mid x^n \in I \ for\ some\ positive\ integer\ n\}$*.*

Definition 2.16. [8] *A proper ideal* I *of a so-ring* R *is said to be primary if* $ab \in I$, $a, b \in R$ *implies* $a \in I$ *or* $b^n \in I$ *for some* $n \in Z^+$*.*

Throughout this paper, R denotes a commutative so-ring.

3. 2-absorbing primary ideals

Following the notion of 2-absorbing primary ideals in [6], we define 2-absorbing primary ideals in so-rings as follows.

Definition 3.1. *Let* I *be a proper ideal of a so-ring* R*. Then* I *is said to be a 2-absorbing primary ideal of* R *if whenever* $a, b, c \in R$ *and* $abc \in I$*, then* $ab \in I$ *or* $bc \in \sqrt{I}$ *or* $ac \in \sqrt{I}$*.*

Proposition 3.2. *Every 2-absorbing ideal of a so-ring* R *is a 2-absorbing primary ideal of* R*.*

Proof. Let $abc \in I$ for some $a, b, c \in R$. Since I is a 2-absorbing ideal of R, then $ab \in I$ or $bc \in I$ or $ac \in I$. Then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$ (Since $I \subseteq \sqrt{I}$). Hence I is a 2-absorbing primary ideal of R. □

The following is an example of a so-ring R in which the converse need not be true.

Example 3.3. *Consider* $R = \mathbb{N}\bigcup\{0\}$ *be a so-ring and* $I = < 8\mathbb{N} >$ *be an ideal of* R. *As in* [6], *it is known that not every 2-absorbing primary ideals of semi-rings is a 2-absorbing ideal. Clearly* I *is a 2-absorbing primary ideal of* R. *Since* $2\cdot2\cdot2 \in < 8 >$, *but* $2 \cdot 2 \notin < 8 >$. *Hence* I *is not a 2-absorbing ideal of* R.

Proposition 3.4. *Every primary ideal of a so-ring* R *is a 2-absorbing primary ideal of* R.

Proof. Let I be a primary ideal of a so-ring R. Suppose $abc \in I$ for some $a, b, c \in R$. Since I is a primary ideal of R, either $a \in I$ or $(bc)^n \in I$ for some positive integer n. Then $ab \in I$ or $bc \in \sqrt{I}$. Hence I is a 2-absorbing primary ideal of R. □

The following is an example of a so-ring R in which the converse need not be true.

Example 3.5. *Define* $R = \{0, u, v, x, y, 1\}$ *with* Σ *defined on* R *by*

$$\Sigma(x_i : i \in I) = \begin{cases} x_j, \ if \ x_i = 0 \ \forall \ i \neq j, \ for \ some \ j, \\ undefined, \ otherwise \end{cases}$$

and $\cdot$ defined by the following table:

.	0	u	v	x	y	1
0	0	0	0	0	0	0
u	0	u	0	0	0	u
v	0	0	v	0	0	v
x	0	0	0	0	0	x
y	0	0	0	0	0	y
1	0	u	v	x	y	1

As in [12]*, clearly R is a so-ring. Let $I = \{0, x\}$ be an ideal of R. One can show that $\{0, x\}$ is a 2-absorbing primary ideal of R. Now $uv = 0 \in I$, but $u \notin I$ and $v^n = v \notin I$ for some $n \in Z^+$. Hence I is not a primary ideal of R.*

Theorem 3.6. *If I is a 2-absorbing primary ideal of a so-ring R, then $\sqrt{I}$ is a 2-absorbing ideal of R.*

Proof. Suppose I is a 2-absorbing primary ideal of R. Let $abc \in \sqrt{I}$ for some $a, b, c \in R$. Then $(abc)^n \in I$ for some positive integer n. We have $a^n b^n c^n \in I$ for some positive integer n. Since I is a 2-absorbing primary ideal of R, then $a^n b^n \in I$ or $b^n c^n \in \sqrt{I}$ or $a^n c^n \in \sqrt{I}$. We have $(ab)^n \in I$ or $(bc)^n \in \sqrt{I}$ or $(ac)^n \in \sqrt{I}$. Then $ab \in \sqrt{I}$ or $bc \in \sqrt{\sqrt{I}}$ or $ac \in \sqrt{\sqrt{I}}$. We have $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$ (as in [11], $\sqrt{\sqrt{I}}$ $= \sqrt{I}$). Hence $\sqrt{I}$ is a 2-absorbing ideal of R. □

