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FUZZY SOFT DIFFERENTIATION IN FUZZY SOFT TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper a concept of fuzzy soft topological vector space has been introduced and neighborhood system of fuzzy soft points are studied. It has also been possible to extend the idea of differentiation of mappings over fuzzy soft topological vector spaces.

1. Introduction

Fuzzy set theory, which was first initiated by Zadeh [24] in 1965, has become a very important tool to solve many complicated problems arising in the fields of economics, social sciences, engineering, medical sciences and so forth, involving uncertainties and provides an appropriate framework for representing vague concepts by allowing partial membership. Many researchers have worked on theoretical aspects and applications of fuzzy set theory over years, such as fuzzy control systems, fuzzy logic, fuzzy automata, fuzzy topology, fuzzy topological groups, fuzzy topological vector spaces, fuzzy differentiation etc. [3, 6, 7, 9, 11, 12, 15, 23]. In 1999, Molodtsov [16] introduced the concept of soft set theory which is a different approach for modeling uncertainty and successfully applied the soft set theory in several directions such as smoothness of functions, operation research, Riemann integration, game theory,

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theory of probability and so on. Maji et al. [13] defined and studied several basic notions of soft set theory in 2003. Afterwords, several mathematical structures such as algebraic, topological, algebraico-topological structures have been developed using soft sets [4, 5, 20]. The notion of fuzzy soft set was introduced by Maji et al. [14] as a fuzzy generalization of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Later Tanay et al. [21] initially gave the concept of fuzzy soft topological spaces and subsequently many authors worked on fuzzy soft topology [1, 19, 22]. Notions of group [17], ring [8], metric space [2] etc. are also extended in fuzzy soft setting. In 2014, Sk. Nazmul [18] introduced a notion of Lowen type fuzzy soft topology and studied fuzzy soft topological groups using this type of fuzzy soft topology. In view of this and also considering the importance of topological vector space in developing the theory of functional analysis, in this paper we introduce for the first time the notion of fuzzy soft topological vector spaces and develop some basic properties of such spaces. Further the concept of derivatives of mappings in linear topological spaces is instrumental in developing general differential geometry and general continuous group theory. In this paper, we introduce a concept of fuzzy soft topological vector space and study some of its basic properties. Also, here we initiate the study of the fuzzy soft differentiation of fuzzy soft continuous mapping in fuzzy soft topological vector spaces and discuss some properties of derivatives briefly.

2. Preliminaries

Unless otherwise stated, X will be assumed to be an initial universal set and A will be taken to be a set of parameters. Let $FP(X)$ denote the set of all fuzzy sets of X , $S(X)$ denote the set of all soft sets over X and $FS(X)$ denote the set of all fuzzy soft sets over X.

Definition 2.1. ([14]) A pair (F, A) is called a fuzzy soft set over X, where F is a mapping given by $F : A \to FP(X)$. In other words, a fuzzy soft set over X is a parametrized family of fuzzy subsets of the universe X.

Let (F_1, A) and (F_2, A) be two fuzzy soft sets over a common universe X. Then

(i) (F_1, A) is said to be fuzzy soft subset of (F_2, A) if $F_1(\alpha) \leq F_2(\alpha), \forall \alpha \in A$. This relation is denoted by $\tilde{\subseteq}$.

(ii) (F_1, A) is said to be fuzzy soft equal to (F_2, A) if $(F_1, A)\tilde{\subseteq}(F_2, A)$ and $(F_2, A)\tilde{\subseteq}(F_1, A)$.

(iii) The complement of a fuzzy soft set (F, A), denoted by (F^C, A), is defined by $F^C(\alpha) = (F(\alpha))^C =$ complement of the fuzzy subset $F(\alpha)$, $\forall \alpha \in A$.

(iv) A fuzzy soft set (F, A) over X is said to be a null fuzzy soft set if $F(\alpha) = \bar{0}$, $\forall \alpha \in A$, and an absolute fuzzy soft set if $F(\alpha) = \bar{1}$, $\forall \alpha \in A$, where $\bar{0}, \bar{1}$ are the null and absolute fuzzy subsets of X respectively. These are denoted by $(\tilde{\Phi}, A)$ and $(\tilde{X}, A)$ respectively.

Definition 2.2. ([14]) Let $\{(F_i, A), i \in \triangle\}$ be a non-empty family of fuzzy soft sets over a common universe X. Then their

(i) Intersection, denoted by $\underset{i\in\triangle}{\tilde{\cap}}$, is defined by $\underset{i\in\triangle}{\tilde{\cap}}(F_i, A) = (\underset{i\in\triangle}{\tilde{\cap}} F_i, A)$, where $(\underset{i\in\triangle}{\tilde{\cap}} F_i)(\alpha) = \underset{i\in\triangle}{\cap}(F_i(\alpha)), \forall \alpha \in A$.

(ii) Union, denoted by $\underset{i\in\triangle}{\tilde{\cup}}$, is defined by $\underset{i\in\triangle}{\tilde{\cup}}(F_i, A) = (\underset{i\in\triangle}{\tilde{\cup}} F_i, A)$, where $(\underset{i\in\triangle}{\tilde{\cup}} F_i)(\alpha) = \underset{i\in\triangle}{\cup}(F_i(\alpha)), \forall \alpha \in A$.

Definition 2.3. ([18]) A fuzzy soft set (F, A) over X is said to be constant fuzzy soft set if $F(\alpha) = \widetilde{c_X^{\alpha}}$, $\forall \alpha \in A$, where $\widetilde{c_X^{\alpha}}(x) = c_\alpha$, $\forall x \in X$ and $c_\alpha \in [0, 1]$. This is denoted by $(\widetilde{c_X}, A)$.

Definition 2.4. ([20]) Let T be a collection of soft sets over X. Then T is said to be a soft topology on X if

(i) the null soft set and the absolute soft set belong to T.

(ii) the intersection of any two soft sets in T belongs to T.

(iii) the union of any number of soft sets in T belongs to T.

The triplet (X, A, T) is called soft topological space over X. The members of T are called open soft sets.

Definition 2.5. ([18]) Let τ be a collection of fuzzy soft sets over X. Then τ is said to be a Lowen type fuzzy soft topology on X if

(i) $(\widetilde{c_X}, A) \in \tau$, where $\widetilde{c_X}(\alpha) = \widetilde{c_X^\alpha}$, $c_\alpha \in [0,1]$, $\forall \alpha \in A$.

(ii) the intersection of any two fuzzy soft sets in τ belongs to τ.

(iii) the union of any number of fuzzy soft sets in τ belongs to τ.

The triplet (X, A, τ) is called Lowen type fuzzy soft topological space over X. The members of τ are said to be Lowen type $\tau-$ fuzzy soft open sets or simply fuzzy soft open sets in X.

Definition 2.6. ([18]) In a fuzzy soft topological space (X, A, τ), a fuzzy soft (G, A) is said to be fuzzy soft closed if its complement is fuzzy soft open.

Proposition 2.1. *([18]) Let (X, A, τ) be a Lowen type fuzzy soft topological space over X. Then the collection $\tau^\alpha = \{F(\alpha) : (F, A) \in \tau\}$ for each $\alpha \in A$, defines a Lowen type fuzzy topology on X.*

Proposition 2.2. *([18]) If (X, A, τ) be a Lowen type fuzzy soft topological space and if $\tau^* = \{(G, A) \in FS(X) : G(\alpha) \in \tau^\alpha, \forall \alpha \in A\}$, then τ^* is a Lowen type fuzzy soft topology on X such that $[\tau^*]^\alpha = \tau^\alpha, \forall \alpha \in A$.*

Definition 2.7. ([18]) Let (X, A, τ) be a fuzzy soft topological space. A sub-collection $\mathcal{B}$ of τ is said to be an open base of τ if every member of τ can be expressed as the union of some members of $\mathcal{B}$.

Definition 2.8. ([18])Let X and Y be two non-empty sets and $f : X \to Y$ be a mapping. Then

(i) the image of a fuzzy soft set $(F, A) \in FS(X)$ under the mapping f is defined by $f[(F, A)] = (f(F), A)$, where $[f(F)](\alpha) = f[F(\alpha)], \forall \alpha \in A$.

(ii) the inverse image of a fuzzy soft set $(G, A) \in FS(Y)$ under the mapping f is defined by $f^{-1}[(G, A)] = (f^{-1}(G), A)$, where $[f^{-1}(G)](\alpha) = f^{-1}[G(\alpha)], \forall \alpha \in A$.

Definition 2.9. ([18]) Let (X, A, τ) and (Y, A, ν) be fuzzy soft topological spaces. A mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is said to be
(i) fuzzy soft continuous if $f^{-1}[(F, A)] \in \tau, \forall (F, A) \in \nu$.
(ii) fuzzy soft homeomorphism if f is bijective and f, f^{-1} are fuzzy soft continuous.

Proposition 2.3. *(*[18]*) Let (X, A, τ), (Y, A, ν) and (Z, A, ω) be fuzzy soft topological spaces. If $f : (X, A, \tau) \to (Y, A, \nu)$ and $g : (Y, A, \nu) \to (Z, A, \omega)$ are fuzzy soft continuous and $f(X) \subseteq Y$, then the mapping $g \circ f : (X, A, \tau) \to (Z, A, \omega)$ is fuzzy soft continuous.*

Definition 2.10. ([18]) Let F, G be two fuzzy subsets of X and Y respectively. Then their product denoted by $F \times G$ and defined by $(F \times G)(x, y) = min\{F(x), G(y)\}, \forall (x, y) \in X \times Y$.

Definition 2.11. ([18]) Let $(F, A), (G, A$ be two fuzzy soft sets over X and Y respectively. Then their product is defined by $(F, A) \tilde{\times} (G, A) = (F \tilde{\times} G, A)$, where $(F \tilde{\times} G)(\alpha) = F(\alpha) \times G(\alpha), \forall \alpha \in A$. It is clear that $(F \tilde{\times} G, A)$ is a fuzzy soft set over $X \times Y$.

Definition 2.12. ([18]) The fuzzy soft topology in $X \times Y$ induced by the open base $\mathcal{F} = \{(F, A) \tilde{\times} (G, A) : (F, A) \in \tau, (G, A) \in \nu\}$ is said to be the product fuzzy soft topology of the fuzzy soft topologies τ and ν. It is denoted by $\tau \tilde{\times} \nu$. The fuzzy soft topological space $[X \times Y, A, \tau \tilde{\times} \nu]$ is said to be the fuzzy soft topological product of the fuzzy soft topological spaces (X, A, τ) and (Y, A, ν).

