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SOME TIGHT POLYNOMIAL-EXPONENTIAL LOWER BOUNDS FOR AN EXPONENTIAL FUNCTION

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Abstract. This note is devoted to new sharp lower bounds for $\exp(x^2)$ over the whole real line. We first introduce and study a new lower bound defined with polynomial of degree 2 and exponential (or hyperbolic) functions. Then we propose two improvements of this lower bound by using two different approaches; the first approach consists in adding well-chosen polynomial term to it, whereas the second approach aims to transform it for large values of $|x|$. We show that they are better to well-known lower bounds. The analytic results are supported by some numerical studies and graphics. A part of the study is devoted to some integral methods having the ability to generate new lower bounds for $\exp(x^2)$.

1. Introduction

Inequalities involving exponential functions are useful in all the areas of mathematics. The most famous of them can be found in [12], [5] and [9]. See also [13], [1], [8] and [2] for current developments on lower and upper bound for $\exp(x)$. The purpose of this note is to provide simple and tight lower bounds for $\exp(x^2)$. Such bounds are important tools to evaluate lower or upper bounds of mathematical terms involving $\exp(x^2)$. Basic examples include the functions $\cosh(x^2)$, $\sinh(x^2)$ and $\tanh(x^2)$, the integral $\int_0^x \exp(y^2)dy$, the sum $\sum_{k=0}^{\infty} \exp(-k^2)$, the cumulative distribution function of

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the Gaussian or Kolmogorov distributions and the Marcum Q function (see [10, 11] and [14]). Well-known lower bounds for $\exp(x^2)$ are $\cosh(\sqrt{2}x)$, $\sinh\left(\sqrt{6}x\right)/(\sqrt{6}x)$, $\exp(x) - x$, $1 + x^2 + x^4/2$ and $(1 + x^2/a)^a$ with $a > 0$. The last one is sharp for $|x|$ large only for large values of a, corresponding to a polynomial with a high degree and large coefficients when a is an integer. Recent sharp lower bounds can be found in [3] for x is a an interval of the form $[0, b]$ with a precise value for b.

The motivation of this paper is to introduce new sharp lower bounds for $\exp(x^2)$ defined with simple functions, at least uniformly better to the two benchmarks: $\cosh(\sqrt{2}x)$ and $1 + x^2 + x^4/2$, for all $x \in \mathbb{R}$. In a first part, a first lower bound is introduced. It is defined as an even function on $\mathbb{R}$ with simple polynomial of degree 2 and exponential functions (without power of x^2). Comparison to $\cosh(\sqrt{2}x)$, $\exp(x) - x$ and $1 + x^2 + x^4/2$ are made analytically and with the used of graphics. Then we propose two significant improvements of this lower bound via two different approaches. The first approach aims to add well chosen polynomial terms to the former lower bound. The second approach adopts the transformation suggested in [6]. It consists in weighting and translating the former lower bound when $|x|$ is large enough. Only polynomial of degree 2 and exponential functions are used. In each case, the theoretical results are supported by a short numerical study and some graphics, illustrating the tightness of the new lower bounds. Finally, we present new integration approaches to generate lower bounds for $\exp(x^2)$ using existing lower bounds. The link existing with this approach and the main lower bound of the study is discussed.

The rest of this note is structured as follows. Section 2 presents our main lower bound. Section 3 is devoted to an improvement of this lower bound. Another improvement is developed in Section 4. Applications are given in Section 5. Section 6 presents some integral approaches to determine lower bounds for $\exp(x^2)$. All the proofs of our results are postponed in Section 7.

2. Lower bound

The main lower bound for $\exp(x^2)$ is presented in the proposition below.

Proposition 2.1. *Let us define the function $f(x)$ by*

$$\begin{aligned} f(x) &= \frac{1}{2}\Bigg[\exp\left(\sqrt{2}x\right)(7-4\sqrt{2}x+2x^2) \\ &+ \exp\left(-\sqrt{2}x\right)(7+4\sqrt{2}x+2x^2)-12\Bigg]. \end{aligned} \tag{2.1}$$

Then, for all $x \in \mathbb{R}$,

$$\exp(x^2) \geq f(x) \geq 1.$$

The proof of Proposition 2.1 is based on the study of the function $g(x) = \exp(x^2) - f(x)$ and the inequality $\exp(x^2) \geq \cosh\left(\sqrt{2}x\right)$ for all $x \in \mathbb{R}$.

Let us observe that $f(x)$ is a continuous even function on $\mathbb{R}$ using simple polynomial-exponential functions. It is of the form $f(x) = (1/2)[G(x) + G(-x) - 12]$, where $G(x) = \exp\left(\sqrt{2}x\right)(7 - 4\sqrt{2}x + 2x^2)$. It can be also express in terms of hyperbolic cosh and sinh functions as:

$$f(x) = (7+2x^2)\cosh\left(\sqrt{2}x\right) - 4\sqrt{2}x\sinh\left(\sqrt{2}x\right) - 6. \tag{2.2}$$

A visual comparison between $\exp(x^2)$ and $f(x)$ is performed in Figure 1, for $x \in [-1, 1]$ for the first graphic and $x \in [-2.5, 2.5]$ for the second graphic. We can see that the two curves are relatively close, specially for small value for $|x|$. This comment is also supported by Table 1 which indicates the numerical values of the error measure: $R(b) = \int_{-b}^{b}[f(x) - \exp(x^2)]^2 dx$, for several values for b. The numerical studies are done with the software Mathematica (version 11), see [15].

The tightness of $f(x)$ is highlighted in Proposition 2.2 below; we proves that $f(x)$ is uniformly greater to most well-known simple lower bounds for $\exp(x^2)$: $\cosh\left(\sqrt{2}x\right)$, $\exp(x) - x$, $1 + x^2 + x^4/2$.

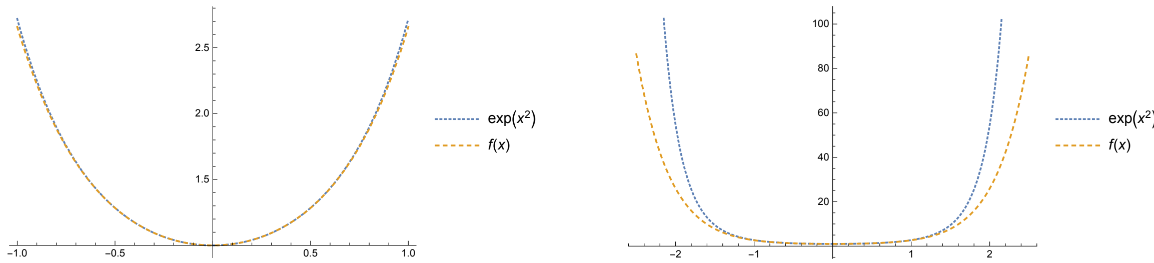

FIGURE 1. Superimposed curves of $\exp(x^2)$ and $f(x)$ for $x \in [-1, 1]$ in the first graphic, and for $x \in [-2.5, 2.5]$ in the second graphic.

