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SOME RESULTS ON Λ-BANACH FRAMES FOR OPERATOR SPACES

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Abstract. In this paper, a new notion of Λ-Banach frame is introduced as a generalization of retro Banach frame for the space of bounded linear operators. Some results regarding characterizations and construction of Λ-Banach frames are presented. We extend results for retro Banach frame to the class of Λ-Banach frame. In the sequel, finite sum of Λ-Banach frames for operator spaces has been discussed. Finally, some perturbation and stability results on Λ-Banach frames have been given.

1. Introduction

In 1952, during the discussion on some deep problems in nonharmonic Fourier series, Duffin and Schaeffer [5] introduced frames for Hilbert spaces. Recall that, a countable system $\{x_n\}$ in a Hilbert space X is called Hilbert frame for X, if there exist positive constants A and B such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |< x, x_n >|^2 \leq B\|x\|^2, \qquad \text{for each } x \in X.$$

The fundamental work done by Daubechies, Grossmann, and Meyer [4] developed and popularized frames for Hilbert spaces. In 1991, Grochenig [6] extended Hilbert

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frames to Banach spaces and introduced Banach frames. Further, many generalizations of frames in Hilbert spaces and Banach spaces have been introduced and studied, namely, p-frames [1], X_d-frames [3], G-frames [15], fusion frames [2] etc.

In 2004, Jain et al. [7], studied frames in conjugate Banach spaces and introduced retro Banach frames. In 2010, fusion Banach frames were introduced in [9] as a generalization of Banach frames in Banach spaces. Later, in 2012, operator frames for Banach spaces were introduced by Chun Yan Li [12]. Recently, Chander Shekhar [13], further studied operator frames and introduced operator Banach frames for Banach spaces. The reconstruction property is an important tool in the study of frame theory as discussed by Kaushik et al. in [11]. In this direction, they introduced the notion of Banach Λ-frames for Banach spaces. The Banach Λ-frame were further studied in [14]. Motivated by the paper [11], we introduce and study the notion of Λ-Banach frames for spaces of bounded linear operators and observe that every retro Banach frame is a Λ-Banach frame but converse need not true as shown in Remark 1.

The paper is organized as follows. Section 2 contains some notations and basic definitions which will be used throughout the paper. In section 3, we define Λ-Banach frames for operator spaces and discuss various types of Λ-Banach frames by giving some illustrative examples. Further, some characterizations of Λ-Banach frames have been given. A sufficient condition for the finite sum of Λ-Banach frames is to be a Λ-Banach frame has been obtained in section 4. In section 5, we prove some perturbation and stability results for Λ-Banach frames. Finally, in section 6, we give an application of Λ-Banach frame in eigenvalue problem.

2. Preliminaries

Throughout this paper, X and Y will denote Banach spaces over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and X^* the conjugate space of X. $B(X,Y)$ be the family of bounded linear operators from X into Y. The closed linear span of $\{x_n\}$ in the norm topology of

X is denoted by $[x_n]$. We denote by $\mathcal{B}_d$, a Banach space of vector valued sequences associated with Y.

In order to make the paper self-contained, we list the following definitions.

Definition 2.1. [6] Let X be a Banach space and X_d be an associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset X^*$ and $S : X_d \to X$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for X with respect to X_d if

(i) $\{f_n(x)\} \in X_d$, for each $x \in X$,

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{f_n(x)\}\|_{X_d} \leq B\|x\|_X, \qquad x \in X, \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \qquad x \in X.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : X_d \to X$ is called the *reconstruction operator* (or, the *pre frame operator*). The inequality (2.1) is called the *frame inequality.*

Definition 2.2. [7] Let X be a Banach space and X^* be its conjugate space. Let $(X^*)_d$ be a Banach space of scalar valued sequences indexed by $\mathbb{N}$ associated with X^*. Let $\{x_n\}$ be a sequence in X and $T : (X^*)_d \to X^*$ be given. A pair $(\{x_n\}, T)$ is called a *retro Banach frame* for X^* with respect to $(X^*)_d$ if

(i) $\{f(x_n)\} \in (X^*)_d$, for each $f \in X^*$

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{X^*} \leq \|\{f(x_n)\}\|_{(X^*)_d} \leq B\|f\|_{X^*}, \qquad \text{for all } f \in X^* \tag{2.2}$$

(iii) T is a bounded linear operator such that

$$T(\{f(x_n)\}) = f, \qquad f \in X^*.$$

The positive constants A and B are called, *lower* and *upper frame bounds* of the retro Banach frame $(\{x_n\}, T)$, respectively. The operator $T : (X^*)_d \to X^*$ called the *reconstruction operator* for the retro Banach frame $(\{x_n\}, T)$. The inequality (2.2) is called the *retro frame inequality.*

Definition 2.3. [11] Let X and Y be Banach spaces and let Y_d be a sequence space associated with Y. A system $\{x_n\} \subset X$ is called a Banach Λ-frame for $B(X, Y)$ if there exist positive constants $0 < A \leq B < \infty$ such that

$$A\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{Y_d} \leq B\|\Lambda\|, \qquad \text{for all } \Lambda \in B(X, Y).$$

We finish this section with the following result proved in [7].

Theorem 2.1 (Jain-Kaushik-Vashisht)**.** *Let X be a Banach space. Then X^* has a retro Banach frame if and only if X is separable.*

3. Λ-Banach frame

Let us begin with the following definition of Λ-Banach frame for operator spaces.

Definition 3.1. Let X and Y be Banach spaces. Let $\{x_n\}$ be a sequence in X, $\Lambda \in B(X, Y)$ and $S : \mathcal{B}_d \to B(X, Y)$ be an operator, where $\mathcal{B}_d$ be a Banach space of vector valued sequences associated with Y. Then $(\{x_n\}, \Lambda, S)$ is called a Λ-*Banach frame* for $B(X, Y)$ with respect to $\mathcal{B}_d$, if

(i) $\{\Lambda(x_n)\} \in \mathcal{B}_d$, $\Lambda \in B(X, Y)$

(ii) there exist constants $0 < A \leq B < \infty$ such that

$$A\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B\|\Lambda\|, \qquad \Lambda \in B(X, Y) \tag{3.1}$$

(iii) S is a bounded linear operator such that

$$S(\{\Lambda(x_n)\}) = \Lambda, \qquad \Lambda \in B(X,Y).$$

The positive constants A and B are called lower and upper frame bounds, respectively, for the Λ-Banach frame $(\{x_n\}, \Lambda, S)$. The inequality (3.1) is called the Λ-frame inequality. The operator $S : \mathcal{B}_d \to B(X,Y)$ is called the reconstruction operator. We shall also use the following keywords.