Theorem 3.7. *Let $f : R \to R'$ be a homomorphism of so-rings, and suppose that I' is a 2-absorbing primary ideal of R'. Then $f^{-1}(I')$ is a 2-absorbing primary ideal of R.*

Proof. Suppose I' is a 2-absorbing primary ideal of R'. Let $abc \in f^{-1}(I')$ for some $a, b, c \in R$. Then $f(abc) \in I'$. i.e., $f(a)f(b)f(c) \in I'$. Since I' is a 2-absorbing primary ideal of R', then $f(a)f(b) \in I'$ or $f(b)f(c) \in \sqrt{I'}$ or $f(a)f(c) \in \sqrt{I'}$. Then $f(ab) \in I'$ or $f(bc) \in (\sqrt{I'})$ or $f(ac) \in (\sqrt{I'})$. We have $ab \in f^{-1}(I')$ or $bc \in f^{-1}(\sqrt{I'})$ or $ac \in f^{-1}(\sqrt{I'}))$. Since $f^{-1}(\sqrt{I'}) \subseteq \sqrt{f^{-1}(I')}$(as in [6]), $ab \in f^{-1}(I')$ or $bc \in \sqrt{f^{-1}(I')}$ or $ac \in \sqrt{f^{-1}(I')}$. Hence $f^{-1}(I')$ is a 2-absorbing primary ideal of R. □

Theorem 3.8. *Let $f : R \to R'$ be a homomorphism of so-rings. If f is an epimorphism and I is a 2-absorbing primary ideal of R then $f(I)$ is a 2-absorbing primary ideal of R'.*

Proof. Let $a', b', c' \in R$ such that $a'b'c' \in f(I)$. Since $f : R \to R'$ is an epimorphism, then there exist $a, b, c \in R$ such that $f(a) = a'$, $f(b) = b'$, and $f(c) = c'$. Now $f(abc)$ $= f(a)f(b)f(c) = a'b'c' \in f(I)$. Then $abc \in I$. Since I is a 2-absorbing primary ideal of R, $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. We have $f(ab) \in f(I)$ or $f(bc) \in f(\sqrt{I})$ or $f(ac) \in f(\sqrt{I})$. Then $f(a)f(b) \in f(I)$ or $f(b)f(c) \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$ or $f(a)f(c) \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$(as in [6]). We have $a'b' \in f(I)$ or $b'c' \in \sqrt{f(I)}$ or $a'c' \in \sqrt{f(I)}$. Hence $f(I)$ is a 2-absorbing primary ideal of R'. □

Theorem 3.9. *If $\sqrt{I}$ is a 2-absorbing ideal of R and if $abc \in I$ with $bc \notin \sqrt{I}$ and $ac \notin \sqrt{I}$. Then I is a 2-absorbing primary ideal of R.*

Proof. Suppose $\sqrt{I}$ is a 2-absorbing ideal of R and if $abc \in I$ with $bc \notin \sqrt{I}$ and $ac \notin \sqrt{I}$ then $ab \in I$. Let $a, b, c \in R$ such that $abc \in I \subseteq \sqrt{I}$. Then $abc \in \sqrt{I}$. Since $\sqrt{I}$ is a 2-absorbing ideal of R, $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Suppose $bc \notin \sqrt{I}$ and $ac \notin \sqrt{I}$, then $ab \in I$. Hence I is a 2-absorbing primary ideal of R. □

Theorem 3.10. *If I is a 2-absorbing primary ideal of a so-ring R, then the following statements hold:*
(i) $(\sqrt{I} : x)$ *is a 2-absorbing ideal of R for all $x \in R \setminus \sqrt{I}$*
(ii) $(\sqrt{I} : x) = (\sqrt{I} : x^2)$ *for all $x \in R \setminus \sqrt{I}$.*