TABLE 1. Numerical evaluations of $R(b)$ for $b \in \{0.5, 1, 1.5, 2, 2.5, 3\}$.

	$b = 0.5$	$b = 1$	$b = 1.5$	$b = 2$	$b = 2.5$	$b = 3$
$R(b)$	1.57828×10^{-8}	0.000465217	0.478101	149.8	34020.2	1.02037×10^7

Proposition 2.2. *Let $f(x)$ be the function given by* (2.1). *We have, for all $x \in \mathbb{R}$,*

$$f(x) \geq \max\left[\cosh\left(\sqrt{2}x\right), \exp(x) - x, 1 + x^2 + \frac{x^4}{2}\right].$$

Hence $f(x)$ can be viewed as the best lower bounds among $\cosh\left(\sqrt{2}x\right)$, $\exp(x) - x$ and $1 + x^2 + x^4/2$. Owing to its simple definition, $f(x)$ provides a simple alternative to the sharp but sophisticated lower bound proved by [4], which involves x^2 in the exponent: $m(x; a) = (1 + x^2/a)^{\sqrt{a(a+x^2)}}$ for $a > 0$.

Figure 2 illustrates this result by considering two intervals of values for x : $[-1, 1]$ and $[-3, 3]$, one for each graphics. It shows that $f(x)$ is closer to $\exp(x^2)$ is comparison to $\cosh\left(\sqrt{2}x\right)$, $\exp(x) - x$ and $1 + x^2 + x^4/2$.

Since $\cosh\left(\sqrt{2}x\right) \geq 1$ or $\exp(x) - x \geq 1$ or $1 + x^2 + x^4/2 \geq 1$, Proposition 2.2 implies that $f(x) \geq 1$, which is the second inequality in Proposition 2.1.

If we consider the polynomial of degree 6: $1+x^2+x^4/2+x^6/6$, which is also a loser bound for $\exp(x^2)$, we have $f(x) \geq 1+x^2+x^4/2+x^6/6$ for some x, but their exists x such that the reverse holds. This motivates the study of some improvements of $f(x)$.

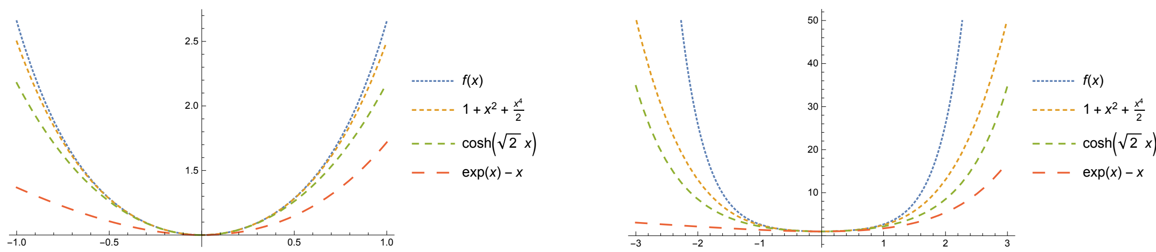

FIGURE 2. Superimposed curves of $f(x)$, $1+x^2+x^4/2$, $\cosh\left(\sqrt{2}x\right)$ and $\exp(x)-x$ for $x \in [-1,1]$ in the first graphic, and for $x \in [-3,3]$ in the second graphic.

In Sections 3 and 4 below, two modifications are proposed: adding a well-chosen polynomial term to $f(x)$ or transforming $f(x)$ for $|x|$ large.

Remark 1. Let us mention that some continuous even upper bounds for $\exp(x^2)$ using hyperbolic cosh and sinh functions can be proved. An example with $|x|$ bounded is the following: For any $a>0$ and $|x| \leq a$, we have $\exp(x^2) \leq \cosh(ax) + (x/a)\sinh(ax)$ ($\leq \exp(a|x|)$). Further details on this inequality is provided in Section 7.

3. First improvement of the lower bound

First of all, let us consider an intermediary result which can be viewed as an improvement of the well-known inequality: for all $x \in \mathbb{R}$, $\exp(x^2) \geq \cosh\left(\sqrt{2}x\right)$.

Lemma 3.1. *For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we define the polynomial $P_n(x)$ by*

$$P_n(x) = 2x^4 \sum_{k=0}^{n} \frac{x^{2k}}{k!(2k+3)(k+2)}$$

Then, for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\exp(x^2) \geq \cosh\left(\sqrt{2}x\right) + P_n(x).$$

Since $P_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, it is clear that $\exp(x^2) \geq \cosh\left(\sqrt{2}x\right) + P_n(x) \geq \cosh\left(\sqrt{2}x\right)$. Let us observe that $P_n(x)$ satisfies the recurrence relation:

$P_n(x) = P_{n-1}(x) + 2x^{2(n+2)}/(n!(2n+3)(n+2))$, with $P_0(x) = x^4/3$. Expressions of $P_n(x)$ for several values of n are given in Table 2.

TABLE 2. Analytic expressions for $P_n(x)$ with $n \in \{0, \ldots, 5\}$.

	$n=0$	$n=1$	$n=2$	$n=3$
$P_n(x)$	$\frac{1}{3}x^4$	$\frac{1}{3}x^4 + \frac{2}{15}x^6$	$\frac{1}{3}x^4 + \frac{2}{15}x^6 + \frac{1}{28}x^8$	$\frac{1}{3}x^4 + \frac{2}{15}x^6 + \frac{1}{28}x^8 + \frac{1}{135}x^{10}$

$n=4$	$n=5$
$\frac{1}{3}x^4 + \frac{2}{15}x^6 + \frac{1}{28}x^8 + \frac{1}{135}x^{10} + \frac{1}{792}x^{12}$	$\frac{1}{3}x^4 + \frac{2}{15}x^6 + \frac{1}{28}x^8 + \frac{1}{135}x^{10} + \frac{1}{792}x^{12} + \frac{1}{5460}x^{14}$

Lemma 3.1 is a key tool to the proof of the following proposition in which we determine a tight lower bound of $\exp(x^2)$ uniformly greater to $f(x)$.