- If condition (i) in definition 3.1 and upper inequality in (3.1) are satisfied, then $\{x_n\}$ is called a Λ-*Bessel sequence* for $B(X,Y)$ with respect to $\mathcal{B}_d$.
- If removal of any x_k from the Λ-Banach frame renders the collection $\{x_n\}_{n\neq k}$ to be a Λ-Banach frame for the underlying space, then $(\{x_n\}, \Lambda, S)$ is said to be an *exact* Λ-*Banach frame* for $B(X,Y)$ with respect to $\mathcal{B}_d$.
- The Λ-Banach frame $(\{x_n\}, \Lambda, S)$ is said to be *tight* if $A = B$, and *normalized tight* if $A = B = 1$.
- A sequence $\{x_n\}$ in X is said to be *total* over $B(X,Y)$ if

$$\{\Lambda \in B(X,Y) : \Lambda(x_n) = 0, \ \forall n \in \mathbb{N}\} = \{0\}.$$

Now, we prove a fundamental result in the form of lemma, which will be used in the subsequent work.

Lemma 3.1. *Let X and Y be Banach spaces and $\Lambda \in B(X,Y)$. If $\{x_n\} \subset X$ is total over $B(X,Y)$, then $\mathcal{B}_d = \{\{\Lambda(x_n)\} : \Lambda \in B(X,Y)\}$ is a Banach space with norm given by $\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = \|\Lambda\|$, $\Lambda \in B(X,Y)$.*

Proof. Clearly, $\mathcal{B}_d$ is a linear space under pointwise addition and scalar multiplication. Also,

$$\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = \|\Lambda\| \geq 0, \qquad \Lambda \in B(X,Y).$$

Let $\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = 0$. Then $\|\Lambda\| = 0 \Rightarrow \Lambda = 0$. This gives $\{\Lambda(x_n)\} = 0$. Also, if $\{\Lambda(x_n)\} = 0$, then $\Lambda(x_n) = 0$, for each $n \in \mathbb{N}$. Since $\{x_n\}$ is total over $B(X, Y)$, $\Lambda = 0$. This gives $\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = \|\Lambda\| = 0$. Further, for any scalar α, we have,

$$\|\alpha\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = |\alpha| \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}$$

$$\text{and } \|\{\Lambda(x_n)\} + \{\Theta(x_n)\}\|_{\mathcal{B}_d} \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} + \|\{\Theta(x_n)\}\|_{\mathcal{B}_d}, \qquad \Theta \in B(X, Y).$$

Thus, the given norm on $\mathcal{B}_d$ is well defined. Let $\{\{\Lambda_n(x_i)\}_i\}_n$ be a Cauchy sequence in $\mathcal{B}_d$. Then $\{\Lambda_n\}$ is a Cauchy sequence in $B(X, Y)$. Thus, completeness of $B(X, Y)$ ensure that, there is a $\Lambda \in B(X, Y)$ such that $\Lambda_n \to \Lambda$ as $n \to \infty$. So that, the Cauchy sequence $\{\{\Lambda_n(x_i)\}_i\}_n$ converges in $\mathcal{B}_d$. Hence, $\mathcal{B}_d$ is a Banach space with norm given by $\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = \|\Lambda\|$, $\Lambda \in B(X, Y)$. □

In order to show the existence of various kinds of Λ-Banach frame, we furnish the following examples.

Example 3.1. *Let $X = Y = l_1$ and $\{x_n\}$ be the sequence of standard unit vectors in X. Since $\{x_n\}$ is total over $B(X, Y)$, by Lemma 3.1, there exists an associated Banach space $\mathcal{B}_d = \{\{\Lambda(x_n)\} : \Lambda \in B(X, Y)\}$ of vector valued sequences and with norm given by $\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = \|\Lambda\|$, $\Lambda \in B(X, Y)$.*

Define $S : \mathcal{B}_d \to B(X, Y)$ by $S(\{\Lambda(x_n)\}) = \Lambda$, $\Lambda \in B(X, Y)$. Then S is a bounded linear operator such that $(\{x_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$.

(1) ***Tight and Exact:*** *Clearly, $(\{x_n\}, \Lambda, S)$ is a normalized tight Λ-Banach frame for $B(X, Y)$ and there exists no reconstruction operator S_0 such that $(\{x_n\}_{n \neq k}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X, Y)$, which shows that the Λ-Banach frame $(\{x_n\}, \Lambda, S)$ is exact.*

(2) ***Non-tight and Exact:*** *Let $\{y_n\}$ be a sequence in X defined by*

$$y_1 = \mu x_1, \ (0 < \mu < 1) \qquad \text{and} \qquad y_n = x_n, \ (n \geq 2).$$

Then, for each $\Lambda \in B(X,Y)$, $\{\Lambda(y_n)\} \in \mathcal{B}_d$ *and*

$$\mu\|\Lambda\| \leq \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \leq \|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Define operators $P, Q : B(X,Y) \to \mathcal{B}_d$ *such that for each* $\Lambda \in B(X,Y)$,

$$P(\Lambda) = \{\Lambda(x_n)\} \qquad \text{and} \qquad Q(\Lambda) = \{\Lambda(y_n)\}.$$

Then $\|P - Q\| < 1$ *and* $SP = I$. *Also,* $\|I - SQ\| < 1$. *Thus,* SQ *is invertible. Put* $T = (SQ)^{-1}S$. *Then* $T : \mathcal{B}_d \to B(X,Y)$ *is such that* $T(\{\Lambda(x_n)\}) = \Lambda$, $\Lambda \in B(X,Y)$. *Therefore,* $(\{y_n\}, \Lambda, T)$ *is a* Λ*-Banach frame for* $B(X,Y)$ *with respect to* $\mathcal{B}_d$ *which is non-tight and exact.*