Proposition 2.4. *(*[18]*) Let (X, A, τ) be the product space of two fuzzy soft topological spaces (X_1, A, τ_1) and (X_2, A, τ_2) respectively.Then the projection mappings*

$\pi_i : (X, A, \tau) \to (X_i, A, \tau_i), i = 1, 2$ *are fuzzy soft continuous and fuzzy soft open. Also* $\tau_1 \tilde{\times} \tau_2$ *is the smallest fuzzy soft topology in* $X \times Y$ *for which the projection mappings are fuzzy soft continuous.*
If further, (Y, A, ν) *be any fuzzy soft topological space then the mapping* $f : (Y, A, \nu) \to (X, A, \tau)$ *is fuzzy soft continuous iff the mappings* $\pi_i f : (Y, A, \nu) \to (X_i, A, \tau_i), i = 1, 2$ *are fuzzy soft continuous.*

Proposition 2.5. *(*[18]*) Let* (X, A, τ) *be a fuzzy soft topological space. Then the mapping* $f : (X, A, \tau) \to (X, A, \tau)$ *defined by* $f(x) = x$, $\forall x \in X$ *is fuzzy soft continuous.*

Proposition 2.6. *(*[18]*) Let* (X, A, τ) *and* (Y, A, ν) *be two fuzzy soft topological spaces. Then the mapping* $f : (X, A, \tau) \to (Y, A, \nu)$ *defined by* $f(x) = y_{0,}$ $\forall x \in X$, *where* y_0 *is a fixed element of* Y *is fuzzy soft continuous.*

Proposition 2.7. *(*[18]*) Let* (X, A, τ) *be the product space of two fuzzy soft topological spaces* (X_1, A, τ_1) *and* (X_2, A, τ_2) *respectively, Let* $a \in X_1$ *(or* X_2*) Then the mapping* $f : (X_2, A, \tau_2) \to (X, A, \tau)$ *(or* $f : (X_1, A, \tau_1) \to (X, A, \tau)$*) defined by* $f(x_2) = (a, x_2)$ *(or* $f(x_1) = (x_1, a)$*) is fuzzy soft continuous* $\forall x_2 \in X_2$*(or* $\forall x_1 \in X_1$*).*

Proposition 2.8. *(*[18]*) Let* (X, A, τ) *be the product space of two fuzzy soft topological spaces* (X_1, A, τ_1) *and* (X_2, A, τ_2) *and* (Y, A, ν) *be the product space of two fuzzy soft topological spaces* (Y_1, A, ν_1) *and* (Y_2, A, ν_2)*. If the mappings* f_j *of* (X_j, A, τ_j) *into* (Y_j, A, τ_j), $j = 1, 2$ *are fuzzy soft continuous(open), then the product mapping* $f = f_1 \times f_2$ *of* (X, A, τ) *into* (Y, A, ν) *defined by* $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ *is fuzzy soft continuous (open).*

Definition 2.13. ([1]) A fuzzy soft point is a fuzzy soft set, denoted by $\tilde{x}_e^{\lambda_e}$, defined by $[\tilde{x}_e^{\lambda_e}(\alpha)](y) = \begin{cases} \lambda_e, \ if\ \alpha = e \ and \ y = x \\ 0, \ if \ \alpha \neq e \ or \ y \neq x \end{cases}$.

A fuzzy soft point $\tilde{x}_e^{\lambda_e}$ belongs to a fuzzy soft set (F, A), denoted by $\tilde{x}_e^{\lambda_e}\tilde{\in}(F, A)$, whenever $0 < \lambda_e \leq [F(e)](x)$.

Proposition 2.9. ([1]) *Every non-null fuzzy soft set (F, A) can be expressed as the union of all fuzzy soft points which belongs to (F, A).*

Definition 2.14. ([6]) A fuzzy topological vector space is a vector space X over the field K, equipped with a Lowen type fuzzy topology ξ such that the two maps
(1) $f : (X, \xi) \times (X, \xi) \to (X, \xi)$, defined by $f(x, y) = x + y$ and
(2) $g : (K, \zeta) \times (X, \xi) \to (X, \xi)$, defined by $g(k, x) = kx$
are fuzzy continuous, where ζ is the usual topology on K.

3. Some properties of fuzzy soft topology

In this section we develop some results of fuzzy soft topology which will be used in subsequent sections.

Definition 3.1. A fuzzy soft set (G, A) in a fuzzy soft topological space (X, A, τ) is called a fuzzy soft neighborhood of a fuzzy soft set (F, A) if there exists a fuzzy soft open set (H, A) such that $(F, A)\tilde{\subseteq}(H, A)\tilde{\subseteq}(G, A)$.

Proposition 3.1. *(G, A) is a fuzzy soft open if and only if for each fuzzy soft set (F, A) contained in (G, A), (G, A) is a fuzzy soft nbd of (F, A).*

Proposition 3.2. *Let (X, A, τ) and (Y, A, ν) be two fuzzy soft topological spaces. For a mapping $f : (X, A, \tau) \to (Y, A, \nu)$ the followings are equivalent:*
(i) f is fuzzy soft continuous.
(ii) for each fuzzy soft set (F, A) in (X, A, τ), the inverse image of every fuzzy soft neighborhood of $f(F, A)$ is a fuzzy soft neighborhood of (F, A).
(iii) for each fuzzy soft set (F, A) in (X, A, τ) and each fuzzy soft neighborhood (H, A) of $f(F, A)$, there is a fuzzy soft neighborhood (G, A) of (F, A) such that $f(G, A)\tilde{\subseteq}(H, A)$.

Definition 3.2. A fuzzy soft set (F, A) in a fuzzy soft topological space (X, A, τ) is said to be a fuzzy soft neighborhood of a fuzzy soft point $\tilde{x}_{\alpha}^{\lambda_\alpha}$ in X if there is $(U, A) \in \tau$ such that $\tilde{x}_{\alpha}^{\lambda_\alpha} \tilde{\in} (U, A) \tilde{\subseteq} (F, A)$.

Proposition 3.3. *Let (X, A, τ) be a fuzzy soft topological space and $\mathcal{F}(\tilde{x}_{\alpha}^{\lambda_\alpha})$ denote the set of all fuzzy soft neighborhoods of a fuzzy soft point $\tilde{x}_{\alpha}^{\lambda_\alpha}$ in X. Then every member of $\mathcal{F}(\tilde{x}_{\alpha}^{\lambda_\alpha})$ has the following properties:*
(i) Every fuzzy soft set which contains a fuzzy soft set belonging to $\mathcal{F}(\tilde{x}_{\alpha}^{\lambda_\alpha})$ itself belongs to $\mathcal{F}(\tilde{x}_{\alpha}^{\lambda_\alpha})$.
(ii) Every finite intersection of sets of $\mathcal{F}(\tilde{x}_{\alpha}^{\lambda_\alpha})$ belongs to $\mathcal{F}(\tilde{x}_{\alpha}^{\lambda_\alpha})$.

Definition 3.3. In a fuzzy soft topological space (X, A, τ) a fundamental system of fuzzy soft neighborhoods of a fuzzy soft point $\tilde{x}_{\alpha}^{\lambda_\alpha}$ is a collection $\mathcal{B}(\tilde{x}_{\alpha}^{\lambda_\alpha})$ of fuzzy soft neighborhoods of $\tilde{x}_{\alpha}^{\lambda_\alpha}$ such that for each fuzzy soft neighborhood (F, A) of $\tilde{x}_{\alpha}^{\lambda_\alpha}$ there is a $(U, A) \in \mathcal{B}(\tilde{x}_{\alpha}^{\lambda_\alpha})$ such that $(U, A) \tilde{\subseteq} (F, A)$.

Definition 3.4. Let (X, A, τ) and (Y, A, ν) be two fuzzy soft topological spaces. A mapping $f : X \to Y$ is called fuzzy soft continuous at a point $x \in X$ if for each $\alpha \in A$ and $(V, A) \in \nu$ containing the fuzzy soft point $f(\tilde{x}_{\alpha}^{\lambda_\alpha}) = \tilde{y}_{\alpha}^{\lambda_\alpha}, 0 < \lambda_\alpha \leq 1$, the inverse image $f^{-1}(V, A) \in \tau$ and contains $\tilde{x}_{\alpha}^{\delta_\alpha}$, $0 < \delta_\alpha \leq \lambda_\alpha$.

Definition 3.5. A fuzzy soft topological space (X, A, τ) is called a fuzzy soft T_1 space if every fuzzy soft point is fuzzy soft closed set.

Proposition 3.4. *A fuzzy soft topological space (X, A, τ) is a fuzzy soft T_1 space if and only if for each $x \in X$, $\lambda \in [0, 1]$ and $\alpha \in A$, there exists $(U, A) \in \tau$ such that $[U(\alpha)](x) = 1 - \lambda$ and $[U(\alpha)](y) = 1$, for $y \neq x$, $U(\beta) = \bar{1}$, for $\beta \neq \alpha$.*

Proof. When $\lambda = 0$, it suffices to take $(U, A) = (\tilde{X}, A)$. When $\lambda > 0$, $\tilde{x}_{\alpha}^{\lambda}$ being a fuzzy soft point is a fuzzy soft closed set by hypothesis, then $(U, A) = (\tilde{X}, A) \setminus \tilde{x}_{\alpha}^{\lambda}$ is the required fuzzy soft open set.

Conversely let $\tilde{x}_\alpha^\lambda$ be an arbitrary fuzzy soft point. Then, by hypothesis, there exists a $(U, A) \in \tau$ such that $[U(\alpha)](x) = 1 - \lambda$ and $[U(\alpha)](y) = 1$ for $y \neq x$ and $U(\beta) = \bar{1}$, for $\beta \neq \alpha$. It follows that $\tilde{x}_\alpha^\lambda = (\tilde{X}, A) \setminus (U, A)$ is fuzzy soft closed set. □

Lemma 3.1. *If (F, A) and (G, A) are fuzzy soft sets over X and Y respectively, then* $\left[(F, A)\tilde{\times}(G, A)\right]^C = \left[(F^C, A)\tilde{\times}(\tilde{Y}, A)\right] \tilde{\cup} \left[(\tilde{X}, A)\tilde{\times}(G^C, A)\right]$.

Proof. For each $\alpha \in A$, $\left[(F, A)\tilde{\times}(G, A)\right]^C(\alpha) = \bar{1}_{X\times Y} \setminus [F(\alpha) \times G(\alpha)]$.