Proposition 3.1. *For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we define the polynomial $Q_n(x)$ by*

$$Q_n(x) = 4x^6 \sum_{k=0}^{n} \frac{x^{2k}}{k!(2k+3)(k+2)} \left(\frac{1}{2(2k+5)(k+3)} + \frac{x^2}{(k+4)(2k+7)} \right).$$

Let $f(x)$ be the function given by (2.1). *We define the function $f_*(x;n)$ by*

$$f_*(x;n) = f(x) + Q_n(x).$$

Then, for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\exp(x^2) \geq f_*(x;n) \geq f(x).$$

Thus $f_*(x;n)$ is a better lower to $f(x)$ for $\exp(x^2)$, for all $x \in \mathbb{R}$. Remark that $Q_n(x)$ satisfies the recurrence relation: $Q_n(x) = Q_{n-1}(x) + 4x^{2(n+3)}/(n!(2n+3)(n+2))[1/(2(2n+5)(n+3)) + x^2/((n+4)(2n+7))]$, with $Q_0(x) = x^6/45 + x^8/42$. Expressions of $Q_n(x)$ for several values of n are given in Table 3.

The result of Proposition 3.1 is illustrated in Figure 3. The two graphics consider the two intervals for x: $[-1,1]$ and $[-2.5, 2.5]$.

TABLE 3. Analytic expressions for $Q_n(x)$ with $n \in \{0, \dots, 4\}$.

	$n = 0$	$n = 1$	$n = 2$
$Q_n(x)$	$\frac{1}{45}x^6 + \frac{1}{42}x^8$	$\frac{1}{45}x^6 + \frac{1}{35}x^8 + \frac{4}{675}x^{10}$	$\frac{1}{45}x^6 + \frac{1}{35}x^8 + \frac{127}{18900}x^{10} + \frac{1}{924}x^{12}$

$n = 3$
$\frac{1}{45}x^6 + \frac{1}{35}x^8 + \frac{127}{18900}x^{10} + \frac{149}{124740}x^{12} + \frac{2}{12285}x^{14}$

$n = 4$
$\frac{1}{45}x^6 + \frac{1}{35}x^8 + \frac{127}{18900}x^{10} + \frac{149}{124740}x^{12} + \frac{191}{1081080}x^{14} + \frac{1}{47520}x^{16}$

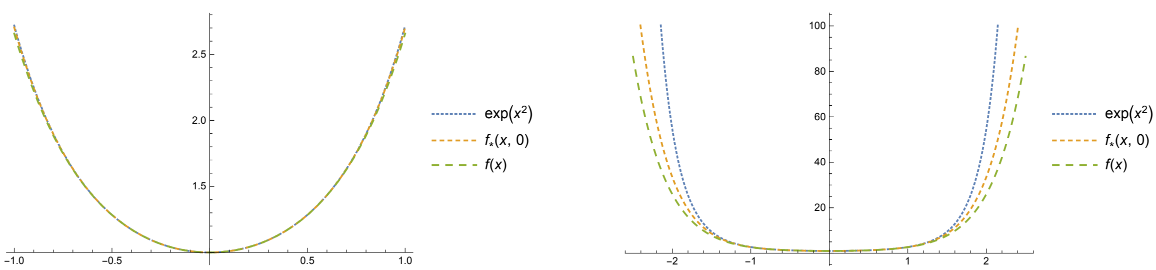

FIGURE 3. Superimposed curves of $\exp(x^2)$, $f_*(x;n)$ for $n = 0$ and $f(x)$ for $x \in [-1, 1]$ in the first graphic, and for $x \in [-2.5, 2.5]$ in the second graphic.

Table 4 shows the numerical values of the error measure:
$R_*(b) = \int_{-b}^{b} [f_*(x;n) - \exp(x^2)]^2 \, dx$, for $n = 0$ and several values for b.

TABLE 4. Numerical evaluations of $R_*(b)$ for $b \in \{0.5, 1, 1.5, 2, 2.5, 3\}$.

	$b = 0.5$	$b = 1$	$b = 1.5$	$b = 2$	$b = 2.5$	$b = 3$
$R_*(b)$	4.89107×10^{-11}	0.0000226587	0.1001	73.0819	26273.5	9.5921×10^{6}

4. Second improvement of the lower bound

We now investigate a transformation of $f(x)$ for large $|x|$, based on a multiplicative exponential weight and translation. It is an adaptation of the method developed by [6] to $f(x)$.

Proposition 4.1. *Let $f(x)$ be the function given by (2.1). For any $a \geq 0$, we define the function $f(x;a)$ by*

$$f_\circ(x;a) = f(x)\mathbf{1}_{\{|x|<a/2\}}(x) + f(|x|-a)\exp(2a|x|-a^2)\mathbf{1}_{\{|x|\geq a/2\}}(x),$$

where $\mathbf{1}_A(x)$ denotes the indicator function over A, i.e. $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 elsewhere. Then, for all $a \geq 0$ and $x \in \mathbb{R}$, we have

$$\exp(x^2) \geq f_\circ(x;a) \geq f(x).$$

Another look of the function $f_\circ(x;a)$ is given by

$$f_\circ(x;a) = \max\left[f(x), f(|x|-a)\exp(2a|x|-a^2)\right].$$

Note that $f_\circ(x;a)$ is an even continuous function on $\mathbb{R}$. It follows from Proposition 4.1 that $f_\circ(x;a)$ is a better lower to $f(x)$ for $\exp(x^2)$, for all $a \geq 0$ and $x \in \mathbb{R}$. Figure 4 proposes a graphical illustration of Proposition 4.1. The two graphics consider the two intervals respectively: $[-1,1]$ and $[-3,3]$. We see that $\exp(x^2)$ and $f_\circ(x,a)$ with $a = 1$ are near confounded for the considered values for x, showing the sharpness of the lower bound.

Owing to Proposition 2.2, defining with the same a, this lower bound is sharper to the lower bounds exhibited in [6]. However, due to the complexity of $f(x)$, it is more complicated from a mathematical point of view.

Table 5 shows the numerical values of the error measure:
$R_\circ(b) = \int_{-b}^{b} \left[f_\circ(x;a) - \exp(x^2)\right]^2 dx$, for $a = 1$ and several values for b.