(3) ***Tight and non-exact:*** *Let* $\{y_n\}$ *be a sequence in* X *defined by*

$$y_1 = x_1, \qquad y_n = x_{n-1}, \qquad n \geq 2.$$

Then there exists a reconstruction operator $S_0 : \mathcal{B}_{d_0} = \{\{\Lambda(y_n)\} : \Lambda \in B(X,Y)\} \to B(X,Y)$ *such that* $(\{y_n\}, \Lambda, S_0)$ *is a* Λ*-Banach frame for* $B(X,Y)$ *with respect to* $\mathcal{B}_{d_0}$ *and having bounds* $A = B = 1$. *Further, note that* $\{y_n\}_{n\neq 1}$ *is total over* $B(X,Y)$, *therefore there exists an associated Banach space* $\mathcal{B}_{d_1} = \{\{\Lambda(y_n)\}_{n\neq 1} : \Lambda \in B(X,Y)\}$ *with norm given by* $\|\{\Lambda(y_n)\}\|_{\mathcal{B}_{d_1}} = \|\Lambda\|$, $\Lambda \in B(X,Y)$ *and a reconstruction operator* $S_1 : \mathcal{B}_{d_1} \to B(X,Y)$ *such that* $(\{y_n\}, \Lambda, S_1)$ *is a* Λ*-Banach frame for* $B(X,Y)$ *with respect to* $\mathcal{B}_{d_1}$. *Hence,* $(\{y_n\}, \Lambda, S_0)$ *is tight and non-exact* Λ*-Banach frame for* $B(X,Y)$.

(4) ***Non-tight and non-exact:*** *Let* $\{y_n\}$ *be a sequence in* X *defined by*

$$y_1 = y_2 = x_1, \qquad y_3 = \mu x_2,\ (0 < \mu < 1), \qquad y_n = x_{n-1},\ (n \geq 4).$$

Then $\{\Lambda(y_n)\} \in \mathcal{B}_d$, $\Lambda \in B(X,Y)$ *and*

$$\mu\|\Lambda\| \leq \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \leq \|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Define a bounded linear operator $S_0 : \mathcal{B}_d \to B(X,Y)$ *by* $S_0(\{\Lambda(y_n)\}) = \Lambda$, $\Lambda \in B(X,Y)$. *Then,* $(\{y_n\}, \Lambda, S_0)$ *is a* Λ*-Banach frame for* $B(X,Y)$ *with respect to* $\mathcal{B}_d$. *Further, there exists an associated Banach space* $\mathcal{B}_{d_1} = \{\{\Lambda(y_n)\}_{n\neq 1} : \Lambda \in B(X,Y)\}$ *with norm given by* $\|\{\Lambda(y_n)\}_{n\neq 1}\|_{\mathcal{B}_{d_1}} = \|\Lambda\|$, $\Lambda \in B(X,Y)$. *Define* $S_1 : \mathcal{B}_{d_1} \to B(X,Y)$ *by* $S_1(\{\Lambda(y_n)\}_{n\neq 1}) = \Lambda$, $\Lambda \in B(X,Y)$. *Then,* $(\{y_n\}_{n\neq 1}, \Lambda, S_1)$ *is a* Λ*-Banach frame for* $B(X,Y)$ *with respect to* $\mathcal{B}_{d_1}$. *Hence,* $(\{y_n\}, \Lambda, S_0)$ *is a non-tight and non-exact* Λ*-Banach frame for* $B(X,Y)$ *with respect to* $\mathcal{B}_d$.

(5) ***Not a*** Λ***-Banach frame:*** *Let* $\{y_n\} \subset X$ *be a sequence defined by* $y_1 = x_2$, $y_n = x_n$, $n \geq 2$. *Then there exists no reconstruction operator* S *such that* $(\{y_n\}, \Lambda, S)$ *is a* Λ*-Banach frame for* $B(X,Y)$.

Remark 1. *Every retro Banach frame is a* Λ*-Banach frame.* Indeed, let X be a Banach space and X^* be a dual of X. Let $(X^*)_d$ be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$ and let $(\{x_n\}, S)$ (where $\{x_n\} \subset X$ and $S : (X^*)_d \to X^*$) is a retro Banach frame for X^* with respect to $(X^*)_d$. Let Y be a field of scalars. Define $\Lambda_f \in B(X,Y)$, for any $f \in X^*$, by $\Lambda_f(x_n) = f(x_n)$. Then $(\{x_n\}, \Lambda_f, S)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $(X^*)_d$. However, the converse need not true as shown in Example 3.2.

Example 3.2. *Let* $X = Y = l_\infty$. *Let* $\{x_n\}$ *be a sequence of unit vectors in* X. *Then, by Lemma 3.1, there exists an associated Banach space* $\mathcal{B}_d = \{\{\Lambda(x_n)\} : \Lambda \in B(X,Y)\}$ *with norm* $\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = \|\Lambda\|$, $\Lambda \in B(X,Y)$ *and a reconstruction operator* $S : \mathcal{B}_d \to B(X,Y)$ *defined by* $S(\{\Lambda(x_n)\}) = \Lambda$, $\Lambda \in B(X,Y)$. *Hence,* $(\{x_n\}, \Lambda, S)$ *is a* Λ*-Banach frame for* $B(X,Y)$ *with respect to* $\mathcal{B}_d$. *On the other hand, by Theorem 2.1,* X^* *has no retro Banach frame.*

For the rest part of this section, the system $(\{x_n\}, \Lambda, S)$ will be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$, where $\{x_n\} \subset X$, $\Lambda \in B(X,Y)$ and $S : \mathcal{B}_d \to B(X,Y)$

be a reconstruction operator. Now, in order to give some characterizations of Λ-Banach frames, we extend some results for retro Banach frame, proved in [7], to the class of Λ-Banach frame. In the sequel, we first prove that the coefficient mapping associated with a Λ-Banach frame for $B(X,Y)$ is a topological isomorphism onto a closed subspace of $\mathcal{B}_d$.

Theorem 3.1. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and A, B are Λ-frame bounds. Then the coefficient mapping $T : B(X,Y) \to \mathcal{B}_d$ defined by $T(\Lambda) = \{\Lambda(x_n)\}$, $\Lambda \in B(X,Y)$ is a topological isomorphism onto a closed subspace of $\mathcal{B}_d$ with $\|T\| \le B$ and $\|T^{-1}\| \le \frac{1}{A}$, where T^{-1} is defined on the range of T, denoted by $R(T)$.*

Proof. The Λ-frame inequality for the Λ-Banach frame $(\{x_n\}, \Lambda, S)$ is given by

$$A\|\Lambda\| \le \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \le B\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Thus, we see that $\|T\| \le B$. Now, let $\Lambda \in \ker T$. Then $T(\Lambda) = 0$, and so $\Lambda(x_n) = 0$, $\forall n$. Then, by Λ-frame inequality, $\Lambda = 0$. Thus, T is injective. So that, T^{-1} exists on $R(T)$ and $\|T^{-1}\| \le \frac{1}{A}$.