Now, $[\bar{1}_{X\times Y} \setminus [F(\alpha) \times G(\alpha)]](x, y) = max\{1 - [F(\alpha)](x), 1 - [G(\alpha)](y)\}$

$= max\{\left[F^C(\alpha) \times \bar{1}_Y\right](x, y), \left[\bar{1}_X \times G^C(\alpha)\right](x, y)\}$

$= \left(\left[F^C(\alpha) \times \bar{1}_Y\right] \cup \left[\bar{1}_X \times G^C(\alpha)\right]\right)(x, y)$

$= \left(\left[(F^C, A)\tilde{\times}(\tilde{Y}, A)\right](\alpha) \cup \left[(\tilde{X}, A)\tilde{\times}(G^C, A)\right](\alpha)\right)(x, y)$, for each $(x, y) \in X \times Y$.

Therefore, $\left[(F, A)\tilde{\times}(G, A)\right]^C(\alpha) = \left[(F^C, A)\tilde{\times}(\tilde{Y}, A)\right](\alpha) \cup \left[(\tilde{X}, A)\tilde{\times}(G^C, A)\right](\alpha)$, $\forall \alpha \in A$. Hence proved. □

Proposition 3.5. *Let (X, A, τ), and (Y, A, ν) be two fuzzy soft topological spaces. If (F, A) and (G, A) are fuzzy soft closed sets in (X, A, τ) and (Y, A, ν) respectively, then $(F, A)\tilde{\times}(G, A)$ is fuzzy soft closed set in $(X \times Y, A, \tau\tilde{\times}\nu)$.*

Proof. It follows from the Lemma 3.1.

□

Proposition 3.6. *If (X_i, A, τ_i), $i = 1, 2, ...n$ is a finite family of fuzzy soft topological spaces, each of which is a fuzzy soft T_1 space, then the product fuzzy topological space* $(X, A, \tau) = \left(\prod_{i=1}^{n} X_i, A, \prod_{i=1}^{n} \tau_i\right)$ *is a fuzzy soft T_1 space.*

Proof. Every fuzzy soft point $\tilde{y}_\alpha^{\lambda_\alpha}, 0 < \lambda_\alpha \leq 1$, in (X, A, τ) can be thought as the product of fuzzy soft points $\widetilde{(y_j)}_\alpha^{\lambda_\alpha}$, $j = 1, 2, .., n$. By hypothesis, each $\widetilde{(y_j)}_\alpha^{\lambda_\alpha}$ is fuzzy soft closed. Therefore the product is also fuzzy soft closed by Proposition 3.5. □

Proposition 3.7. *Let X be a non-empty set, A be the set of parameters and for each $\alpha \in A$, τ^α is a Lowen type fuzzy topology on X. Then $\tau^* = \{(G, A) \in FS(X) : G(\alpha) \in \tau^\alpha, \forall \alpha \in A\}$ is a Lowen type fuzzy soft topology on X.*

Proof. Any constant fuzzy soft set $\widetilde{c_X} \in \tau^*$, as $\widetilde{c_X}(\alpha) = \widetilde{c_X^\alpha} \in \tau^\alpha$.
Let $(G, A), (H, A) \in \tau^*$. Then $G(\alpha), H(\alpha) \in \tau^\alpha$, $\forall \alpha \in A$. Since for each $\alpha \in A$, τ^α is a Lowen type fuzzy topology, $G(\alpha) \cap H(\alpha) \in \tau^\alpha$, $\forall \alpha \in A$. Hence $(G, A)\tilde{\cap}(H, A) \in \tau^*$. Similarly it can be shown that $(F_i, A) \in \tau^*$, for $i \in \Delta \implies \tilde{\cup}_{i\in\Delta}(F_i, A) \in \tau^*$. Hence τ^* is a Lowen type fuzzy soft topology. □

Proposition 3.8. *Let (X, A, τ) and (Y, A, ν) be two fuzzy soft topological spaces. Then for each $\alpha \in A$, $(\tau\tilde{\times}\nu)^\alpha = \tau^\alpha \times \nu^\alpha$.*

Proof. Let $\alpha \in A$ and $U \in (\tau\tilde{\times}\nu)^\alpha$. Then $\exists(F, A) \in \tau\tilde{\times}\nu$ such that $U = F(\alpha)$. Since $(F, A) \in \tau\tilde{\times}\nu$, it follows that $\exists\{(U_i, A) \in \tau, (V_i, A) \in \nu, i \in \Delta\}$ such that $(F, A) = \tilde{\cup}_{i\in\Delta}[(U_i, A)\tilde{\times}(V_i, A)]$. So, $U = F(\alpha) = \cup_{i\in\Delta}[U_i(\alpha)\times V_i(\alpha)] \in \tau^\alpha \times \nu^\alpha$. Therefore $(\tau\tilde{\times}\nu)^\alpha \subseteq \tau^\alpha \times \nu^\alpha$. Next let $U \in \tau^\alpha \times \nu^\alpha$. Then we have $U_i \in \tau^\alpha, V_i \in \nu^\alpha, i \in \Delta$ such that $U = \cup_{i\in\Delta}[U_i \times V_i]$ and hence there exist $(F_i, A) \in \tau, (G_i, A) \in \nu$ such that $F_i(\alpha) = U_i$, $G_i(\alpha) = V_i, i = I$. Thus $\tilde{\cup}_{i\in\Delta}[(F_i, A)\tilde{\times}(G_i, A)] \in \tau\tilde{\times}\nu$ and $U = \cup_{i\in\Delta}[F_i(\alpha) \times G_i(\alpha)] \in (\tau\tilde{\times}\nu)^\alpha$. So, $\tau^\alpha \times \nu^\alpha \subseteq (\tau\tilde{\times}\nu)^\alpha$. Therefore for each $\alpha \in A$, $(\tau\tilde{\times}\nu)^\alpha = \tau^\alpha \times \nu^\alpha$. □

Proposition 3.9. *Let (X, A, τ) and (Y, A, ν) be two fuzzy soft topological spaces and define $T^* = \{(F, A) \in FS(X \times Y) : F(\alpha) \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A\}$. Then T^* is a fuzzy soft topology over $X \times Y$. and $T^* = \tau^*\tilde{\times}\nu^*$ where $\tau^* = \{(G, A) \in FS(X) : G(\alpha) \in \tau^\alpha, \forall \alpha \in A\}$ and $\nu^* = \{(H, A) \in FS(Y) : H(\alpha) \in \nu^\alpha, \forall \alpha \in A\}$.*

Proof. Since all fuzzy sets of the type $\widetilde{c_X^\alpha} \times \widetilde{c_Y^\alpha} \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A$. we have $(\tilde{c}_{X\times Y}, A) \in T^*$.
Again, let $(F_1, A), (F_2, A) \in T^*$. Then $F_1(\alpha), F_2(\alpha) \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A$.

So, $(F_1 \tilde{\bigcap} F_2)(\alpha) = F_1(\alpha) \bigcap F_2(\alpha) \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A$.

Thus $(F_1, A) \tilde{\bigcap} (F_2, A) \in T^*$.

Next let $(F_i, A) \in T^*, i \in \Delta$. So $(\tilde{\bigcup}_{i \in \Delta} F_i)(\alpha) = \bigcup_{i \in \Delta} [F_i(\alpha)] \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A$. Thus $\tilde{\bigcup}_{i \in \Delta} (F_i, A) \in T^*$.

Therefore T^* is a fuzzy soft topology over $X \times Y$.

Now let $(F, A) \in T^*$ and $\alpha \in A$. Then $F(\alpha) \in \tau^\alpha \times \nu^\alpha$ and hence $\exists\ U_i \in \tau^\alpha, V_i \in \nu^\alpha, i \in \Delta_\alpha$ such that $F(\alpha) = \bigcup_{i \in \Delta_\alpha} U_i \times V_i$. for each pair $U_i \in \tau^\alpha, V_i \in \nu^\alpha$ take fuzzy soft sets (F_{U_i}, A) and (F_{V_i}, A) such that $F_{U_i}(\alpha) = U_i, F_{U_i}(\beta) = \bar{0}$, $\forall \beta (\neq \alpha) \in A$ and $F_{V_i}(\alpha) = V_i, F_{V_i}(\beta) = \bar{0}$, $\forall \beta (\neq \alpha) \in A$.

So, $(F_{U_i}, A) \in \tau^*$ and $(F_{V_i}, A) \in \nu^*$ and hence $(F_{U_i} \tilde{\times} F_{V_i}, A) \in \tau^* \times \nu^*$.

Also $(F_{U_i} \tilde{\times} F_{V_i})(\alpha) = F_{U_i}(\alpha) \times F_{V_i}(\alpha) = U_i \times V_i$ and $(F_{U_i} \tilde{\times} F_{V_i})(\beta) = F_{U_i}(\beta) \times F_{V_i}(\beta) = \overline{0}, \forall \beta (\neq \alpha) \in A$.

Let $(G_\alpha, A) = \tilde{\bigcup}_{i \in \Delta_\alpha} (F_{U_i} \tilde{\times} F_{V_i}, A)$. Then $(G_\alpha, A) \in \tau^* \tilde{\times} \nu^*$ and $G_\alpha(\alpha) = \bigcup_{i \in \Delta_\alpha} U_i \times V_i = F(\alpha)$, $G_\alpha(\beta) = \overline{0}, \forall \beta (\neq \alpha) \in A$.

Again, let $(G, A) = \tilde{\bigcup}_{\alpha \in A} (G_\alpha, A)$. Then $(G, A) \in \tau^* \tilde{\times} \nu^*$ and $G(\alpha) = F(\alpha)$, $\forall \alpha \in A$.

Thus $(F, A) = (G, A) \in \tau^* \tilde{\times} \nu^*$.

$\therefore T^* \subseteq \tau^* \tilde{\times} \nu^*$.

Also, let $(F, A) \in \tau^* \tilde{\times} \nu^*$. Then $\exists \{(U_i, A) \in \tau^*, (V_i, A) \in \nu^*, i \in \Delta_\alpha\}$ such that $(F, A) = \tilde{\bigcup}_{i \in \Delta_\alpha} [(U_i, A) \tilde{\times} (V_i, A)]$. Also $F(\alpha) = \bigcup_{i \in \Delta_\alpha} [U_i(\alpha) \times V_i(\alpha)] \in \tau^\alpha \times \nu^\alpha, \forall \alpha \in A$. Hence $(F, A) \in T^*$.