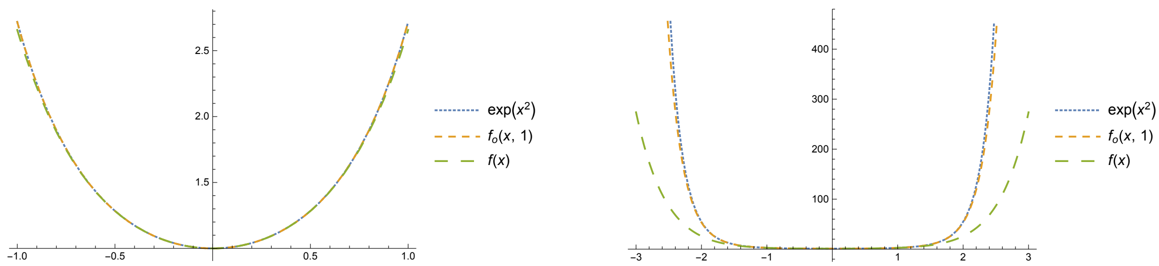

FIGURE 4. Superimposed curves of $\exp(x^2)$, $f_\circ(x;a)$ for $a=1$ and $f(x)$ for $x\in[-1,1]$ in the first graphic, and for $x\in[-2.5,2.5]$ in the second graphic.

TABLE 5. Numerical evaluations of $R_\circ(b)$ for $b\in\{0.5,1,1.5,2,2.5,3\}$.

	$b=0.5$	$b=1$	$b=1.5$	$b=2$	$b=2.5$	$b=3$
$R_\circ(b)$	1.57828×10^{-8}	3.39908×10^{-8}	7.93014×10^{-7}	0.151264	1081.86	2.42278×10^{6}

The value $a=1$ in the numerical study is arbitrary chosen; one can find a more optimal value for the problem for a given criteria of optimization.

A comparison of the two lower bounds $f_*(x;n)$ and $f_\circ(x;a)$ is now discussed. When $|x|$ is small, say $|x|<a/2$ for a fixed a, we have $f_*(x;n)\geq f(x)=f_\circ(x;a)$, so $f_*(x;n)$ is better. When $|x|$ is large, in view of Figure 4, and Tables 4 and 5, we claim that $f_\circ(x;n)$ is better for some a and n.

5. Applications

Let us now present some direct applications of our results. First of all, sharp polynomial-exponential lower bounds for $\exp(x^2)$ give sharp polynomial-exponential lower bounds for $\cosh(x^2)$ or $\sinh(x^2)$; using the inequality $\exp(y)\geq 1+y$, for all $y\in\mathbb{R}$, for any $\psi(x)\in\{f(x),f_*(x;n),f_\circ(x;a)\}$, for all $x\in\mathbb{R}$, we have

$$\cosh(x^2)=\frac{\exp(x^2)+\exp(-x^2)}{2}\geq\frac{1}{2}\left(\psi(x)+\max(1-x^2,0)\right).$$

On the other hand, using the inequality: $\exp(-y) \leq 1 - y + y^2/2$, for all $y \geq 0$, for any $x \in \mathbb{R}$, we have

$$\sinh(x^2) = \frac{\exp(x^2) - \exp(-x^2)}{2} \geq \frac{1}{2}\left(\psi(x) - \max\left(1 - x^2 + \frac{x^4}{2}, 1\right)\right).$$

These inequalities can be useful in various mathematical settings.

Another example concerns integrals involving $\exp(-x^2)$: for any positive integrable function $\phi(x)$ and any $\psi(x) \in \{f(x), f_*(x;n), f_\circ(x;a)\}$, for all $x \in \mathbb{R}$ and $a > 0$, we have

$$\int_x^{\infty} \phi(t)\exp(-at^2)dt \leq \int_x^{\infty} \phi(t)[\psi(\sqrt{a}t)]^{-1}dt.$$

For instance, this can be used to bound the error function $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} \exp(-t^2)dt$, or functions involving $\operatorname{erfc}(x)$ since

$$\operatorname{erfc}(x) \leq \min\left(\exp(-x^2), \frac{1}{6}\exp(-x^2) + \frac{1}{2}\exp(-\frac{4}{3}x^2)\right)$$

(see [7]), or the Marcum Q function (see [10, 11] and [14]).

6. On some generators of lower bounds

We now present and discuss some general approaches based on integration to generate new lower bounds for $\exp(x^2)$ from existing lower bounds. When it is possible, conditions are mentioned to improved the tightness of the former lower bounds.

6.1. **First integral approach.** The main result is described in the proposition below.

Proposition 6.1. *Let $\theta(x)$ be a positive function on $\mathbb{R}$ and $\omega(x)$ be the function defined by*

$$\omega(x) = \int_0^{|x|}\left(\int_0^{y} 2(1+2t^2)\theta(t)dt\right)dy + 1.$$

- *Suppose that $\exp(x^2) \geq \theta(x)$ for all $x \in \mathbb{R}$. Then $\exp(x^2) \geq \omega(x)$ for all $x \in \mathbb{R}$.*

- *Suppose that $\theta(x)$ is even, two times differentiable with $\theta(0) = 1$, $\theta'(0) = 0$ and $2(1 + 2x^2)\theta(x) - \theta''(x) \geq 0$ for all $x \geq 0$. Then, for all $x \in \mathbb{R}$, we have*

$$\omega(x) \geq \theta(x).$$

Under the assumptions of the two points above, we have

$$\exp(x^2) \geq \omega(x) \geq \theta(x).$$

So $\omega(x)$ is a better lower bound to $\theta(x)$.

Connections between Proposition 6.1 and 2.1 exist. Indeed, let us consider the well-known lower bound for $\exp(x^2)$: $\theta(x) = \cosh\left(\sqrt{2}x\right)$. We have $\theta'(x) = \sqrt{2}\sinh\left(\sqrt{2}x\right)$ and $\theta''(x) = 2\cosh\left(\sqrt{2}x\right)$. Therefore $\theta(0) = 1$, $\theta'(0) = 0$ and $2(1+2x^2)\theta(x) - \theta''(x) = 4x^2\cosh\left(\sqrt{2}x\right) \geq 0$. It follows from Proposition 6.1 that a better lower bound of $\theta(x) = \cosh\left(\sqrt{2}x\right)$ is given by

$$\begin{aligned}
\omega(x) &= \int_0^{|x|}\left(\int_0^y 2(1+2t^2)\theta(t)dt\right)dy + 1 \\
&= \int_0^{|x|}\left(-4y\cosh\left(\sqrt{2}y\right) + 3\sqrt{2}\sinh\left(\sqrt{2}y\right) + 2\sqrt{2}y^2\sinh\left(\sqrt{2}y\right)\right)dy + 1 \\
&= (7+2x^2)\cosh\left(\sqrt{2}x\right) - 4\sqrt{2}x\sinh\left(\sqrt{2}x\right) - 6.
\end{aligned}$$

We thus obtain the hyperbolic expression of the lower bound $f(x)$ given by (2.2).