Now, we shall show that $R(T)$ is closed. Let $\{\sigma_n\}$ be a sequence in $R(T)$ converging to $\sigma \in \mathcal{B}_d$ and $\{\Theta_n\}$ be a sequence in $B(X,Y)$ such that $T(\Theta_n) = \sigma_n$, $n \in \mathbb{N}$. Then $\{T(\Theta_n)\} = \{\sigma_n\}$ is a Cauchy sequence in $\mathcal{B}_d$ and so by continuity of T^{-1}, $\{\Theta_n\}$ is a Cauchy sequence in $B(X,Y)$. Since $B(X,Y)$ is complete, there exists a $\Theta \in B(X,Y)$ such that $\lim_{n\to\infty} \Theta_n = \Theta$. Therefore, by continuity of T, $\lim_{n\to\infty} T(\Theta_n) = T(\Theta)$. Hence, $R(T)$ is a closed subspace of $\mathcal{B}_d$. □

Remark 2. For a sequence $\{x_n\} \subset X$, if the coefficient mapping $T : B(X,Y) \to \mathcal{B}_d$ defined by $T(\Lambda) = \{\Lambda(x_n)\}$, $\Lambda \in B(X,Y)$ is a topological isomorphism onto $\mathcal{B}_d$, then there exists a reconstruction operator $S : \mathcal{B}_d \to B(X,Y)$ such that $(\{x_n\}, \Lambda, S)$ is a

Λ-Banach frame for $B(X,Y)$ and with frame bounds $\|T^{-1}\|^{-1}$ and $\|T\|$. Indeed, for each $\Lambda \in B(X,Y)$,

$$\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = \|T(\Lambda)\| \leq \|T\|\|\Lambda\|$$

and

$$\|T^{-1}\|^{-1}\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}.$$

Hence,

$$\|T^{-1}\|^{-1}\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq \|T\|\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Put $S = T^{-1}$. Then $(\{x_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$.

In the next result, we prove that the linear homeomorphic image of a Λ-Banach frame is a Λ-Banach frame.

Theorem 3.2. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and with best bounds A and B. Let $Q : X \to X$ be a linear homeomorphism, then there is a bounded linear operator $\Lambda_0 \in B(X,Y)$ and a reconstruction operator S_0 such that $(\{Qx_n\}, \Lambda_0, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and with best bounds C and D satisfying,*

$$A\|Q^{-1}\|^{-1} \leq C \leq A\|Q\|,$$

$$B\|Q^{-1}\|^{-1} \leq D \leq B\|Q\|.$$

Proof. The Λ-frame inequality for the Λ-Banach frame $(\{x_n\}, \Lambda, S)$ is given by

$$A\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B\|\Lambda\|, \qquad \Lambda \in B(X,Y). \tag{3.2}$$

Take $\Lambda_0 = \Lambda Q^{-1}$. Then $\Lambda_0 \in B(X,Y)$ such that

$$\{\Lambda(x_n)\} = \{\Lambda(Q^{-1}Qx_n)\} = \{\Lambda_0(Qx_n)\}. \tag{3.3}$$

So, $\{\Lambda_0(Qx_n)\} \in \mathcal{B}_d$, $\Lambda_0 \in B(X,Y)$. Also,

$$\|\Lambda_0\| = \|\Lambda Q^{-1}\| \leq \|\Lambda\|\|Q^{-1}\|. \tag{3.4}$$

Therefore, by (3.2), (3.3) and (3.4) we have

$$A\|Q^{-1}\|^{-1}\|\Lambda_0\| \leq A\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} = \|\{\Lambda_0(Qx_n)\}\|_{\mathcal{B}_d} \leq B\|\Lambda\| \leq B\|Q\|\|\Lambda_0\|.$$

Define $S_0 : \mathcal{B}_d \to B(X,Y)$ by $S_0(\{\Lambda_0(Qx_n)\}) = \Lambda_0$, $\Lambda_0 \in B(X,Y)$. Then, $(\{Qx_n\}, \Lambda_0, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$, with bounds $A\|Q^{-1}\|^{-1}$ and $B\|Q\|$.

Since, the constants C and D are best bounds for Λ-Banach frame $(\{Qx_n\}, \Lambda_0, S_0)$, we have

$$A\|Q^{-1}\|^{-1} \leq C \qquad \text{and} \qquad D \leq B\|Q\|. \tag{3.5}$$

Further, $\|\Lambda\| = \|\Lambda_0 Q\| \leq \|\Lambda_0\|\|Q\|$. So,

$$\begin{aligned} C\|Q\|^{-1}\|\Lambda\| &\leq C\|\Lambda_0\| \\ &\leq \|\{\Lambda_0(Qx_n)\}\|_{\mathcal{B}_d} \qquad (= \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}) \\ &\leq D\|\Lambda_0\| \\ &\leq D\|\Lambda\|\|Q^{-1}\|. \end{aligned}$$

Thus, we obtain

$$C\|Q\|^{-1}\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq D\|Q^{-1}\|\|\Lambda\|. \tag{3.6}$$

Hence,

$$A \geq C\|Q\|^{-1} \qquad \text{and} \qquad B \leq D\|Q^{-1}\| \tag{3.7}$$

Hence, from (6.1) and (3.7) we conclude that

$$A\|Q^{-1}\|^{-1} \leq C \leq A\|Q\|,$$

$$B\|Q^{-1}\|^{-1} \leq D \leq B\|Q\|.$$

□

Now, we construct a normalized tight Λ-Banach frame associated with a given Λ-Banach frame. This kind of construction of Λ-Banach frame is motivated by Theorem 2.9 [13] given for operator Banach frame.