Proof. As in [11], clearly $\sqrt{I}$ and $(\sqrt{I} : x)$ are ideals of R. Let $abc \in (\sqrt{I} : x)$ for some $a, b, c \in R$ and $x \in R \setminus \sqrt{I}$, then $(abc)x \in \sqrt{I}$. Since I is a 2-absorbing primary ideal, $\sqrt{I}$ is a 2-absorbing ideal of R (since by theorem 2.6.). Then $(ab)c \in \sqrt{I}$ or $(ab)x \in \sqrt{I}$ or $cx \in \sqrt{I}$. If $(ab)c \in \sqrt{I}$ then $ab \in \sqrt{I}$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Then $(ab)x \in \sqrt{I}$ or $(bc)x \in \sqrt{I}$ or $(ac)x \in \sqrt{I}$. We have $ab \in (\sqrt{I} : x)$ or $bc \in (\sqrt{I} : x)$ or $ac \in (\sqrt{I} : x)$. If $(ab)x \in \sqrt{I}$ then $ab \in (\sqrt{I} : x)$. If $cx \in \sqrt{I}$ then $a(cx) \in \sqrt{I}$. Then $ac \in (\sqrt{I} : x)$. Therefore $(\sqrt{I} : x)$ is a 2-absorbing ideal of R.
(ii) Let $x \in R \setminus \sqrt{I}$. It is clear that $(\sqrt{I} : x) \subseteq (\sqrt{I} : x^2)$. Now let $y \in (\sqrt{I} : x^2)$, $yx^2 \in \sqrt{I}$. Since $\sqrt{I}$ is a 2-absorbing ideal of R, either $x^2 \in \sqrt{I}$ or $yx \in \sqrt{I}$. If $x^2 \in \sqrt{I}$ then $x \in \sqrt{I}$, a contradiction. So let $yx \in \sqrt{I}$, $y \in (\sqrt{I} : x)$. Therefore $(\sqrt{I} : x^2) \subseteq (\sqrt{I} : x)$. Hence $(\sqrt{I} : x) = (\sqrt{I} : x^2)$. □

Theorem 3.11. *Let I be a proper ideal of a so-ring R. Then I is a 2-absorbing primary ideal of R if and only if for any $a, b \in R$ and any ideal J of R with $abJ \subseteq I$ and $ab \notin I$ then either $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$.*

Proof. Suppose I is a 2-absorbing primary ideal of R. Let J be an ideal of R and $a, b \in R$ such that $abJ \subseteq I$ and $ab \notin I$. Suppose $aJ \not\subseteq \sqrt{I}$ and $bJ \not\subseteq \sqrt{I}$. Then there exist $x, y \in J$ such that $ax \notin \sqrt{I}$ and $by \notin \sqrt{I}$. Since $abJ \subseteq I$, $abx \in I$ and $aby \in I$. Then $abx + aby \in I$. We have $ab(x + y) \in I$. Since $ab(x + y) \in I$, $ab \notin I$ and I is a 2-absorbing primary ideal of R, $a(x + y) \in \sqrt{I}$ or $b(x + y) \in \sqrt{I}$. Since $ax \leq a(x + y) \in \sqrt{I}$ and $by \leq b(x + y) \in \sqrt{I}$, we have $ax \in \sqrt{I}$ or $by \in \sqrt{I}$. Hence either $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$.

Conversly, we assume that for any $a, b \in R$ and for any ideal J of R, $abJ \subseteq I$ and $ab \notin I$ then either $aJ \subseteq \sqrt{I}$ or $bJ \subseteq \sqrt{I}$. Let $x, y, z \in R$ such that $xyz \in I$ and $xy \notin I$. Take $K = < z >$, the principal ideal of R generated by z. Let $p \in xyK = xy < z >$. Then $p \leq xy[\Sigma_n z + \Sigma_i z r_i]$ for some $r_i \in R$. We have $p \leq \Sigma_n xyz + \Sigma_i (xyz) r_i$ for some $r_i \in R$. Since $xyz \in I$, $\Sigma_n xyz$ and $\Sigma_i (xyz) r_i \in I$. Then $p \in I$. Therefore $xyK \subseteq I$

and $xy \notin I$. By assumption, $xK \subseteq \sqrt{I}$ or $yK \subseteq \sqrt{I}$. Since $z \in< z >= K$, $xz \in \sqrt{I}$ or $yz \in \sqrt{I}$. Hence I is a $2-$absorbing primary ideal of R. □