Naturally, the first point of Proposition 6.1 can be used to generate new lower bounds for $\exp(x^2)$. For instance, using the inequality $\exp(y) \geq 1 + y$ for all $y \in \mathbb{R}$, we have $\exp(x^2) = \exp(x^2 - |x|)\exp(|x|) \geq (1 + x^2 - |x|)\exp(|x|)$. Let us set $\theta(x) = (1 + x^2 - |x|)\exp(|x|)$. Hence a new lower bound for $\exp(x^2)$ is given by

$$\begin{aligned}
\omega(x) &= \int_0^{|x|}\left(\int_0^y 2(1+2t^2)\theta(t)dt\right)dy + 1 \\
&= \int_0^{|x|}\left[\exp(y)\left(136 - 134y + 66y^2 - 20y^3 + 4y^4\right) - 136\right]dy + 1 \\
&= \exp(|x|)\left(618 - 482|x| + 174x^2 - 36|x|^3 + 4x^4\right) - 617 - 136|x|.
\end{aligned}$$

However, note that this lower bound is not better to $\theta(x)$. In particular, the assumption $2(1+2x^2)\theta(x)-\theta''(x)\geq 0$ for all $x\in\mathbb{R}$ is not satisfied. This bound is tight but not shaper than $f(x)$ for all $x\in\mathbb{R}$. Moreover, from a mathematical point of view, it is more difficult to manipulate to $f(x)$.

Let us mention that the well-known lower bounds: $1+x^2+x^4/2$ and $\exp(|x|)-|x|$, also satisfy $\theta(0)=1$, $\theta'(0)=0$ and $2(1+2x^2)\theta(x)-\theta''(x)\geq 0$ for all $x\geq 0$, yielding more sharp lower bounds $\omega(x)$ for $\exp(x^2)$. However, one can show that they are not better to $f(x)$ for all $x\in\mathbb{R}$ (and the presented improvements).

6.2. **Generalization.** Proposition 6.2 below presents a generalization of Proposition 6.1. From two lower bounds $\theta_1(x)$ and $\theta_2(x)$ of $\exp(x^2)$, one can construct a lower bound better to $\theta_1(x)$ or $\theta_2(x)$, under some assumptions.

Proposition 6.2. *Let $\theta_1(x)$ and $\theta_2(x)$ be two positive functions on $\mathbb{R}$ and $\kappa(x)$ be the function defined by*

$$\kappa(x)=\int_0^{|x|}\left(\int_0^y 2(\theta_1(t)+2t^2\theta_2(t))dt\right)dy+1.$$

- *Suppose that $\exp(x^2)\geq\max\left[\theta_1(x),\theta_2(x)\right]$ for all $x\in\mathbb{R}$. Then $\exp(x^2)\geq\kappa(x)$ for all $x\in\mathbb{R}$.*
- *Suppose that $\theta_1(x)$ and $\theta_2(x)$ are even, two times differentiable with*
  - *$\theta_1(0)=1$, $\theta_1'(0)=0$, $2(\theta_1(x)+2x^2\theta_2(x))-\theta_1''(x)\geq 0$ for all $x\geq 0$. Then, for all $x\in\mathbb{R}$, we have*

    $$\kappa(x)\geq\theta_1(x).$$

    This inequality holds with $\theta_2(x)$ by inverting the role of $\theta_1(x)$ and $\theta_2(x)$ in the definition of $\kappa(x)$ and the conditions.
  - *$\theta_1(0)=1$, $\theta_2(0)=1$, $\theta_1'(0)=0$, $\theta_2'(0)=0$, $2(\theta_1(x)+2x^2\theta_2(x))-\theta_1''(x)\geq 0$ and $2(\theta_1(x)+2x^2\theta_2(x))-\theta_2''(x)\geq 0$ for all $x\geq 0$. Then, for all $x\in\mathbb{R}$,*

we have

$$\kappa(x) \geq \max\left[\theta_1(x), \theta_2(x)\right].$$

Under the assumptions of the first point and the second item of the second point, we have

$$\exp(x^2) \geq \kappa(x) \geq \max\left[\theta_1(x), \theta_2(x)\right].$$

So $\kappa(x)$ *is a better lower bound to* $\theta_1(x)$ *for* $\exp(x^2)$*, or both of them.*

Taking $\theta_1(x) = \theta_2(x)$ in Proposition 6.2, we obtain Proposition 6.1 with $\theta(x) = \theta_1(x)$.

Note that, taking $\theta_1(x) = \cosh\left(\sqrt{2}x\right)$ and $\theta_2(x) = \sum_{k=0}^{n} x^{2k}/(k!)$, the first point and the first item of the second point of Proposition 6.2 are satisfied; we thus obtain Lemma 3.1.

Another example is given by chosing $\theta_1(x) = f(x)$ and $\theta_2(x) = \cosh\left(\sqrt{2}x\right)$. After some calculus, we have

$$\kappa(x) = 27\cosh\left(\sqrt{2}x\right) - 2x^2\left(3 - 2\cosh\left(\sqrt{2}x\right)\right) - 12\sqrt{2}x\sinh\left(\sqrt{2}x\right) - 26.$$

Also, one can show that the first point and the second item of the second point of Proposition 6.2 are satisfied; $\kappa(x)$ is a better lower bound to $f(x)$. However, the comparison with the proposed improvements for $f(x)$ need further investigations.

6.3. **Another integral approach.** We conclude this section by presenting another generator of lower bounds for $\exp(x^2)$.

Proposition 6.3. *Let* $\theta(x)$ *be a positive function on* $\mathbb{R}$ *and* $\gamma(x)$ *be the function defined by*

$$\gamma(x) = \exp(-|x|)\int_0^{|x|} \exp(t)\left[(1+2t)\theta(t) - 1\right]dt + 1.$$

If $\exp(x^2) \geq \theta(x)$ *for all* $x \in \mathbb{R}$*. Then* $\exp(x^2) \geq \omega(x)$ *for all* $x \in \mathbb{R}$*.*

For instance, if $\theta(x) = \cosh\left(\sqrt{2}x\right)$, we have

$$\gamma(x) = 8\exp(-|x|) - (7 + 2|x|)\cosh\left(\sqrt{2}x\right) + \sqrt{2}(5 + 2|x|)\sinh\left(\sqrt{2}|x|\right).$$

However, one can show that $\gamma(x)$ is not better to $f(x)$ for all $x \in \mathbb{R}$.