Theorem 3.3. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X, Y)$ with respect $\mathcal{B}_d$. Let $\{y_n\}$ be a sequence in X such that $\{x_n + y_n\}$ is a Λ-Bessel sequence for $B(X, Y)$ with respect to $\mathcal{B}_d$ and with bound $M < \|S\|^{-1}$. Then there exists a reconstruction operator $S_0 : \mathcal{B}_d \to B(X, Y)$ such that $(\{y_n\}, \Lambda, S_0)$ is a normalized tight Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$.*

Proof. Let A and B be the Λ-frame bounds for the Λ-Banach frame $(\{x_n\}, \Lambda, S)$. Then the Λ-frame inequality is given by

$$A\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B\|\Lambda\|, \qquad \Lambda \in B(X, Y).$$

Since $\{x_n + y_n\}$ is a Λ-Bessel sequence for $B(X, Y)$ with respect to $\mathcal{B}_d$ with bound M, we have,

$$\|\{\Lambda(x_n + y_n)\}\|_{\mathcal{B}_d} \leq M\|\Lambda\|, \qquad \Lambda \in B(X, Y).$$

Since, the operator $S : \mathcal{B}_d \to B(X, Y)$ is given by $S(\{\Lambda(x_n)\}) = \Lambda$, $\Lambda \in B(X, Y)$, we obtain

$$\begin{aligned}(\|S\|^{-1} - M)\|\Lambda\| \leq& \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} - \|\{\Lambda(x_n + y_n)\}\|_{\mathcal{B}_d} \\ \leq& \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}.\end{aligned}$$

Thus, it is easy to check that, $\{y_n\}$ is total over $B(X, Y)$. Therefore, by Lemma 3.1 there exists an associated Banach space $\mathcal{B}_{d_0} = \{\{\Lambda(y_n)\} : \Lambda \in B(X, Y)\}$ with norm given $\|\{\Lambda(y_n)\}\|_{\mathcal{B}_{d_0}} = \|\Lambda\|$, $\Lambda \in B(X, Y)$.

Define $S_0 : \mathcal{B}_{d_0} \to B(X,Y)$ by $S_0(\{\Lambda(y_n)\}) = \Lambda$, $\Lambda \in B(X,Y)$. Then S_0 is a bounded linear operator such that $(\{y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_{d_0}$. □

Regarding the converse of Theorem 3.3, we prove the next result.

Theorem 3.4. *Let $(\{x_n\}, \Lambda, S)$ and $(\{y_n\}, \Lambda, T)$ be Λ-Banach frames for $B(X,Y)$ with respect to $\mathcal{B}_d$. Let $W : \mathcal{B}_d \to \mathcal{B}_d$ be a linear homeomorphism such that*

$$W(\{\Lambda(x_n)\}) = \{\Lambda(y_n)\}, \qquad \Lambda \in B(X,Y).$$

Then $\{x_n + y_n\}$ is a Λ-Bessel sequence for $B(X,Y)$ with respect to $\mathcal{B}_d$ and with bound $M = \min\{\|U\|\|I+W\|, \|V\|\|I+W^{-1}\|\}$, where $U, V : B(X,Y) \to \mathcal{B}_d$ are the mapping given by $U(\Lambda) = \{\Lambda(x_n)\}$ and $V(\Lambda) = \{\Lambda(y_n)\}$, $\Lambda \in B(X,Y)$ and I denote the identity mapping on $\mathcal{B}_d$.

Proof. For each $\Lambda \in B(X,Y)$, we have

$$\begin{aligned}\|\{\Lambda(x_n+y_n)\}\|_{\mathcal{B}_d} =& \|\{\Lambda(x_n)\} + \{\Lambda(y_n)\}\| \\ =& \|I(\{\Lambda(x_n)\}) + W(\{\Lambda(x_n)\})\| \\ =& \|(I+W)(\{\Lambda(x_n)\})\| \\ =& \|(I+W)U(\Lambda)\| \\ \le& \|I+W\|\|U\|\|\Lambda\|.\end{aligned}$$

Similarly, we compute

$$\begin{aligned}\|\{\Lambda(x_n + y_n)\}\|_{\mathcal{B}_d} =& \|\{\Lambda(x_n)\} + \{\Lambda(y_n)\}\| \\ =& \|W^{-1}(\{\Lambda(y_n)\}) + I(\{\Lambda(y_n)\})\| \\ =& \|(I + W^{-1})(\{\Lambda(y_n)\})\| \\ =& \|(I + W^{-1})V(\Lambda)\| \\ \leq& \|I + W^{-1}\|\|V\|\|\Lambda\|.\end{aligned}$$

Hence, $\{x_n + y_n\}$ is a Λ-Bessel sequence for $B(X,Y)$ with respect to $\mathcal{B}_d$ with bound $M = \min\{\|U\|\|I + W\|, \|V\|\|I + W^{-1}\|\}$.

□

In the following theorem, we obtain a necessary and sufficient condition for a sequence in a Banach space to be a Λ-Banach frame.

Theorem 3.5. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$. Let $\{y_n\}$ be a sequence in X. Then, there exists a reconstruction operator $S_0 : \mathcal{B}_d \to B(X,Y)$ such that $(\{y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ if and only if there exists a constant $\beta > 0$ such that*

$$\|W(\{\Lambda(x_n)\})\|_{\mathcal{B}_d} \geq \beta\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}, \qquad \Lambda \in B(X,Y)$$

where $W : \mathcal{B}_d \to \mathcal{B}_d$ is a bounded linear operator given by $W(\{\Lambda(x_n)\}) = \{\Lambda(y_n)\}, \Lambda \in B(X,Y)$.

Proof. Let A_1 and B_1 be the Λ-frame bounds for the Λ-Banach frame $(\{x_n\}, \Lambda, S)$. Then the frame inequality is given by,

$$A_1\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B_1\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Assume that there is a reconstruction operator $S_0 : \mathcal{B}_d \to B(X,Y)$ such that $(\{y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and having frame bounds A_2 and B_2. Then the Λ-frame inequality for the Λ-Banach frame $(\{y_n\}, \Lambda, S_0)$ is given by

$$A_2\|\Lambda\| \leq \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \leq B_2\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Since $W : \mathcal{B}_d \to \mathcal{B}_d$ is given by $W(\{\Lambda(x_n)\}) = \{\Lambda(y_n)\}$, $\Lambda \in B(X,Y)$, using above frame inequalities, we compute,

$$\begin{aligned}\|W(\{\Lambda(x_n)\})\|_{\mathcal{B}_d} =& \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}\\ \geq& A_2\|\Lambda\|\\ \geq& \frac{A_2}{B_1}\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}.\end{aligned}$$

This yields, $\|W(\{\Lambda(x_n)\})\|_{\mathcal{B}_d} \geq \beta\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}$, where $\beta = \frac{A_2}{B_1}$. Conversely, for each $\Lambda \in B(X,Y)$, we have,

$$\|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \geq \beta\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \geq \beta A_1\|\Lambda\|.$$

Also,

$$\|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} = \|W(\{\Lambda(x_n)\})\| \leq \|W\|\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B_1\|W\|\|\Lambda\|.$$

Hence, we conclude that

$$\beta A_1\|\Lambda\| \leq \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \leq B_1\|W\|\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Put $S_0 = SW$. Then $S_0 : \mathcal{B}_d \to B(X,Y)$ is a bounded linear operator such that $S_0(\{\Lambda(y_n)\}) = \Lambda$, $\Lambda \in B(X,Y)$. Hence, $(\{y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and having frame bounds βA_1 and $B_1\|W\|$. □

Now, we construct Λ-Banach frames in the product space. In fact, we prove that if $B(X_1,Y_1)$ and $B(X_2,Y_2)$ have Λ-Banach frames then $B(X_1 \times X_2, Y_1 \times Y_2)$ also has a Λ-Banach frame.