$\therefore \tau^* \tilde{\times} \nu^* \subseteq T^*$. Thus $T^* = \tau^* \tilde{\times} \nu^*$. □

4. Sum and scalar product of fuzzy soft sets

Definition 4.1. ([9]) Let X be a vector space over the field K, the field of real and complex numbers, $k \in K$, A and B be two fuzzy sets of X. Then

(1) the sum of A and B is defined to be the fuzzy set $A + B$ of X given by

$(A + B)(x) = \underset{x=a+b}{sup} \{A(a) \wedge B(b)\}$

(2) kA is defined to be the fuzzy set kA of X, where

$$(kA)(x) = \begin{cases} A(k^{-1}x) & if\ k \neq 0 \\ \sup\limits_{y \in X} A(y) & if\ k = 0, x = \theta \\ 0 & if\ k = 0, x \neq \theta. \end{cases}$$

Definition 4.2. Let (F, A) and (G, A) be two fuzzy soft sets over the vector space X over the field K. Then
(i) $(F, A) + (G, A) = (F + G, A)$ where $(F + G)(\alpha) = F(\alpha) + G(\alpha)$, $\forall \alpha \in A$.
(ii) $k(F, A) = (kF, A)$ where $(kF)(\alpha) = kF(\alpha)$ $\forall \alpha \in A, \forall k \in K$.
(iii) $x + (F, A) = (x + F, A)$ where $(x + F)(\alpha) = x + F(\alpha)$, $\forall \alpha \in A, \forall x \in X$.

Proposition 4.1. *Let X and Y be two vector spaces over K and let f be a linear map from X into Y. Then for all fuzzy soft sets (F, A), (G, A) in X and all scalars $k \in K$,*
(i) $f[(F, A) + (G, A)] = f(F, A) + f(G, A)$.
(ii) $f[k(F, A)] = kf(F, A)$.

Corollary 4.1. $k[(F, A) + (G, A)] = k(F, A) + k(G, A)$, *for all fuzzy soft sets (F, A), (G, A) in X and all scalars k.*

Proposition 4.2. *If $(F, A), (G, A)$ are two fuzzy soft sets in X and $k \in K$, $k \neq 0$, then $k(F, A) \tilde{\subseteq} (G, A) \Rightarrow (F, A) \tilde{\subseteq} \frac{1}{k}(G, A)$.*

Proposition 4.3. *If $(F, A), (G, A)$ are two fuzzy soft sets in X, then $(F, A) + (G, A) = \tilde{\bigcup}_{\alpha \in A} \left[\tilde{\bigcup}_{x \in X} \left(\tilde{x}_{\alpha}^{\lambda_\alpha} + (G, A) \right) \right]$, where $[F(\alpha)](x) = \lambda_\alpha$.*

Proof. Let $(F, A), (G, A)$ be two fuzzy soft sets in X and $[F(\alpha)](x) = \lambda_\alpha$. Set $(S, A) = \tilde{x}_{\alpha}^{\lambda_\alpha} + (G, A)$. Then for $y \in X$, $[S(\alpha)](y) = [F(\alpha)](x) \wedge [G(\alpha)](y - x)$ and $[S(\beta)](y) = 0, \beta \neq \alpha$.
Let $\tilde{\bigcup}_{x \in X} \left(\tilde{x}_{\alpha}^{\lambda_\alpha} + (G, A) \right) = (T^\alpha, A)$. Then for $y \in X$, $[T^\alpha(\alpha)](y) = \bigvee_{x \in X} \{[F(\alpha)](x) \wedge$

$[G(\alpha)](y-x)\}$ and $[T^{\alpha}(\beta)](y)=0, \beta \neq \alpha$.
Then for any $\gamma \in A$, $\left[\left(\underset{\alpha\in A}{\tilde{\cup}}[(T^{\alpha},A)]\right)(\gamma)\right](y) = \underset{x\in X}{\vee}\{[F(\gamma)](x) \wedge [G(\gamma)](y-x)\} =$
$[F(\gamma)+G(\gamma)](y), y \in X$.
Hence proved. □

5. Convex and balanced fuzzy soft sets

Throughout the rest of the paper we use the notation X for the vector space $(X,+,\cdot)$ over the scalar field K, where K is the field of real or complex numbers, A is the parameter set.

Definition 5.1. A fuzzy soft set (F,A) in a vector space X over the scalar field K is said to be
(a) convex if $k(F,A)+(1-k)(F,A) \subseteq (F,A)$, $\forall k \in [0,1]$.
(b) balanced if $k(F,A) \subseteq (F,A)$ for all scalars k with $\mid k \mid \leq 1$.
(c) absolutely convex if it is balanced and convex.

Note 5.1. *It is to be noted that, (F,A) is convex (balanced) iff for each $\alpha \in A$, $F(\alpha)$ is convex (balanced) fuzzy set.*

Example 5.1. *Consider the vector space $\mathbb{R}$ over the field $\mathbb{R}$ and let $\mathbb{N}$ be the parameter set. Then if we define a fuzzy soft set $(F,\mathbb{N})$ such that for each $n \in \mathbb{N}$, if $F(n)$ is a fuzzy real number then $F(n)$ is a fuzzy convex set over $\mathbb{R}$. Hence by Note 5.1, $(F,\mathbb{N})$ is a fuzzy soft convex set.*

Proposition 5.1. *If (F,A) and (G,A) are two convex (balanced) fuzzy soft sets in a vector space X over the scalar field K, then $k_1(F,A)+k_2(G,A)$ is a convex (balanced) fuzzy soft set in X for all scalars $k_1,k_2 \in K$.*

Proposition 5.2. *If $\{(F_i,A)\}_{i\in I}$ is a family of convex (balanced) fuzzy soft sets in a vector space X, then $(F,A)=\tilde{\cap}_{i\in I}(F_i,A)$ is a convex (balanced) fuzzy soft set in X.*

Proposition 5.3. *Let X and Y be two vector spaces over the scalar field K and let $f : X \to Y$ be a linear map.*
(a) If (F, A) is a convex (balanced) fuzzy soft set in X, then $f[(F, A)]$ is a convex (balanced) fuzzy soft set in Y.
(b) $f^{-1}[(G, A)]$ is a convex (balanced) fuzzy soft set in X whenever (G, A) is a convex (balanced) fuzzy soft set in Y.

Proof. (a) We will prove the result for the convex case. The proof for the balanced case is similar. Let $k \in [0, 1]$ and (F, A) be a convex fuzzy soft set in X. Then $\forall \alpha \in A$,
$[kf[(F, A)] + (1 - k)f[(F, A)]](\alpha)$
$= [kf[(F, A)]](\alpha) + [(1 - k)f[(F, A)]](\alpha)$
$= kf(F(\alpha)) + (1 - k)f(F(\alpha))$
$= f(kF(\alpha) + (1 - k)F(\alpha))$
$\subseteq f(F(\alpha))$.
$\therefore k[(F, A)] + (1 - k)f[(F, A)] \tilde{\subseteq} f[(F, A)]$, which proves that $f[(F, A)]$ is a convex fuzzy soft set.
(b) Assume next that (G, A) is a convex fuzzy soft set in Y and let $k \in [0, 1]$. Set $(M, A) = kf^{-1}[(G, A)] + (1 - k)f^{-1}[(G, A)]$. Then $\forall \alpha \in A$, $[f(M)](\alpha) = kf[[f^{-1}(G)](\alpha)] + (1 - k)f[[f^{-1}(G)](\alpha)] = kf[f^{-1}[G(\alpha)]] + (1 - k)f[f^{-1}[G(\alpha)]]$ $\subseteq kG(\alpha) + (1 - k)G(\alpha) \subseteq G(\alpha)$, and hence $M(\alpha) \subseteq f^{-1}[G(\alpha)]$.
Therefore $(M, A) \tilde{\subseteq} f^{-1}[(G, A)]$. □

Definition 5.2. Let (F, A) be a fuzzy soft set over a vector space X. The balanced hull of (F, A) is the intersection of all balanced fuzzy soft sets in X which contain (F, A).

Proposition 5.4. *Let (F, A) be a fuzzy soft set over a vector space X. The balanced hull of (F, A) is the fuzzy soft set $\tilde{\bigcup}_{|k| \leq 1} k(F, A)$.*

Proof. Obviously, the fuzzy soft set $(G, A) = \underset{|k|\leq 1}{\tilde{\cup}} k(F, A)$ is contained in any balanced fuzzy soft set which contains (F, A). Since $(F, A)\tilde{\subseteq}(G, A)$, it suffices to show that (G, A) is balanced. Now $G(\alpha) = \underset{|k|\leq 1}{\cup} kF(\alpha)$ is a balanced fuzzy set for all $\alpha \in A$. Hence by Note 5.1, (G, A) is balanced fuzzy soft set. □

6. Fuzzy soft topological vector spaces

Definition 6.1. Let K be the field of real or complex numbers, A be the parameter set and ν^α be the usual topology on K, $\forall \alpha \in A$. Then the soft topology ν defined by $\nu = \{(F, A) \in S(X) : F(\alpha) \in \nu^\alpha\}$ is called the soft usual topology on K.

Definition 6.2. Let X be a vector space over the scalar field K endowed with the soft usual topology ν, A be the parameter set and τ be a fuzzy soft topology on X. Then τ is said to be a fuzzy soft linear topology on X if the mappings:

(1) $f : (X \times X, A, \tau\tilde{\times}\tau) \to (X, A, \tau)$, defined by $f(x, y) = x + y$ and

(2) $g : (K \times X, A, \nu\tilde{\times}\tau) \to (X, A, \tau)$, defined by $g(k, x) = kx$

are fuzzy soft continuous $\forall x, y \in X$ and $\forall k \in K$.

The fuzzy soft topological space (X, A, τ) is called fuzzy soft topological vector space.

Proposition 6.1. *Let τ be a fuzzy soft linear topology on a vector space X over the field K, A be the parameter set and ν be the fuzzy soft usual topology on K. Then τ^α is a fuzzy linear topology on X, $\forall \alpha \in A$.*

Proof. Let $U \in \tau^\alpha$. Then $\exists (F, A) \in \tau$ such that $F(\alpha) = U$. Since τ is a fuzzy soft linear topology, we have the mappings

$f : (X \times X, A, \tau\tilde{\times}\tau) \to (X, A, \tau)$, defined by $f(x, y) = x + y$ and

$g : (K \times X, A, \nu\tilde{\times}\tau) \to (X, A, \tau)$, defined by $g(k, x) = kx$

are fuzzy soft continuous and hence$f^{-1}((F, A)) \in \tau\tilde{\times}\tau$ and $g^{-1}((F, A)) \in \nu\tilde{\times}\tau$.