7. Proofs

Proof of Proposition 2.1. Let us set $g(x) = \exp(x^2) - f(x)$. We aim to study this function. It follows from several algebraic manipulations that

$$\begin{aligned} g'(x) &= 2x\exp(x^2) \\ &- \frac{1}{2}\left[\exp\left(\sqrt{2}x\right)(3\sqrt{2} - 4x + 2\sqrt{2}x^2) - \exp\left(-\sqrt{2}x\right)(3\sqrt{2} + 4x + 2\sqrt{2}x^2)\right]. \end{aligned} \tag{7.1}$$

In order to study the sign of $g'(x)$, let us investigate $g''(x)$. Algebraic manipulations and simplifications give

$$g''(x) = 2(1 + 2x^2)\left[\exp(x^2) - \cosh\left(\sqrt{2}x\right)\right]. \tag{7.2}$$

Owing to the elementary inequality: $\exp(x^2) \geq \cosh\left(\sqrt{2}x\right)$, we have $g''(x) \geq 0$. Thus $g'(x)$ is increasing. Since $g'(0) = 0$ and $g(0) = 0$, we have $g(x) \geq 0$ for all $x \in \mathbb{R}$.

The second inequality can be prove in a similar manner. Let us set $h(x) = f(x) - 1$. It follows from several algebraic manipulations that

$$h'(x) = \frac{1}{2}\left[\exp\left(\sqrt{2}x\right)(3\sqrt{2} - 4x + 2\sqrt{2}x^2) - \exp\left(-\sqrt{2}x\right)(3\sqrt{2} + 4x + 2\sqrt{2}x^2)\right]$$

and

$$h''(x) = 2(1 + 2x^2)\cosh\left(\sqrt{2}x\right).$$

So $h''(x) \geq 0$. Since $h'(0) = 0$ and $h(0) = 0$, we have $h(x) \geq 0$ for all $x \in \mathbb{R}$. This ends the proof of Proposition 2.1. □

Proof of Proposition 2.2. Let us prove that, for all $x \in \mathbb{R}$, $f(x) \geq \cosh\left(\sqrt{2}x\right)$, $f(x) \geq \exp(x) - x$ and $f(x) \geq 1 + x^2 + x^4/2$, in turn.

- Proof for $f(x) \geq \cosh\left(\sqrt{2}x\right)$. Let us set $k(x) = f(x) - \cosh\left(\sqrt{2}x\right)$. After calculus and simplifications, we obtain

$$k'(x) = (\sqrt{2} - 2x + \sqrt{2}x^2) \exp\left(\sqrt{2}x\right) - (\sqrt{2} + 2x + \sqrt{2}x^2) \exp\left(-\sqrt{2}x\right)$$

and

$$k''(x) = 4 \cosh\left(\sqrt{2}x\right) x^2.$$

Since $k''(x) \geq 0$, $k'(0) = 0$ and $k(0) = 0$, we have $k(x) \geq 0$ for all $x \in \mathbb{R}$, implying the desired inequality.

- Proof for $f(x) \geq \exp(x) - x$. Let us set $\ell(x) = f(x) - \exp(x) + x$. We have

$$\begin{aligned} \ell'(x) &= \frac{1}{2}\Bigg[\exp\left(\sqrt{2}x\right)(3\sqrt{2} - 4x + 2\sqrt{2}x^2) \\ &- \exp\left(-\sqrt{2}x\right)(3\sqrt{2} + 4x + 2\sqrt{2}x^2) - 2\exp(x) + 2\Bigg] \end{aligned}$$

and

$$\ell''(x) = 2(1 + 2x^2) \cosh\left(\sqrt{2}x\right) - \exp(x).$$

Observe that $\ell''(x) \geq 2\cosh\left(\sqrt{2}x\right) - \exp(x) \geq 0$. Since $\ell'(0) = 0$ and $\ell(0) = 0$, we have $\ell(x) \geq 0$ for all $x \in \mathbb{R}$, ending the proof of this point.

- Proof for $f(x) \geq 1 + x^2 + x^4/2$. Let us set $m(x) = f(x) - (1 + x^2 + x^4/2)$. After calculus and simplifications, we obtain

$$\begin{aligned} m'(x) &= \frac{1}{2}\Bigg[\exp\left(\sqrt{2}x\right)(3\sqrt{2} - 4x + 2\sqrt{2}x^2) \\ &- \exp\left(-\sqrt{2}x\right)(3\sqrt{2} + 4x + 2\sqrt{2}x^2) - 4x(1 + x^2)\Bigg] \end{aligned}$$

and

$$m''(x) = 2(1 + 2x^2) \cosh\left(\sqrt{2}x\right) - 2 - 6x^2.$$

Since $\cosh\left(\sqrt{2}x\right) = \sum_{k=0}^{+\infty}\left(\sqrt{2}x\right)^{2k}/(2k)! > 1 + x^2$ (see [12]), we have $m''(x) \geq 2(1+2x^2)(1+x^2) - 2 - 6x^2 = 4x^4 \geq 0$. Since $m'(0) = 0$ and $m(0) = 0$, we have $m(x) \geq 0$ for all $x \in \mathbb{R}$. This implies the result.

The proof of Proposition 2.2 is complete. □

On Remark 1. Let us set $\rho(x) = (1/2)[1 + (x/a)]$. Since $|x| \leq a$, observe that $\rho(x) \in [0,1]$. Also, we can write $x^2 = \rho(x)(ax) + (1-\rho(x))(-ax)$. Owing to the convexity of the function $\exp(y)$, we have

$$\begin{aligned}\exp(x^2) &= \exp\left[\rho(x)(ax) + (1-\rho(x))(-ax)\right] \\ &\leq \rho(x)\exp(ax) + (1-\rho(x))\exp(-ax) = \cosh(ax) + \frac{x}{a}\sinh(ax).\end{aligned}$$

This completes Remark 1. □

Proof of Lemma 3.1. Let us set $p(x) = \exp(x^2) - \cosh\left(\sqrt{2}x\right) - P_n(x)$. Then we have

$$p'(x) = 2x\exp(x^2) - \sqrt{2}\sinh\left(\sqrt{2}x\right) - 4\sum_{k=0}^{n}\frac{x^{2k+3}}{k!(2k+3)}$$

and

$$\begin{aligned}p''(x) &= 2\exp(x^2) + 4x^2\exp(x^2) - 2\cosh\left(\sqrt{2}x\right) - 4x^2\sum_{k=0}^{n}\frac{x^{2k}}{k!} \\ &= 2\left(\exp(x^2) - \cosh\left(\sqrt{2}x\right)\right) + 4x^2\left(\exp(x^2) - \sum_{k=0}^{n}\frac{x^{2k}}{k!}\right).\end{aligned}$$