Theorem 3.6. *Let X_1, X_2, Y_1, Y_2, be Banach spaces. Let $(\{x_n\}, \Lambda_1, S_1)$ (where $\{x_n\} \subset X_1$, $\Lambda_1 \in B(X_1, Y_1)$, $S_1 : \mathcal{B}_{d_1} \to B(X_1, Y_1)$) and $(\{y_n\}, \Lambda_2, S_2)$ (where $\{y_n\} \subset X_2$, $\Lambda_2 \in B(X_2, Y_2)$, $S_2 : \mathcal{B}_{d_2} \to B(X_2, Y_2)$) be Λ-Banach frames for $B(X_1, Y_1)$ and $B(X_2, Y_2)$ with respect to $\mathcal{B}_{d_1}$ and $\mathcal{B}_{d_2}$, respectively. Then there exists a sequence $\{z_n\}$ in $X_1 \times X_2$, an operator $\Lambda \in B(X_1 \times X_2, Y_1 \times Y_2)$, an associated Banach space $\mathcal{B}_d$ and a reconstruction operator $U : \mathcal{B}_d \to B(X_1 \times X_2, Y_1 \times Y_2)$ such that $(\{z_n\}, \Lambda, U)$ is a Λ-Banach frame for $B(X_1 \times X_2, Y_1 \times Y_2)$ with respect to $\mathcal{B}_d$.*

Proof. Let A_1, B_1 be a couple of bounds for Λ-Banach frame $(\{x_n\}, \Lambda_1, S_1)$ and A_2, B_2 be a couple of bounds for Λ-Banach frame $(\{y_n\}, \Lambda_2, S_2)$. Then, the Λ-frame inequalities for $B(X_1, Y_1)$ and $B(X_2, Y_2)$ are given by

$$A_1\|\Lambda_1\| \leq \|\{\Lambda_1(x_n)\}\|_{\mathcal{B}_{d_1}} \leq B_1\|\Lambda_1\|, \qquad \Lambda_1 \in B(X_1, Y_1), \tag{3.8}$$

$$A_2\|\Lambda_2\| \leq \|\{\Lambda_2(y_n)\}\|_{\mathcal{B}_{d_2}} \leq B_2\|\Lambda_1\|, \qquad \Lambda_2 \in B(X_2, Y_2). \tag{3.9}$$

Let $\{z_n\}$ be a sequence in $X_1 \times X_2$, where $z_n = (x_n, y_n)$, $n \in \mathbb{N}$ and $\Lambda \in B(X_1 \times X_2, Y_1 \times Y_2)$ defined by

$$\Lambda(x_n, y_n) = (\Lambda_1(x_n), \Lambda_2(y_n)).$$

Suppose $\Lambda(x_n, y_n) = 0$, $\forall n \in \mathbb{N}$. Then $\Lambda_1(x_n) = \Lambda_2(y_n) = 0$, $\forall n \in \mathbb{N}$. By (3.8) and (3.9) we obtain $\Lambda = (\Lambda_1, \Lambda_2) = 0$. Hence, $\{z_n\}$ is total over $B(X_1 \times X_2, Y_1 \times Y_2)$. Therefore, by Lemma 3.1, there exits an associated Banach space

$$\mathcal{B}_d = \{\{\Lambda(z_n)\} : \Lambda \in B(X_1 \times X_2, Y_1 \times Y_2)\}$$

with norm given by

$$\|\{\Lambda(z_n)\}\|_{\mathcal{B}_d} = \|\Lambda\|, \qquad \Lambda \in B(X_1 \times X_2, Y_1 \times Y_2).$$

Define $U : \mathcal{B}_d \to B(X_1 \times X_2, Y_1 \times Y_2)$ by

$$U(\{\Lambda(z_n)\}) = \Lambda, \qquad \Lambda \in B(X_1 \times X_2, Y_1 \times Y_2).$$

Then, $(\{z_n\}, \Lambda, U)$ is a Λ-Banach frame for $B(X_1 \times X_2, Y_1 \times Y_2)$ with respect to $\mathcal{B}_d$. □

4. Finite sum of Λ-Banach frame

Let $(\{x_{1,i}\}, \Lambda, S_1)$ and $(\{x_{2,i}\}, \Lambda, S_2)$ be two Λ-Banach frames for $B(X, Y)$ with respect to $\mathcal{B}_d$. Then there exists, in general, no reconstruction operator S_0 such that $(\{x_{1,i} + x_{2,i}\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X, Y)$, as shown in the following example.

Example 4.1. *Let $X = Y = l_p$ $(1 \le p < \infty)$. Let $\{x_n\}$ be a sequence of unit vectors in X. Since $\{x_n\}$ is total over $B(X, Y)$, by Lemma 3.1, there exists an associated Banach space $\mathcal{B}_{d_0} = \{\{\Lambda(x_n)\} : \Lambda \in B(X, Y)\}$ with norm given by $\|\{\Lambda(x_n)\}\|_{\mathcal{B}_{d_0}} = \|\Lambda\|$, $\Lambda \in B(X, Y)$ and a bounded linear operator $S_0 : \mathcal{B}_{d_0} \to B(X, Y)$ such that $S_0(\{\Lambda(x_n)\}) = \Lambda$, $\Lambda \in B(X, Y)$. Hence, $(\{x_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_{d_0}$.*

Now, let $\{y_n\}$ be a sequence in X defined by by

$$y_1 = x_1, \qquad y_2 = x_1 - x_2, \qquad y_n = x_n, \ (n \ge 3).$$

Then, by Lemma 3.1, there exists an associated Banach space $\mathcal{B}_{d_1} = \{\{\Lambda(y_n)\} : \Lambda \in B(X, Y)\}$ with norm given by $\|\{\Lambda(y_n)\}\|_{\mathcal{B}_{d_1}} = \|\Lambda\|$, $\Lambda \in B(X, Y)$ and a bounded linear operator $S_1 : \mathcal{B}_{d_1} \to B(X, Y)$ such that $S_1(\{\Lambda(y_n)\}) = \Lambda$, $\Lambda \in B(X, Y)$. Hence, $(\{y_n\}, \Lambda, S_1)$ is a Λ- Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_{d_1}$.