Theorem 3.12. *Let I be a proper ideal of a so-ring R. Then I is a 2-absorbing primary ideal of R if and only if for any $a \in R$, for any ideals J, K of R, $aJK \subseteq I$ and $JK \nsubseteq I$ then $aJ \subseteq \sqrt{I}$ or $aK \subseteq \sqrt{I}$.*

Proof. Suppose I is a 2-absorbing primary ideal of R. Let $a \in R$ and J, K be any ideals of R such that $aJK \subseteq I$ and $JK \nsubseteq I$. Since $JK \nsubseteq I$, there exist $p \in JK$ such that $p \notin I$. Since $p \in JK$, $p \leq \Sigma_i x_i y_i$ for some $x_i \in J$, $y_j \in K$. Since $p \notin I$, $\Sigma_i x_i y_i \notin I$. Then $x_k y_k \notin I$ for some $x_k \in J$, $y_k \in K$. Now we prove that either $aJ \subseteq \sqrt{I}$ or $aK \subseteq \sqrt{I}$. Suppose $aJ \not\subset \sqrt{I}$ and $aK \not\subset \sqrt{I}$. Then $\exists\, x \in J$ *and* $y \in K$ *such that* $ax \notin \sqrt{I}$ *and* $ay \notin \sqrt{I}$. Since $x, x_k \in J$ and $y, y_k \in K$, we have $x + x_k \in J$ and $y + y_k \in K$. Since $aJK \subseteq I$, $a(x + x_k)(y + y_k) \in I$. Since I is a 2-absorbing primary ideal of R, $a(x + x_k)(y + y_k) \in I$,$(x + x_k)(y + y_k) \in I$ or $a(x+x_k) \in \sqrt{I}$ or $a(y+y_k) \in \sqrt{I}$. Then $xy+xy_k+x_k y+x_k y_k \in I$ or $ax+ax_k \in \sqrt{I}$ or $ay + ay_k \in \sqrt{I}$. We have $x_k y_k \in I$ or $ax \in \sqrt{I}$ or $ay \in \sqrt{I}$, a contradiction. Hence either $aJ \subseteq \sqrt{I}$ or $aK \subseteq \sqrt{I}$.

Conversly, we assume that for any $a \in R$, for any ideals J, K of R, $aJK \subseteq I$ and $JK \nsubseteq I$ we have $aJ \subseteq \sqrt{I}$ or $aK \subseteq \sqrt{I}$. Let $x, y, z \in R$ such that $xyz \in I$ and $yz \notin I$. Take $J =< y >$, $K =< z >$, the principal ideal of R generated by y, z respectively. First we prove that $xJK \subseteq I$. Let $p \in xJK$. Then $p \in x < y >< z >$. We have $p \leq x(\Sigma_n y + \Sigma_i y r_i)(\Sigma_m z + \Sigma_j z s_j)$ for some $r_i,\ s_j \in R$, positive integers n, m. Then $p \leq \Sigma_n \Sigma_m xyz + \Sigma_n \Sigma_j (xyz)s_j + \Sigma_i \Sigma_m (xyz)r_i + \Sigma_i \Sigma_j (xyz)(r_i s_j)$. Since $xyz \in I$, all terms on right side are in I. Then $p \in I$. Therefore $xJK \subseteq I$. Since $y \in< y >= J$, $z \in< z >= K$ and $yz \notin I$, $JK \nsubseteq I$. By assumption either $xJ \subseteq \sqrt{I}$ or $xK \subseteq \sqrt{I}$. We have $xy \in \sqrt{I}$ or $xz \in \sqrt{I}$. Hence I is a 2-absorbing primary ideal of R. □

Now we generalise the definition of 2-absorbing primary ideals of so-rings in terms of ideals in the following theorem:

Theorem 3.13. *Let P be a proper ideal of a so-ring R. Then P is a 2-absorbing primary ideal of R if and only if whenever $I_1I_2I_3 \subseteq P$ for some ideals I_1, I_2, I_3 of R then $I_1I_2 \subseteq P$ or $I_2I_3 \subseteq \sqrt{P}$ or $I_1I_3 \subseteq \sqrt{P}$.*