So, $f^{-1}(F(\alpha)) \in (\tau\tilde{\times}\tau)^\alpha = \tau^\alpha \times \tau^\alpha, \forall \alpha \in A$ and

$g^{-1}(F(\alpha)) \in (\nu\tilde{\times}\tau)^\alpha = \nu^\alpha \times \tau^\alpha, \forall \alpha \in A$.

$\therefore f : (X \times X, \tau^\alpha \times \tau^\alpha) \to (X, \tau^\alpha)$ and $g : (K \times X, \nu^\alpha \times \tau^\alpha) \to (X, \tau^\alpha)$ are fuzzy continuous, $\forall \alpha \in A$. Hence, τ^α is a fuzzy linear topology on X, $\forall \alpha \in A$. □

Proposition 6.2. *Let X be a vector space over the scalar field K where K is endowed with the soft usual topology ν, A be the parameter set and for each $\alpha \in A$, τ^α be a fuzzy linear topology on X. Then τ^* is a fuzzy soft linear topology on X, where τ^* is defined as in Proposition 3.7.*

Proof. Let $(F, A) \in \tau^*$. Then $F(\alpha) \in \tau^\alpha$, $\forall \alpha \in A$.
Since $\forall \alpha \in A$, τ^α is a fuzzy linear topology on X whereas ν^α is a usual topology on K, we have the mappings $f : (X \times X, \tau^\alpha \times \tau^\alpha) \to (X, \tau^\alpha)$, defined by $f(x, y) = x + y$ and
$g : (K \times X, \nu^\alpha \times \tau^\alpha) \to (X, \tau^\alpha)$, defined by $g(k, x) = kx$ are fuzzy continuous, $\forall x, y \in X$ and $\forall k \in K$, $\forall \alpha \in A$.
So, $f^{-1}(F(\alpha)) \in \tau^\alpha \times \tau^\alpha, \forall \alpha \in A$ and $g^{-1}(F(\alpha)) \in \nu^\alpha \times \tau^\alpha, \forall \alpha \in A$ and hence by Proposition 3.9, $f^{-1}((F, A)) \in \tau^* \tilde{\times} \tau^*$ and $g^{-1}((F, A)) \in \nu \tilde{\times} \tau^*$.
Thus, the mappings $f : (X \times X, A, \tau^* \tilde{\times} \tau^*) \to (X, A, \tau^*)$, defined by $f(x, y) = x + y$ and $g : (K \times X, A, \nu \tilde{\times} \tau^*) \to (X, A, \tau^*)$, defined by $g(k, x) = kx$ are fuzzy soft continuous $\forall x, y \in X$ and $\forall k \in K$. Therefore, τ^* is a fuzzy soft linear topology on X. □

Proposition 6.3. *In a fuzzy soft topological vector space (X, A, τ), the map $M_k : (X, A, \tau) \to (X, A, \tau)$, defined by $M_k(x) = kx$ is fuzzy soft continuous, $\forall k \in K$ and M_k is a fuzzy soft homeomorphism for $k \neq 0$.*

Proof. In (X, A, τ), the map $g : (K \times X, A, \nu \tilde{\times} \tau) \to (X, A, \tau)$, defined by $g(k, x) = kx$ is fuzzy soft continuous. Also by Proposition 2.7, the map $h : (X, A, \tau) \to (K \times X, A, \nu \tilde{\times} \tau)$, defined by $h(x) = (k, x)$ is fuzzy soft continuous for a fixed $k \in K$. Hence $M_k = g \circ h$ is fuzzy soft continuous. In case $k \neq 0$, $M_k^{-1}(x) = x/k$ is fuzzy soft continuous. Therefore, M_k is a fuzzy soft homeomorphism. □

Proposition 6.4. *Let (X, A, τ) be a fuzzy soft topological vector space. Then the map $T_a : (X, A, \tau) \to (X, A, \tau)$, defined by $T_a(x) = a + x$ is fuzzy soft homeomorphism for any $a \in X$.*

Proof. The proof is similar as above. □

Proposition 6.5. *Let (X, A, τ) be a fuzzy soft topological vector space. Then the mapping $h : (X \times X, A, \tau \tilde{\times} \tau) \to (X \times X, A, \tau \tilde{\times} \tau)$, defined by $h(x, y) = (ax, by)$ is fuzzy soft continuous for all scalars $a, b \in K$ and $x, y \in X$.*

Proof. We know that the mappings $\pi_i : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau), i = 1, 2$; defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are fuzzy soft continuous. Also, $M_k : (X, A, \tau) \to (X, A, \tau)$, defined by $M_k(x) = kx$ is fuzzy soft continuous, $\forall k \in K$.
Now, $\pi_1 h : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau)$, defined by $\pi_1 h(x, y) = \pi_1(ax, by) = ax = M_a \pi_1(x, y)$.
$\therefore \pi_1 h\,(= M_a \pi_1)$ is fuzzy soft continuous.
Similarly, $\pi_2 h\,(= M_b \pi_2)$ is fuzzy soft continuous.
Then from Proposition 2.4, we get that the mapping $h : (X \times X, A, \tau \tilde{\times} \tau) \to (X \times X, A, \tau \tilde{\times} \tau)$, defined by $h(x, y) = (ax, by)$ is fuzzy soft continuous for all scalars $a, b \in K$ and $x, y \in X$. □

Proposition 6.6. *A fuzzy soft topology τ on a vector space X over the field K, where K is endowed with the soft usual topology ν, is a fuzzy soft linear topology iff the mapping $L_{(a,b)} : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau)$, defined by $L_{(a,b)}(x, y) = ax + by$, is fuzzy soft continuous $\forall a, b \in K$ and $\forall x, y \in X$.*

Proof. Let τ be a fuzzy soft linear topology. Therefore $f : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau)$, defined by $f(x, y) = x + y$ is fuzzy soft continuous, $\forall x, y \in X$. Also, from Proposition 6.5, the mapping $h : (X \times X, A, \tau \tilde{\times} \tau) \to (X \times X, A, \tau \tilde{\times} \tau)$, defined by $h(x, y) = (ax, by)$ is fuzzy soft continuous for all scalars $a, b \in K$ and $x, y \in X$.

Therefore, $L_{(a,b)} = f \circ h : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau)$, defined by

$L_{(a,b)}(x, y) = f(h(x, y)) = f(ax, by) = ax + by$, is fuzzy soft continuous $\forall a, b \in K$ and $\forall x, y \in X$.

Conversely, let the mapping $L_{(a,b)} : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau)$, defined by $L_{(a,b)}(x, y) = ax + by$, is fuzzy soft continuous $\forall a, b \in K$ and $\forall x, y \in X$.

We know that the mappings $\pi_1 : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau)$, defined by $\pi_1(x, y) = x$ and $\pi_2 : (K \times X, A, \nu \tilde{\times} \tau) \to (X, A, \tau)$, defined by $\pi_2(k, y) = y$ are fuzzy soft continuous $\forall k \in K$ and $\forall x, y \in X$. Also, by Proposotion 2.7, $h_0 : (X, A, \tau) \to (X \times X, A, \tau \tilde{\times} \tau)$, defined by $h_0(x) = (x, \theta)$ is fuzzy soft continuous $\forall x \in X$, where θ is the zero element of X.

Therefore, $g = L_{(a,0)} \circ h_0 \circ \pi_2 : (K \times X, A, \nu \tilde{\times} \tau) \to (X, A, \tau)$,defined by

$$g(a, x) = L_{(a,0)} \circ h_0 \circ \pi_2(a, x)$$
$$= L_{(a,0)}(h_0(\pi_2(a, x)))$$
$$= L_{(a,0)}(h_0(x))$$
$$= L_{(a,0)}(x, \theta)$$
$$= ax$$

is fuzzy soft continuous, $\forall a \in K$ and $\forall x \in X$.

Since, $L_{(a,b)} : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau)$, defined by $L_{(a,b)}(x, y) = ax + by$, is fuzzy soft continuous $\forall a, b \in K$ and $\forall x, y \in X$, taking $a = 1, b = 1$; we can define $f = L_{(1,1)} : (X \times X, A, \tau \tilde{\times} \tau) \to (X, A, \tau)$, such that $f(x, y) = L_{(1,1)}(x, y) = x + y$. Then f is fuzzy soft continuous.

Thus τ is a fuzzy soft linear topology on X. □

Proposition 6.7. *Let (X, A, τ) be a fuzzy soft topological vector space. If (F, A) is a fuzzy soft open set and t is a non-zero scalar, then (tF, A) is also a fuzzy soft open set.*

Proof. Since for $t \neq 0$, $M_t : (X, A, \tau) \to (X, A, \tau)$ is a fuzzy soft homeomorphism, $(tF, A) = M_t(F, A)$ is fuzzy soft open whenever (F, A) is so. □

Proposition 6.8. *Let (X, A, τ) be a fuzzy soft topological vector space. If (F, A) is fuzzy soft open, then $[x_0 + (F, A)]$ is fuzzy soft open for all $x_0 \in X$.*

Proof. Since the map $T_{x_0} : (X, A, \tau) \to (X, A, \tau)$, defined by $T_{x_0}(F, A) = x + x_0$ is a fuzzy soft homeomorphism and $T_{x_0}(F, A) = [x_0 + (F, A)]$, $[x_0 + (F, A)]$ is fuzzy soft open for all $x_0 \in X$ and $\forall (F, A) \in \tau$. □

Proposition 6.9. *Let $\{(X_i, A, \tau_i)\}_{i \in \triangle}$ be a family of fuzzy soft topological vector spaces and let $(X, A, \tau) = (\prod_{i \in \triangle} X_i, A, \prod_{i \in \triangle} \tau_i)$ be their product fuzzy soft topological space. Then (X, A, τ) is a fuzzy soft topological vector space.*

Proof. Let $\pi_i : X \to X_i$ denote the i-th projection map and $f : X \times X \to X$ be defined by $f(x, y) = x + y$ and $f_i : X_i \times X_i \to X_i$ be defined by $f_i(x, y) = x + y$. Now the map $f_i \circ (\pi_i \times \pi_i) = \pi_i \circ f$. Since $\forall i \in \triangle$, $f_i \circ (\pi_i \times \pi_i)$ is fuzzy soft continuous, each $\pi_i \circ f$ is fuzzy soft continuous. By Proposition 2.4, f is fuzzy soft continuous. Next, let $g : K \times X \to X$, be defined by $g(k, x) = kx$, $g_i : K \times X_i \to X_i$, defined by $g_i(k, x) = kx$ and $I : K \to K$ be defined by $I(k) = k, \forall k \in K$. We see that the map $g_i \circ (I \times \pi_i) = \pi_i \circ g$. Since $\forall i \in \triangle$, $g_i \circ (I \times \pi_i)$ is fuzzy soft continuous, each $\pi_i \circ g$ is fuzzy soft continuous. By Proposition 2.4, g is fuzzy soft continuous. □