Put $z_n = x_n + y_n$, $n \in \mathbb{N}$. Then, there exists no associated Banach space $\mathcal{B}_d$ and hence no reconstruction operator $S : \mathcal{B}_d \to B(X, Y)$ such that $(\{z_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$. Indeed, if $(\{z_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$. Then there exist positive constants A_z, B_z such that

$$A_z\|\Lambda\| \le \|\{\Lambda(z_n)\}\|_{\mathcal{B}_d} \le B_z\|\Lambda\|, \qquad \Lambda \in B(X, Y). \tag{4.1}$$

Now, $\Lambda = (0,1,0,0,...)$ *is a nonzero operator in* $B(X,Y)$ *such that* $\Lambda(z_n) = 0$, $\forall n \in \mathbb{N}$. *So, by inequality (4.1),* $\Lambda = 0$, *a contradiction.*

In view of above example, in the next result, we provide a condition under which the finite sum of Λ-Banach frames for $B(X,Y)$ is a Λ-Banach frame for $B(X,Y)$. This result extend Theorem 2.15 [13] to the case of Λ-Banach frame.

Theorem 4.1. *Let* $(\{x_{i,n}\}, \Lambda, S_i)$ $(i = 1,2,...,k)$ *be a* Λ*-Banach frame for* $B(X,Y)$ *with respect to* $\mathcal{B}_d$. *Then there exists a reconstruction operator* S_0 *such that* $(\{\sum_{i=1}^k x_{i,n}\}, \Lambda, S_0)$ *is a normalized tight* Λ*-Banach frame for* $B(X,Y)$, *provided*

$$\|\{\Lambda(x_{i,n})\}\|_{\mathcal{B}_d} \leq \left\|\left\{\Lambda\left(\sum_{i=1}^k x_{i,n}\right)\right\}\right\|, \qquad \Lambda \in B(X,Y)$$

for some $i \in \{1,2,3,...,k\}$.

Proof. For each $i \in \{1,2,...,k\}$, $(\{x_{i,n}\}, \Lambda, S_i)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$. Thus we have

$$\|\Lambda\| = \|S_i(\{\Lambda(x_{i,n})\})\| \leq \|S_i\| \left\|\left\{\Lambda\left(\sum_{i=1}^k x_{i,n}\right)\right\}\right\|_{\mathcal{B}_d}.$$

This gives

$$\|S_i\|^{-1}\|\Lambda\| \leq \left\|\left\{\Lambda\left(\sum_{i=1}^k x_{i,n}\right)\right\}\right\|_{\mathcal{B}_d}, \qquad \Lambda \in B(X,Y).$$

Therefore, $\{\Lambda(\sum_{i=1}^k x_{i,n})\}$ is total over $B(X,Y)$. Hence, by Lemma 3.1, there exists an associated Banach space $\mathcal{B}_{d_0} = \{\{\Lambda(\sum_{i=1}^k x_{i,n})\} : \Lambda \in B(X,Y)\}$ with norm given by

$$\left\|\left\{\Lambda\left(\sum_{i=1}^k x_{i,n}\right)\right\}\right\|_{\mathcal{B}_{d_0}} = \|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Define $S_0 : \mathcal{B}_{d_0} \to B(X,Y)$ by $S_0(\{\Lambda(\sum_{i=1}^k x_{i,n})\}) = \Lambda$, $\Lambda \in B(X,Y)$. Then S_0 is a bounded linear operator such that $(\{\Lambda(\sum_{i=1}^k x_{i,n})\}, \Lambda, S_0)$ is a normalized tight Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_{d_0}$. □

5. Perturbation of Λ-Banach frames

In this section, we discuss some perturbation and stability results of Λ-Banach frames. In fact, we extend some perturbation results [8] to the class of Λ-Banach frame. Let us begin this section with the following result.

Theorem 5.1. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and let $\{y_n\}$ be a sequence in X such that $\{\Lambda(y_n)\} \in \mathcal{B}_d$, $\Lambda \in B(X,Y)$ and let $W : \mathcal{B}_d \to \mathcal{B}_d$ be a bounded linear operator such that $W(\{\Lambda(y_n)\}) = \{\Lambda(x_n)\}$, $\Lambda \in B(X,Y)$. Then there exists a bounded linear operator $S_0 : \mathcal{B}_d \to B(X,Y)$ such that $(\{y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ if and only if there exists a constant $M > 0$ such that*

$$\|\{\Lambda(x_n - y_n)\}\|_{\mathcal{B}_d} \leq M \min\{\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}, \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}\}, \qquad \Lambda \in B(X,Y).$$

Proof. Let A_1, B_1 be the Λ-frame bounds for Λ-Banach frame $(\{x_n\}, \Lambda, S)$ and A_2, B_2 be the Λ-frame bounds for Λ-Banach frame $(\{y_n\}, \Lambda, S_0)$. Then, the Λ-frame inequalities are given by,

$$A_1\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B_1\|\Lambda\|, \qquad \Lambda \in B(X,Y)$$

$$A_2\|\Lambda\| \leq \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \leq B_2\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Using above inequalities, we obtain

$$\begin{aligned}\|\{\Lambda(x_n - y_n)\}\|_{\mathcal{B}_d} &\leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} + \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \\ &\leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} + B_2\|\Lambda\| \\ &\leq \left(1 + \frac{B_2}{A_1}\right)\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|\{\Lambda(x_n - y_n)\}\|_{\mathcal{B}_d} \leq& \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} + \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}\\ \leq& B_1\|\Lambda\| + \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}\\ \leq& \left(1 + \frac{B_1}{A_2}\right)\|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}\end{aligned}$$

Choose $M = \max\{\left(1 + \frac{B_2}{A_1}\right), \left(1 + \frac{B_1}{A_2}\right)\}$. Then we get

$$\|\{\Lambda(x_n - y_n)\}\|_{\mathcal{B}_d} \leq M \min\{\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}, \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}\}, \qquad \Lambda \in B(X, Y).$$