It follows from the well-know inequalities: $\exp(x^2) \geq \cosh\left(\sqrt{2}x\right)$ and $\exp(x^2) = \sum_{k=0}^{+\infty} x^{2k}/k! > \sum_{k=0}^{n} x^{2k}/k!$ (see [12]) that $p''(x) \geq 0$. Noticing that $p'(0) = 0$ and $p(0) = 0$, we have $p(x) \geq 0$ for all $x \in \mathbb{R}$. This concludes the proof of Lemma 3.1. □

Proof of Proposition 3.1. Set $g(x) = \exp(x^2) - f(x)$ and $q(x) = \exp(x^2) - f_*(x;n) = g(x) - Q_n(x)$. Let us recall that $g'(x)$ and $g''(x)$ have been determined in (7.1) and

(7.2) respectively. We have

$$q'(x) = g'(x) - 4x^5 \sum_{k=0}^{n} \frac{x^{2k}}{k!(2k+3)(k+2)} \left(\frac{1}{2k+5} + \frac{2x^2}{2k+7} \right)$$

and, by (7.2),

$$\begin{aligned} q''(x) &= g''(x) - 4(1+2x^2)x^4 \sum_{k=0}^{n} \frac{x^{2k}}{k!(2k+3)(k+2)} \\ &= 2(1+2x^2) \left[\exp(x^2) - \cosh\left(\sqrt{2}x\right) - 2x^4 \sum_{k=0}^{n} \frac{x^{2k}}{k!(2k+3)(k+2)} \right]. \end{aligned}$$

It follows from Lemma 3.1 that $q''(x) \geq 0$. Owing to $q'(0) = 0$ and $q(0) = 0$, we have $q(x) \geq 0$ for all $x \in \mathbb{R}$. So $\exp(x^2) \geq f_*(x; n)$. Since $Q_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, it is clear that $f_*(x; n) \geq f(x)$. The proof of Proposition 3.1 is complete. □

Proof of Proposition 4.1. We follow the approach to the proof of [6, Proposition 1]. Using $x^2 = (|x| - a)^2 + 2a|x| - a^2$ and Proposition 2.1, we have

$$\exp(x^2) = \exp((|x| - a)^2) \exp(2a|x| - a^2) \geq f(|x| - a) \exp(2a|x| - a^2).$$

Let us set $v(x; a) = f(|x| - a) \exp(2a|x| - a^2)$. Hence we have $\exp(x^2) \geq \max[f(x), v(x; a)]$. We will now show that $v(x; a)$ can be a better lower bound to $f(x)$ for some x. Let us study it according to the variable a. After several algebraic calculus and simplifications, we obtain

$$\frac{\partial}{\partial a} v(x; a) = \exp\left(2a|x| - a^2\right) w(|x| - a),$$

where $w(y)$ is the function defined by

$$\begin{aligned} w(y) &= \frac{1}{2} \left[\exp\left(-\sqrt{2}y\right) (3\sqrt{2} + 18y + 10\sqrt{2}y^2 + 4y^3) \right. \\ &+ \left. \exp\left(\sqrt{2}y\right) (-3\sqrt{2} + 18y - 10\sqrt{2}y^2 + 4y^3) \right] - 12y, \end{aligned}$$

Since $\exp(2a|x| - a^2) > 0$, we need to determine the sign of $w(|x| - a)$. Let us show that $w(y)$ is increasing on $\mathbb{R}$. After several algebraic calculus and simplifications, we have

$$\begin{aligned} w'(y) &= \exp\left(-\sqrt{2}y\right)(6 + \sqrt{2}y - 4y^2 - 2\sqrt{2}y^3) \\ &+ \exp\left(\sqrt{2}y\right)(6 - \sqrt{2}y - 4y^2 + 2\sqrt{2}y^3) - 12, \end{aligned}$$

$$\begin{aligned} w''(y) &= \exp\left(-\sqrt{2}y\right)(-5\sqrt{2} - 10y - 2\sqrt{2}y^2 + 4y^3) \\ &+ \exp\left(\sqrt{2}y\right)(5\sqrt{2} - 10y + 2\sqrt{2}y^2 + 4y^3) \end{aligned}$$

and

$$w'''(y) = 2y\left[\exp\left(-\sqrt{2}y\right)(3\sqrt{2} + 8y - 2\sqrt{2}y^2) + \exp\left(\sqrt{2}y\right)(-3\sqrt{2} + 8y + 2\sqrt{2}y^2)\right].$$

Let us observe that this last function is even and of the form $w'''(y) = 2y\,(R(y) - R(-y))$, with $R(y) = \exp\left(\sqrt{2}y\right)(-3\sqrt{2} + 8y + 2\sqrt{2}y^2)$. So it is enough to study its sign on $[0, \infty)$, which corresponds to the sign of $R(y) - R(-y)$ on $[0, \infty)$. We have, for all $y \geq 0$,

$$\begin{aligned} &(R(y) - R(-y))' \\ &= 2\left[\exp\left(-\sqrt{2}y\right)(1 - 6\sqrt{2}y + 2y^2) + \exp\left(\sqrt{2}y\right)(1 + 6\sqrt{2}y + 2y^2)\right] \\ &= 4(1 + 2y^2)\cosh\left(\sqrt{2}y\right) + 24\sqrt{2}y\sinh\left(\sqrt{2}y\right) \geq 0. \end{aligned}$$

So $R(y) - R(-y)$ is increasing on $[0, \infty)$, we have $R(y) - R(-y) \geq R(0) - R(-0) = 0$ for all $y \geq 0$, and, a fortiori, $w'''(y) = 2y\,(R(y) - R(-y)) \geq 0$ for all $y \geq 0$. Since $w'''(y)$ is even, we have $w'''(y) \geq 0$ for all $y \in \mathbb{R}$. This implies that $w''(y)$ is increasing on $\mathbb{R}$. Since $w''(0) = 0$, $w'(y)$ is decreasing $(-\infty, 0]$ and increasing on $(0, \infty)$. So $w'(y) \geq w'(0) = 0$ implying that $w(y)$ is increasing on $\mathbb{R}$.