Lemma 6.1. *Let $\tilde{\theta}_\alpha^{\lambda_\alpha}$ be a fuzzy soft point in a fuzzy soft topological vector space (X, A, τ) and (F, A) be any fuzzy soft set containing $\tilde{\theta}_\alpha^{\lambda_\alpha}$. If there is a point $a \in X$ such that $[F(\alpha)](ka) = 0$, for all non-zero scalar $k \in K$, then (F, A) is not a fuzzy soft neighborhood of $\tilde{\theta}_\alpha^{\lambda_\alpha}$.*

Proof. Suppose that (F, A) be a fuzzy soft open nbd of $\tilde{\theta}_\alpha^{\lambda_\alpha}$. Consider the function $g : (k, a) \to ka$ and let $\tilde{a}_\alpha^{\delta_\alpha}$ be any fuzzy soft point where $0 < \delta_\alpha \leq \lambda_\alpha$. For $k = 0$, the point $k\tilde{a}_{\alpha}^{\delta_\alpha}\tilde{\in}(F,A)$. Since g is fuzzy soft continuous, there exists a fuzzy soft neighborhood(G,A) of $\tilde{a}_{\alpha}^{\delta_\alpha}$ such that $\varepsilon\tilde{a}_{\alpha}^{\delta_\alpha}\tilde{\in}\varepsilon(H,A)\tilde{\subseteq}(F,A)$, for a non-zero scalar ε. Then $\left[\tilde{a}_{\alpha}^{\delta_\alpha}(\alpha)\right](a) \leq [F(\alpha)](\varepsilon a)$. But this is a contradiction to our assumption. Hence proved. □

Lemma 6.2. *If (F,A) and (G,A) are fuzzy soft neighborhoods of a fuzzy soft point $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ in a fuzzy soft topological vector space (X,A,τ), then the sum $(F,A)+(G,A)$ and the scalar product $k(F,A)$, $k\in K$, $k\neq 0$, are fuzzy soft neighborhoods of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$.*

Proof. If (F,A) and (G,A) are fuzzy soft neighborhoods of a fuzzy soft point $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$, then there exist two fuzzy soft open sets (U,A) and (V,A) in X such that $\tilde{\theta}_{\alpha}^{\lambda_\alpha}\tilde{\in}(U,A)\tilde{\subseteq}(F,A)$ and $\tilde{\theta}_{\alpha}^{\lambda_\alpha}\tilde{\in}(V,A)\tilde{\subseteq}(G,A)$.

By Proposition 4.3, the sum $(U,A)+(V,A) = \underset{\alpha\in A}{\tilde{\cup}}\left[\underset{x\in X}{\tilde{\cup}}\left(\tilde{x}_{\alpha}^{\delta_\alpha}+(V,A)\right)\right]$, where $[U(\alpha)](x)=\delta_\alpha$.

Now $\left[\left(\tilde{x}_{\alpha}^{\delta_\alpha}+(V,A)\right)(\alpha)\right](y) = min\{\widetilde{c_{\delta_\alpha}}(y), [T_x(V,A)(\alpha)](y)\}, y\in X$, and $\left[\left(\tilde{x}_{\alpha}^{\delta_\alpha}+(V,A)\right)(\beta)\right](y) = min\{\bar{0}, [T_x(V,A)(\beta)](y)\},\ y\in X,\ \beta\neq\alpha$,

where $\widetilde{c_X^{\delta_\alpha}}$ is the constant fuzzy set $\widetilde{c_X^{\delta_\alpha}}(y)=\delta_\alpha$ and $T_x(y)=y+x,\ \forall y\in X$.

Let (C,A) be the constant fuzzy soft set such that $(C,A)(\alpha)=\widetilde{c_X^{\delta_\alpha}}$ and $(C,A)(\beta)=\bar{0},\beta\neq\alpha$. Then $\left(\tilde{x}_{\alpha}^{\delta_\alpha}+(V,A)\right) = (C,A)\tilde{\cap}[T_x(V,A)]$ and hence $\tilde{x}_{\alpha}^{\delta_\alpha}+(V,A)$ is fuzzy soft open set.

Since $(U,A)+(V,A)$ is union of fuzzy soft open sets, it is also fuzzy soft open. Obviously, $(U,A)+(V,A)\tilde{\subseteq}(F,A)+(G,A)$. Next we show that $\tilde{\theta}_{\alpha}^{\lambda_\alpha}\tilde{\in}(U,A)+(V,A)$. Since $\tilde{\theta}_{\alpha}^{\lambda_\alpha}\tilde{\in}(U,A)$ and $\tilde{\theta}_{\alpha}^{\lambda_\alpha}\tilde{\in}(V,A)$, $min\{[U(\alpha)](\theta),[V(\alpha)](\theta)\}\geq\lambda_\alpha$.

Hence $[U(\alpha)+V(\alpha)](\theta) = \underset{x_1+x_2=\theta}{sup}\ min\{[U(\alpha)](x_1),[V(\alpha)](x_2)\}\geq\lambda_\alpha$.

Therefore, $\tilde{\theta}_{\alpha}^{\lambda_\alpha}\tilde{\in}(U,A)+(V,A)$. The second statement is obvious. □

Lemma 6.3. *If (F,A) is a fuzzy soft neighborhood of a fuzzy soft point $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ in a fuzzy soft topological vector space (X,A,τ), then there exists a fuzzy soft neighborhood (G,A) of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ such that $(G,A)+(G,A)\tilde{\subseteq}(F,A)$.*

Proof. Since sum function is fuzzy soft continuous, by Proposition 3.2, for every fuzzy soft neighborhood (F, A) of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$, there exist fuzzy soft neighborhoods $(G_1, A), (G_2, A)$ of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ such that $(G_1, A) + (G_2, A)\tilde{\subseteq}(F, A)$. Let $(G, A) = (G_1, A)\tilde{\cap}(G_2, A)$. Then (G, A) is a fuzzy soft neighborhood of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ such that $(G, A) + (G, A)\tilde{\subseteq}(F, A)$. □

Lemma 6.4. *If (F, A) is a fuzzy soft neighborhood of a fuzzy soft point $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ in a fuzzy soft topological vector space (X, A, τ), then there exists a fuzzy soft neighborhood (G, A) of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ such that $k(G, A)\tilde{\subseteq}(F, A)$ for every $k \in K, \mid k \mid \leq 1$.*

Proof. Let (F, A) be a fuzzy soft neighborhood of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$. Since the scalar product is fuzzy soft continuous, by Proposition 3.2, there exists an $\varepsilon > 0$ and a fuzzy soft neighborhood (U, A) of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ such that $\xi \in K, \mid \xi \mid < \varepsilon$, $\xi(U, A)\tilde{\subseteq}(F, A)$. By hypothesis, $\mid k \mid \leq 1$. Hence $\mid k\xi \mid < \varepsilon$ and $k\xi(U, A)\tilde{\subseteq}(F, A)$. Set $\xi(U, A) = (G, A)$. Thus the result follows. □

Proposition 6.10. *Let (X, A, τ) be a fuzzy soft topological vector space. For every fuzzy soft point $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$, $0 < \lambda_\alpha \leq 1$, $\alpha \in A$, there exists a fundamental system of fuzzy soft neighborhoods $\mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ in (X, A, τ) for which the following results hold:*
(i) For each $(U, A) \in \mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ there is a $(V, A) \in \mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ with $(V, A) + (V, A)\tilde{\subseteq}(U, A)$.
(ii) For each $(U, A) \in \mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ there is a $(V, A) \in \mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ such that $k(U, A)\tilde{\subseteq}(V, A)$ for every $k \in K, \mid k \mid \leq 1$.
(iii) Every $(U, A) \in \mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ is balanced fuzzy soft set.

Proof. Let $\mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ be any fundamental system of fuzzy soft neighborhoods of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}, 0 < \lambda_\alpha \leq 1, \alpha \in A$. Then (i) and (ii) holds by Lemma 6.3 and Lemma 6.4 respectively. For (iii) we show that the set of balanced hulls of fuzzy soft sets in $\mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ is itself a fundamental system of fuzzy soft neighborhoods of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$. Let (W, A) be a fuzzy soft neighborhood of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$. Then there exists a fuzzy soft set $(V, A) \in \mathcal{B}(\tilde{\theta}_{\alpha}^{\lambda_\alpha})$ such that $k(V, A)\tilde{\subseteq}(W, A)$, for every $k \in K, \mid k \mid \leq 1$. Then the balanced hull (U, A) of (V, A)

is of the form $\underset{|k|\leq 1}{\tilde{\cup}} k(V,A)$. Hence $(U,A)\tilde{\subseteq}(W,A)$ and also (U,A) is balanced. Hence proved. □

Proposition 6.11. *Let (X,A,τ) be a fuzzy soft topological vector space. Then*
(i) If (W,A) be any open fuzzy soft nbd of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}, 0<\lambda_\alpha\leq 1, \alpha\in A$, then $x+(W,A)$ is an open fuzzy soft nbd of $\tilde{x}_{\alpha}^{\lambda_\alpha}$.
(ii) If (W,A) be any open fuzzy soft nbd of $\tilde{x}_{\alpha}^{\lambda_\alpha}, 0<\lambda_\alpha\leq 1, \alpha\in A$, then $t(W,A)$ is an open fuzzy soft nbd of $\widetilde{tx}_{\alpha}^{\lambda_\alpha}$, for any non-zero scalar t.

7. Fuzzy soft tangent

Definition 7.1. [10] A real valued function of a real variable t, defined on some nbd of 0 is said to be $o(t)$ if $\lim_{t\to 0}\frac{o(t)}{t}=0$.

Definition 7.2. Let (X_1,A,τ_1) and (X_2,A,τ_2) be two fuzzy soft topological vector spaces and θ,θ' are null vectors of X_1 and X_2 respectively. Then a function $\phi : (X_1,A,\tau_1)\to(X_2,A,\tau_2)$ such that $\phi(\theta)=\theta'$ is said to be fuzzy soft tangent to θ, if for each $\alpha\in A$ and any fuzzy soft neighborhood (G,A) of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}, 0<\delta_\alpha\leq 1$ in (X_2,A,τ_2), there exists a fuzzy soft neighborhood (F,A) of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}, 0<\lambda_\alpha\leq\delta_\alpha$ in (X_1,A,τ_1) such that $\phi(t(F,A))\tilde{\subseteq}o(t)(G,A)$, for some function $o(t)$.