Proof. Suppose P is a 2-absorbing primary ideal of R. Let I_1, I_2, I_3 be ideals of R such that $I_1I_2I_3 \subseteq P$. Suppose $I_1I_2 \nsubseteq P$, $I_2I_3 \nsubseteq \sqrt{P}$ and $I_1I_3 \nsubseteq \sqrt{P}$. Then $\exists\ p \in I_1I_2$, $q \in I_2I_3$, $r \in I_1I_3$ such that $p \notin P$, q, $r \notin \sqrt{P}$. Now $p \in I_1I_2$. Then $p \leq \Sigma_i x_i y_i$ for some $x_i \in I_1,\ y_i \in I_2,\ i \in I$. Since $p \notin P$, $x_k y_k \notin P$ for some $x_k \in I_1$, $y_k \in I_2$. Now $q \in I_2I_3$. Then $q \leq \Sigma_j y_j z_j$ for some $y_j \in I_2,\ z_j \in I_3,\ j \in I$. Since $q \notin \sqrt{P}$, $q \notin P$ (since $P \subseteq \sqrt{P}$). i.e., $y_l z_l \notin P$ for some $y_l \in I_2$, $z_l \in I_3$. Now $r \in I_1I_3$. Then $r \leq \Sigma_s x_s z_s$ for some $x_s \in I_1,\ z_s \in I_3,\ s \in I$. Since $r \notin \sqrt{P}$, $r \notin P$. i.e., $x_m z_m \notin P$ for some $x_m \in I_1,\ z_m \in I_3$. Now $x_k + x_m \in I_1,\ y_k + y_l \in I_2,\ z_l + z_m \in I_3$. Then $(x_k + x_m)(y_k + y_l)(z_l + z_m) \in I_1I_2I_3 \subseteq P$. Since P is a 2-absorbing primary ideal of R, $(x_k + x_m)(y_k + y_l) \in P$ or $(y_k + y_l)(z_l + z_m) \in \sqrt{P}$ or $(x_k + x_m)(z_l + z_m) \in \sqrt{P}$. Since P is an ideal of R, $x_k y_k \in P$ or $y_l z_l \in \sqrt{P}$ or $x_m z_m \in \sqrt{P}$, a contradiction. Hence $I_1I_2 \subseteq P$ or $I_2I_3 \subseteq \sqrt{P}$ or $I_1I_3 \subseteq \sqrt{P}$.

Conversly, we assume that for any ideals I_1, I_2, I_3 of R, $I_1I_2I_3 \subseteq P$ then either $I_1I_2 \subseteq P$ or $I_2I_3 \subseteq \sqrt{P}$ or $I_1I_3 \subseteq \sqrt{P}$. Let $x, y, z \in R$ such that $xyz \in I$. Take $I_1 =< x >$, $I_2 =< y >$, $I_3 =< z >$, the principal ideals of R generated by x, y, z respectively. First we prove that $I_1I_2I_3 \subseteq P$. Let $p \in I_1I_2I_3 = < x >< y >< z >$. Then $p \leq (\Sigma_n x + \Sigma_i x r_i)(\Sigma_m y + \Sigma_j y s_j)(\Sigma_l z + \Sigma_k z s_k^1)$ for some $r_i, s_j, s_k^1 \in R$ and some positive integers n, m, l. Then $p \leq \Sigma_n\Sigma_m\Sigma_l(xyz) + \Sigma_n\Sigma_j\Sigma_l(xyz)s_j + \Sigma_n\Sigma_m\Sigma_k(xyz)s_k^1 + \Sigma_i\Sigma_m\Sigma_l(xyz)r_i + \Sigma_i\Sigma_j\Sigma_l(xyz)(r_i s_j) + \Sigma_i\Sigma_j\Sigma_k(xyz)(r_i s_j s_k^1) + \Sigma_n\Sigma_j\Sigma_k(xyz)(s_j s_k^1) + \Sigma_i\Sigma_m\Sigma_k(xyz)(r_i s_k^1)$ (since R is commutative). Since $xyz \in P$, all terms on right side are in P. We have $p \in P$. Therefore $I_1I_2I_3 \subseteq P$. By assumption, $I_1I_2 \subseteq P$ or

$I_2I_3 \subseteq \sqrt{P}$ or $I_1I_3 \subseteq \sqrt{P}$. Since $x \in < x > = I_1$, $y \in < y > = I_2$, $z \in < z > = I_3$, $xy \in P$ or $yz \in \sqrt{P}$ or $xz \in \sqrt{P}$. Hence P is a 2-absorbing primary ideal of R. $\square$

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