It follows from this result that, if $a < |x|$, we have $w(|x| - a) > w(0) = 0$, implying that $\frac{\partial}{\partial a} v(x; a) > 0$, so $v(x; a)$ is strictly increasing according to a. If $a > |x|$, we have

$w(|x|-a) < w(0) = 0$, implying that $\frac{\partial}{\partial a}v(x;a) < 0$, so $v(x;a)$ is strictly decreasing according to a. Clearly, we have $\frac{\partial}{\partial a}v(x;a) = 0$ if and only if $a = |x|$. Since $f(x)$ is an even function and $v(x;0) = v(x;2|x|) = f(x)$, for all $|x| \geq a/2$, we have

$$\begin{aligned}\exp(x^2) &= v(x;|x|) = \sup_{a\in[0,2|x|]} v(x;a) \geq v(x;a) \geq \inf_{a\in[0,2|x|]} v(x;a)\\ &= \inf\left[v(x;0), v(x;2|x|)\right] = f(x).\end{aligned}$$

For $|x| < a/2$, we have

$$f(x) = \inf_{a\in[0,2|x|]} v(x;a) \geq \sup_{a>2|x|} v(x;a) \geq v(x;a).$$

Hence, for all $x \in \mathbb{R}$, we have

$$\begin{aligned}\exp(x^2) &\geq \max\left[f(x), v(x;a)\right] = f(x)\mathbf{1}_{\{|x|<a/2\}}(x) + v(x;a)\mathbf{1}_{\{|x|\geq a/2\}}(x)\\ &= f_\circ(x;a) \geq f(x).\end{aligned}$$

Proposition 4.1 is proved. □

Proof of Proposition 6.1. Let us prove the two points in turn.

- Let us remark that, by two successive integrations, we have

$$\int_0^{|x|}\left(\int_0^y 2(1+2t^2)\exp(t^2)dt\right)dy = \int_0^{|x|} 2y\exp(y^2)dy = \exp(x^2) - 1,$$

so

$$\exp(x^2) = \int_0^{|x|}\left(\int_0^y 2(1+2t^2)\exp(t^2)dt\right)dy + 1. \tag{7.3}$$

Owing to $\exp(x^2) \geq \theta(x)$, we have

$$\exp(x^2) - \omega(x) = \int_0^{|x|}\left(\int_0^y 2(1+2t^2)\left[\exp(t^2) - \theta(t)\right]dt\right)dy \geq 0.$$

Hence $\exp(x^2) \geq \omega(x)$.

- Let us define the function $\beta(x)$ by

$$\beta(x) = \int_0^x \left(\int_0^y 2(1+2t^2)\theta(t)dt \right) dy + 1 - \theta(x).$$

Then we have

$$\beta'(x) = \int_0^x 2(1+2t^2)\theta(t)dt - \theta'(x), \qquad \beta''(x) = 2(1+2x^2)\theta(x) - \theta''(x).$$

Thanks to the assumption $2(1+2x^2)\theta(x) - \theta''(x) \geq 0$, we have $\beta''(x) \geq 0$. Since $\theta'(0) = 0$, we have $\beta'(0) = 0$ and, using $\theta(0) = 1$, we have $\beta(0) = 0$. So $\beta(x) \geq 0$ for all $x \geq 0$. Therefore $\beta(|x|) \geq 0$ for all $x \in \mathbb{R}$. Since $\theta(x)$ is even, we have

$$\omega(x) = \int_0^{|x|} \left(\int_0^y 2(1+2t^2)\theta(t)dt \right) dy + 1 \geq \theta(x).$$

The proof of Proposition 6.1 is complete. □

Proof of Proposition 6.2. We proceed as the proof of Proposition 6.1.

- If follows from the equality (7.3), $\exp(x^2) \geq \theta_1(x)$ and $\exp(x^2) \geq \theta_2(x)$ that

$$\begin{aligned} &\exp(x^2) - \kappa(x) \\ &= \int_0^{|x|} \left(\int_0^y 2\left[(\exp(t^2) - \theta_1(t)) + 2t^2(\exp(t^2) - \theta_2(t)) \right] dt \right) dy \geq 0. \end{aligned}$$

The first point is proved.

- Let us define the function $\phi(x)$ by

$$\phi(x) = \int_0^x \left(\int_0^y 2(\theta_1(t) + 2t^2\theta_2(t))dt \right) dy + 1 - \theta_1(x).$$

Two differentiations give

$$\phi'(x) = \int_0^x 2(\theta_1(t) + 2t^2\theta_2(t))dt - \theta_1'(x), \qquad \phi''(x) = 2(\theta_1(x) + 2x^2\theta_2(x)) - \theta_1''(x).$$

Since $2(\theta_1(x) + 2x^2\theta_2(x)) - \theta_1''(x) \geq 0$, we have $\phi''(x) \geq 0$, implying that $\phi'(x)$ is increasing. Since $\theta_1'(0) = 0$, we have $\phi'(0) = 0$, and using $\theta(0) = 1$, we have

$\phi(0) = 0$. Therefore, we have $\phi(x) \geq 0$ for all $x \geq 0$, implying that $\phi(|x|) \geq 0$ for all $x \in \mathbb{R}$. Since $\theta(x)$ is even, we have

$$\kappa(x) = \int_0^{|x|} \left(\int_0^y 2(\theta_1(t) + 2t^2\theta_2(t))dt \right) dy + 1 \geq \theta_1(x).$$

By exchanging the role of $\theta_1(x)$ and $\theta_2(x)$, we obtain

$$\kappa(x) = \int_0^{|x|} \left(\int_0^y 2(\theta_1(t) + 2t^2\theta_2(t))dt \right) dy + 1 \geq \theta_2(x).$$

This ends the proof of Proposition 6.2. □

Proof of Proposition 6.3. We have

$$\int_0^{|x|} \exp(t) \left[(1+2t)\exp(t^2) - 1\right] dt = \left[\exp(t^2+t) - \exp(t)\right]_0^{|x|} = \exp(|x|)\left(\exp(x^2) - 1\right),$$

so

$$\exp(x^2) = 1 + \exp(-|x|) \int_0^{|x|} \exp(t) \left[(1+2t)\exp(t^2) - 1\right] dt.$$

If $\exp(x^2) \geq \theta(x)$ for all $x \in \mathbb{R}$, we have

$$\exp(x^2) \geq 1 + \exp(-|x|) \int_0^{|x|} \exp(t) \left[(1+2t)\theta(t) - 1\right] dt = \gamma(x).$$

Proposition 6.3 is proved. □

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