Example 7.1. *Let (X,ξ) be a fuzzy topological vector space as in Definition 2.14 and θ be the null vector of X. Define fuzzy soft topology τ on X as $\tau=\{(F,A):F(\alpha)\in\xi\}$ and soft topology ν on K such that $\nu=\{(G,A):G(\alpha)\in\zeta\}$. Then ν is the soft usual topology on K. Also τ is a fuzzy soft linear topology on X, by Proposition 6.2. Consider the identity function $I:X\to X$ defined by $I(x)=x$. Choose $\alpha\in A$. Let (G,A) be a fuzzy soft neighborhood of $\tilde{\theta}_{\alpha}^{\delta_\alpha}$ for some $0<\delta_\alpha\leq 1$. Then $t(G,A)$ is also a fuzzy soft neighborhood of $\tilde{\theta}_{\alpha}^{\delta_\alpha}$, for $t\neq 0$. Thus $t(G,A)$ is a fuzzy soft neighborhood of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}, 0<\lambda_\alpha\leq\delta_\alpha$. Let $(F,A)=t(G,A)$. Then $I(t(F,A))=t.t(G,A)=t^2(G,A)$.*

Since $\alpha \in A$ is arbitrary, it holds for all $\alpha \in A$. Hence I is a fuzzy soft tangent to θ in (X, A, τ).

Lemma 7.1. *If the function $\phi : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ is fuzzy soft tangent to θ, then ϕ is fuzzy soft continuous at θ.*

Proof. Choose $\alpha \in A$. By Lemma 6.4, for every fuzzy soft neighborhood (F, A) of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}, 0 < \delta_\alpha \leq 1$ in (X_2, A, τ_2), there exists a fuzzy soft neighborhood (F', A) of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}$ such that $o(t)(F', A)\tilde{\subseteq}(F, A)$ for $\mid o(t) \mid\leq 1$. By Definition 7.2, for each (F', A) there exist fuzzy soft neighborhoods $(G, A), (G', A)$, $(G, A) = t(G', A)$, of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}, 0 < \lambda_\alpha \leq \delta_\alpha$ in (X_1, A, τ_1) such that $\phi(G, A) = \phi(tG', A)\tilde{\subseteq}o(t)(F', A)\tilde{\subseteq}(F, A)$. Since α is arbitrary. It is true for all $\alpha \in A$. Hence ϕ is fuzzy soft continuous at θ. □

Proposition 7.1. *Let (X_1, A, τ_1) and (X_2, A, τ_2) be two fuzzy soft topological vector spaces and θ, θ' be null vectors of X_1 and X_2 respectively. If the functions $\phi, \psi : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ are fuzzy soft tangent to θ, then $\phi + \psi$ is fuzzy soft tangent to θ.*

Proof. Choose $\alpha \in A$. For every fuzzy soft neighborhood of (G, A) of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}, 0 < \delta_\alpha \leq 1$ in (X_2, A, τ_2), there exists a fuzzy soft neighborhood (G', A) of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}$ such that $(G', A) + (G', A)\tilde{\subseteq}(G, A)$. Hence, $o(t)(G', A) + o(t)(G', A)\tilde{\subseteq}o(t)(G, A)$.
Since $\phi, \psi : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ is fuzzy soft tangent to θ, for any fuzzy soft neighborhood (G', A) of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}, 0 < \delta_\alpha \leq 1$ in (X_2, A, τ_2), there exist fuzzy soft neighborhoods $(F', A), (F'', A)$ of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}, 0 < \lambda_\alpha \leq \delta_\alpha$ in (X_1, A, τ_1) such that $\phi(t(F', A))\tilde{\subseteq}o(t)(G', A)$ and $\psi(t(F'', A))\tilde{\subseteq}o(t)(G', A)$.
Let $(F, A) = ((F', A)\tilde{\cap}(F'', A)$. Then $\phi(t(F, A))\tilde{\subseteq}\, o(t)(G', A)$ and $\psi(t(F, A))\, \tilde{\subseteq}\, o(t)(G', A)$.
Now, $(\phi + \psi)\, (t(F, A)) = \phi(t(F, A)) + \psi(t(F, A))\tilde{\subseteq}\, o(t)(G', A) + o(t)(G', A)\, \tilde{\subseteq}\, o(t)(G, A)$.
This is true for all $\alpha \in A$. Hence proved. □

Proposition 7.2. *Let (X_1, A, τ_1), (X_2, A, τ_2) and (X_3, A, τ_3) be three fuzzy soft topological vector spaces and $\theta, \theta', \theta''$ are null vectors of X_1, X_2 and X_3 respectively. If the function $\phi : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ is fuzzy soft tangent to θ and $f : (X_2, A, \tau_2) \to (X_3, A, \tau_3)$ is linear fuzzy soft continuous, then $f \circ \phi$ is fuzzy soft tangent to θ. Conversely, if $f : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ is linear fuzzy soft continuous mapping and $\phi : (X_2, A, \tau_2) \to (X_3, A, \tau_3)$ is fuzzy soft tangent to θ', then $\phi \circ f$ is fuzzy soft tangent to θ.*

Proof. Choose $\alpha \in A$. By fuzzy soft continuity of f, for every fuzzy soft neighborhood (F, A) of $\tilde{\theta''}_{\alpha}^{\xi_\alpha}, 0 < \xi_\alpha \leq 1$ in (X_3, A, τ_3), there exists a fuzzy soft neighborhood (G, A) of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}, 0 < \delta_\alpha \leq \xi_\alpha$ in (X_2, A, τ_2) such that $f(G, A)\tilde{\subseteq}(F, A)$. For every such (G, A) there is a fuzzy soft neighborhood (H, A) of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}, 0 < \lambda_\alpha \leq \delta_\alpha$ such that $\phi(t(H, A))\tilde{\subseteq}o(t)(G, A)$. Then $f(\phi(t(H, A)))\tilde{\subseteq}f(o(t)(G, A)) = o(t)f(G, A)$ $\tilde{\subseteq}o(t)(F, A)$. This is true for all $\alpha \in A$. Hence $f \circ \phi$ is fuzzy soft tangent to θ. The rest part of the *Proposition* proceeds in a similar way. □

8. Fuzzy soft differentiation

Definition 8.1. Let (X_1, A, τ_1) and (X_2, A, τ_2) be two fuzzy soft topological vector spaces endowed with T_1 fuzzy soft topologies and $f : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ be a fuzzy soft continuous function. Then f is said to be fuzzy soft differentiable at a point $x \in X_1$, if there exists a linear fuzzy soft continuous function $u : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ such that we can write

$f(x + y) = f(x) + u(y) + \phi(y)$, $y \in X_1$, where ϕ is a fuzzy soft tangent to θ. The mapping u is called the fuzzy soft derivative of f at x and denoted by $f'(x)$. Here the function u depends on x.

Example 8.1. *Let (X, ξ) be a fuzzy topological vector space as in Definition 2.14 and θ be the null vector of X. Define fuzzy soft topology τ on X as $\tau = \{(F, A) :$*

$F(\alpha) \in \xi\}$ and soft topology ν on K such that $\nu = \{(G, A) : G(\alpha) \in \zeta\}$. Then ν is the soft usual topology on K. Also τ is a fuzzy soft linear topology on X, by Proposition 6.2 i.e. (X, A, τ) is a fuzzy soft topological vector space. Now for any $k \in K$, define the mapping $U_k : X \to X$ by $U_k(x) = kx$, $x \in X$. Then obviously, $U_k : (X, A, \tau) \to (X, A, \tau)$ is linear fuzzy soft continuous mapping. Also, for any $x \in X$, $U_k(x + y) = U_k(x) + U_k(y) + O(y)$, $y \in X$, where $O : X \to X$ is defined by $O(x) = \theta$, $\forall x \in X$, is a fuzzy soft tangent to θ. So, U_k is fuzzy soft differentiable at every $x \in X$.

Proposition 8.1. *The fuzzy soft derivative of a function $f : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ at a point $x \in X_1$ is unique.*

Proof. Suppose that the derivative of f at a point $x \in X_1$ is not unique. Then there exist two linear fuzzy soft continuous function u_1, u_2 such that
$u_1(y) + \phi(y) = u_2(y) + \psi(y)$, $y \in X_1$, where ϕ, ψ are each fuzzy soft tangent to θ. Set $\eta(y) = u_1(y) - u_2(y)$, $y \in X_1$. Then obviously $\eta : X_1 \to X_2$ is linear function. Then by Proposition 7.1, η is fuzzy soft tangent to θ. By hypothesis η is not zero. Let $a \in X_1$ such that $\eta(a) = r \neq \theta'$. Since (X_2, A, τ_2) is fuzzy soft T_1, for $r \neq \theta'$ and for each $\alpha \in A$ there exist fuzzy soft open set (F, A) such that $[F(\alpha)](r) = 0$, $[F(\alpha)](\theta') = 1$ and $F(\beta) = \bar{1}$, for $\beta \neq \alpha$. If $\mathcal{B}(\tilde{\theta}'^{\delta_\alpha}_\alpha)$ is a fundamental system of balanced fuzzy soft neighborhoods of $\tilde{\theta}'^{\delta_\alpha}_\alpha, 0 < \delta_\alpha \leq 1$, in (X_2, A, τ_2), then there is a $(W, A) \in \mathcal{B}(\tilde{\theta}'^{\delta_\alpha}_\alpha)$ such that $\tilde{\theta}'^{\delta_\alpha}_\alpha \tilde{\in} (W, A) \tilde{\subseteq} (F, A)$ with $(\varepsilon W, A) \tilde{\subseteq} (W, A)$, for all $\mid \varepsilon \mid \leq 1$. If $\xi = \frac{1}{\varepsilon}$ for $\varepsilon \neq 0$, then $[W(\alpha)](\xi r) \leq [W(\alpha)](r) = 0$, for $\mid \xi \mid \geq 1$. It follows that for every r' of the form kr, $k \in K$, $k \neq 0$, there exists ξ such that $[W(\alpha)](\xi r') = 0$. Since η is fuzzy soft tangent to θ, there must be a fuzzy soft neighborhood (V, A) of $\tilde{\theta}^{\lambda_\alpha}_\alpha, 0 < \lambda_\alpha \leq \delta_\alpha$ in (X_1, A, τ_1) such that $\eta(t(V, A)) \tilde{\subseteq} o(t)(W, A) \Rightarrow \eta(V, A) \tilde{\subseteq} \frac{o(t)}{t}(W, A)$, as η is linear. Put $\frac{t}{o(t)} = \xi$, then $\eta(V, A) \tilde{\subseteq} \frac{1}{\xi}(W, A) = (\varepsilon W, A) \tilde{\subseteq} (W, A)$. Then $[W(\alpha)](\xi r') = 0 \Rightarrow [(\eta(V, A))(\alpha)](\xi r') = 0 \Rightarrow \sup_{x \in \eta^{-1}(kr)} [V(\alpha)](x) = 0$. Which implies $[V(\alpha)](ka) = 0$.

But, by Lemma 6.1, a fuzzy soft set (V, A) with $V(\alpha)(ka) = 0$, for all scalars $k \neq 0$ is not a fuzzy soft neighborhood of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$. Hence η must be zero. The fuzzy soft derivative of f at $x \in X_1$ is unique. □

Proposition 8.2. *Let (X_1, A, τ_1), (X_2, A, τ_2) be two fuzzy soft topological vector spaces. Then any constant function $f : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ is fuzzy soft differentiable at every $x \in X_1$.*

Proposition 8.3. *The fuzzy soft derivative of a linear fuzzy soft continuous mapping $u : (X_1, A, \tau_1) \to (X_2, A, \tau_2)$ exists at every $x \in X_1$.*

Proposition 8.4. *Suppose that $(Y, A, \nu) = \prod_{j=1}^{n} (Y_j, A, \nu_j)$ is the product fuzzy soft topological vector space of a finite family of fuzzy soft topological vector spaces (Y_j, A, ν_j), $j = 1, 2, ..., n$, and (X, A, τ) is any fuzzy soft topological vector space. Then a fuzzy soft continuous mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is fuzzy soft differentiable at $x \in X$ if and only if $p_j \circ f$ is fuzzy soft differentiable at x.*

Proof. Let f be fuzzy soft differentiable at x. By linearity of projection mapping p_j we can write for every j, $p_j(f(x + y)) - p_j(f(y)) = p_j(f'(x)(y)) + p_j(\phi(y)),\ y \in X$. Since p_j and f' both are linear and fuzzy soft continuous, $p_j \circ f'$ is linear and fuzzy soft continuous and by Proposition 7.2, $p_j \circ \phi$ is fuzzy soft tangent to θ. Since $f'(x)$ is unique, $p_j \circ f'(x)$ is unique.

Let $p_j \circ f$ be fuzzy soft differentiable at x, for every $j \in \{1, 2, .., n\}$. So, for every j we can write, $p_j(f(x+y)) - p_j(f(y)) = u_j(y) + \phi_j(y)$, where u_j is a linear fuzzy soft continuous mapping and ϕ_j is fuzzy soft tangent to θ. Choose $\alpha \in A$. Let (W, A) be a fuzzy soft neighborhood of $\tilde{\theta'}_{\alpha}^{\lambda_\alpha}, 0 < \lambda_\alpha \leq 1$ in (Y, A, ν). By definition of fuzzy soft product topology, $(W, A) \tilde{\supseteq} \prod_{j=1}^{n} (W_j, A)$, where (W_j, A) are fuzzy soft neighborhoods of $\tilde{\theta'}_{j\alpha}^{\lambda_\alpha}, 0 < \lambda_\alpha \leq 1$ in (Y_j, A, ν_j). By hypothesis, for every (W_j, A) there exists a fuzzy soft neighborhood (V_j, A) of $\tilde{\theta}_{\alpha}^{\delta_\alpha}, 0 < \delta_\alpha \leq \lambda_\alpha$ in (X, A, τ) such that $\phi_j(t(V_j, A)) \tilde{\subseteq} o(t).(W_j, A)$.

Setting $(V, A) = \tilde{\cap}(V_j, A)$ we have $\phi_j(t(V, A))\tilde{\subseteq}o(t).(W_j, A)$, $\forall j$.
Again $o(t)(W, A)\tilde{\supseteq}o(t)\prod_{j=1}^{n}(W_j, A) = \prod_{j=1}^{n}o(t)(W_j, A)$. Set $\phi = \prod_{j=1}^{n}\phi_j$. Then $\phi(t(V, A)) = \prod_{j=1}^{n}\phi_j(t(V, A))\tilde{\subseteq}\prod_{j=1}^{n}o(t).(W_j, A)\tilde{\subseteq}o(t)(W, A)$. This is true for all $\alpha \in A$. Therefore ϕ is fuzzy soft tangent to θ.
Define $u = \prod_{j=1}^{n}u_j$. This mapping is linear and fuzzy soft continuous by linearity and fuzzy soft continuity of the functions u_j. The uniqueness of $f'(x)$ follows by the uniqueness of u_j. □

Proposition 8.5. *Let $(X_1, A, \tau_1), (X_2, A, \tau_2), (X_3, A, \tau_3)$ be three fuzzy soft topological vector spaces, $f : (X_1, A, \tau_1) \rightarrow (X_2, A, \tau_2)$ and $g : (X_2, A, \tau_2) \rightarrow (X_3, A, \tau_3)$ be two fuzzy soft continuous mapping. Let $x \in X_1$ and $y = f(x)$. If f is fuzzy soft differentiable at x and g is fuzzy soft differentiable at y, then the composition $h = g\circ f$ is fuzzy soft differentiable at x.*

Proof. By hypothesis f and g are fuzzy soft differentiable. Hence we can write $f(x + r) = f(x) + (f'(x))(r) + \phi(r)$, $r \in X_1$, and $g(y + s) = g(y) + g'(y)(s) + \psi(s)$, $s \in X_2$, where ϕ is a fuzzy soft tangent to θ and ψ is a fuzzy soft tangent to θ'. Defining $h = g \circ f$, we obtain, after substitution, $h(x + r) - h(x) = g'(y)(f'(x)(r)) + g'(y)(\phi(r)) + \psi(f'(x)(r) + \phi(r))$, $r \in X_1$. By Proposition 7.2, $g'(y) \circ \phi$ is fuzzy soft tangent to θ. Consider the mapping $\psi \circ (f'(x) + \phi)$. Choose $\alpha \in A$. For every fuzzy soft neighborhood (G, A) of $\tilde{\theta''}_{\alpha}^{\xi_\alpha}, 0 < \xi_\alpha \leq 1$, in (X_3, A, τ_3), there is a fuzzy soft neighborhood (F, A) in $\tilde{\theta'}_{\alpha}^{\delta_\alpha}, 0 < \delta_\alpha \leq \xi_\alpha$, in (X_2, A, τ_2) such that $\psi(t(F, A))\tilde{\subseteq}o(t)(G, A)$. Given (F, A) in (X_2, A, τ_2) there exists a fuzzy soft neighborhood (F', A) of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}$ such that $(F', A) + (F', A)\tilde{\subseteq}(F, A)$. We can suppose without loss of generality, that both (F, A) and (F', A) belong to a fundamental system of balanced fuzzy soft neighborhoods $\mathcal{B}(\tilde{\theta'}_{\alpha}^{\delta_\alpha})$, of $\tilde{\theta'}_{\alpha}^{\delta_\alpha}$, in (X_2, A, τ_2). By fuzzy soft continuity of $f'(x)$ there is a fuzzy soft neighborhood (H, A) of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}, 0 < \lambda_\alpha \leq \delta_\alpha$ in (X_1, A, τ_1), such that $f'(x)((H, A))\tilde{\subseteq}(F', A)$, which implies that $tf'(x)((H, A))$ $\tilde{\subseteq}t(F', A)$, i.e.

$f'(x)(t(H,A))\tilde{\subseteq}t(F',A)$. For every (F',A) there exists a fuzzy soft neighborhood (J,A) of of $\tilde{\theta}_{\alpha}^{\lambda_\alpha}$ in (X_1,A,τ_1), such that $\phi(t(J,A))\tilde{\subseteq}o(t)(F',A)$ and for $\mid \frac{o(t)}{t} \mid\leq 1$, $o(t)(F',A)\tilde{\subseteq}t(F',A)$.
Setting $(N,A) = (H,A)\tilde{\cap}(J,A)$ we get $f'(x)(t(N,A))+\phi(t(N,A))\tilde{\subseteq}t(F',A)+t(F',A)$ $\tilde{\subseteq}t(F,A)$ and which implies that $\psi[f'(x)(t(N,A))+\phi(t(N,A))]\tilde{\subseteq}\psi(t(F,A))\tilde{\subseteq}o(t)(G,A)$. This is true for all $\alpha\in A$. Hence the mapping $\psi\circ(f'(x)+\phi)$ is fuzzy soft tangent to θ.
Thus we can write $h(x+r)-h(x) = g'(y)\circ f'(x)(r)+\chi(r), r\in X_1$, where $g'(y)\circ f'(x)$ is linear fuzzy soft continuous and $\chi = (g'(y)\circ\phi)+(\psi\circ(f'(x)+\phi))$, the sum of two mappings which are fuzzy soft tangent to θ, is fuzzy soft tangent to θ. Hence proved. □

Proposition 8.6. *Let $(X_1,A,\tau_1),(X_2,A,\tau_2)$ be two fuzzy soft topological vector space and $f,g:(X_1,A,\tau_1)\to(X_2,A,\tau_2)$ be two fuzzy soft continuous mappings. If f and g are fuzzy soft differentiable at x, so are $f+g$ and kf, $k\in K$.*

Proof. The mapping $f+g$ is composition of $x\to(f(x),g(x))$ from (X_1,A,τ_1) into $(X_2\tilde{\times}X_2,A,\tau_2\tilde{\times}\tau_2)$ and of $(u,v)\to u+v$ from $(X_2\tilde{\times}X_2,A,\tau_2\tilde{\times}\tau_2)$ into (X_2,A,τ_2). The first is fuzzy soft differentiable by Proposition 8.4 and second by the definition of sum; the result follows from Proposition 8.5. For kf it is sufficient to note that the mapping $u\to ku$ of (X_2,A,τ_2) into itself is fuzzy soft differentiable by Proposition 8.3. □

9. Conclusion

This paper may be the starting point for the studies on fuzzy soft topological vector spaces and fuzzy soft differentiation in that spaces. There is a wide scope of studying fuzzy soft topological vector space such as separation properties, normability, open mapping theorem, closed graph theorem etc. Higher order derivatives and other properties of fuzzy soft differentiable functions is a problem of another direction.

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