Conversely, suppose that A and B are frame bounds for Λ-Banach frame $(\{x_n\}, \Lambda, S)$ for $B(X, Y)$. Then for each $\Lambda \in B(X, Y)$, we have

$$\begin{aligned}A\|\Lambda\| \leq& \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}\\ \leq& \|\{\Lambda(x_n) - \Lambda(y_n)\}\|_{\mathcal{B}_d} + \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}\\ \leq& (1 + M)\|\{\Lambda(y_n)\}\|_{\mathcal{B}_d}\\ \leq& (1 + M)(\|\{\Lambda(x_n) - \Lambda(y_n)\}\|_{\mathcal{B}_d} + \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d})\\ \leq& (1 + M)^2\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}\\ \leq& (1 + M)^2 B\|\Lambda\|.\end{aligned}$$

Put $S_0 = SW$. Then $S_0 : \mathcal{B}_d \to B(X, Y)$ is a bounded linear operator such that $S_0(\{\Lambda(y_n)\}) = \Lambda$, $\Lambda \in B(X, Y)$. Hence, $(\{y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$. □

Next, we give a condition under which the perturbation of a given Λ-Banach frame by uniformly scaled version of a given Λ-Bessel sequence (by an appropriately chosen scalar number) is still a Λ-Banach frame.

Theorem 5.2. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$ with a couple of frame bounds A and B. Let $\{y_n\}$ be a sequence in X such that*

$\{\Lambda(y_n)\} \in \mathcal{B}_d$, $\Lambda \in B(X,Y)$ and for some constant $M > 0$,

$$\|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \leq M\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Let $Q : \mathcal{B}_d \to \mathcal{B}_d$ be a bounded linear operator defined by

$$Q(\{\Lambda(x_n + \lambda y_n)\}) = \{\Lambda(x_n)\}, \qquad \Lambda \in B(X,Y),$$

where λ is a nonzero constant such that $|\lambda| < \frac{A}{M}$. Then there exists a reconstruction operator $S_0 : \mathcal{B}_d \to B(X,Y)$ such that $(\{x_n + \lambda y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and with a couple of frame bounds $(A - |\lambda|M)$ and $(B + |\lambda|M)$.

Proof. Clearly, $\{\Lambda(x_n + \lambda y_n)\} \in \mathcal{B}_d$, for each $\Lambda \in B(X,Y)$. Using the Λ-frame inequality for Λ-Banach frame $(\{x_n\}, \Lambda, S)$ we compute,

$$\begin{aligned}\|\{\Lambda(x_n + \lambda y_n)\}\|_{\mathcal{B}_d} &\leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} + |\lambda|\|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \\ &\leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} + |\lambda|M\|\Lambda\| \\ &\leq B\|\Lambda\| + |\lambda|M\|\Lambda\| = (B + |\lambda|M)\|\Lambda\|.\end{aligned}$$

Similarly, we compute

$$\begin{aligned}\|\{\Lambda(x_n + \lambda y_n)\}\|_{\mathcal{B}_d} &\geq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} - |\lambda|\|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \\ &\geq A\|\Lambda\| - |\lambda|M\|\Lambda\| = (A - |\lambda|M)\|\Lambda\|.\end{aligned}$$

Hence,

$$(A - |\lambda|M)\|\Lambda\| \leq \|\{\Lambda(x_n + \lambda y_n)\}\|_{\mathcal{B}_d} \leq (B + |\lambda|M)\|\Lambda\|, \qquad \Lambda \in B(X,Y).$$

Put $S_0 = SQ$. Then $S_0 : \mathcal{B}_d \to B(X,Y)$ is a bounded linear operator such that $S_0(\{\Lambda(x_n + \lambda y_n)\}) = \Lambda$, $\Lambda \in B(X,Y)$. Hence, $(\{x_n + \lambda y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$. □

Recall that, a sequence $\{\alpha_n\} \subset \mathbb{R}$ is said to be positively confined if $0 < \inf_{1\leq n<\infty} \alpha_n \leq \sup_{1\leq n<\infty} \alpha_n < \infty$. In the final result, we obtain a sufficient condition for the perturbation of Λ-Banach frame by a sequence of type $\{\alpha_n y_n\}$ (where $\{\alpha_n\} \subset \mathbb{R}$ be a positively confined sequence and $\{y_n\}$ be a sequence in a Banach space X) to be a Λ-Banach frame.

Theorem 5.3. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ with a couple of frame bounds A and B. Let $\{y_n\}$ be a sequence in X and $\{\alpha_n\} \subset \mathbb{R}$ be any positively confined sequence such that $\{\Lambda(\alpha_n y_n)\} \in \mathcal{B}_d$, $\Lambda \in B(X,Y)$. Let $Q : \mathcal{B}_d \to \mathcal{B}_d$ be a bounded linear operator defined by $Q(\{\Lambda(x_n + \alpha_n y_n)\}) = \{\Lambda(x_n)\}$, $\Lambda \in B(X,Y)$. If $U : B(X,Y) \to \mathcal{B}_d$ is a coefficient mapping defined by $U(\Lambda) = \{\Lambda(y_n)\}$, $\Lambda \in B(X,Y)$ with $\|U\| < \frac{A}{(\sup_{1\leq n<\infty} \alpha_n)}$. Then there exists a reconstruction operator $S_0 : B_d \to B(X,Y)$ such that $(\{x_n + \alpha_n y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and with frame bounds $[A - (\sup_{1\leq n<\infty} \alpha_n)\|U\|]$ and $[B + (\sup_{1\leq n<\infty} \alpha_n)\|U\|]$.*

Proof. Note that for each $\Lambda \in B(X,Y)$, $\{\Lambda(x_n + \alpha_n y_n)\} \in \mathcal{B}_d$. Now, we compute

$$\begin{aligned}
\|\{\Lambda(x_n + \alpha_n y_n)\}\|_{\mathcal{B}_d} \leq & \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} + \|\{\Lambda(\alpha_n y_n)\}\|_{\mathcal{B}_d} \\
\leq & B\|\Lambda\| + \left(\sup_{1\leq n<\infty} \alpha_n\right) \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \\
\leq & \left[B + \left(\sup_{1\leq n<\infty} \alpha_n\right)\|U\|\right] \|\Lambda\|.
\end{aligned}$$

Again,

$$\begin{aligned}
\|\{\Lambda(x_n + \alpha_n y_n)\}\|_{\mathcal{B}_d} \geq & \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} - \|\{\Lambda(\alpha_n y_n)\}\|_{\mathcal{B}_d} \\
\geq & A\|\Lambda\| - \left(\sup_{1\leq n<\infty} \alpha_n\right) \|\{\Lambda(y_n)\}\|_{\mathcal{B}_d} \\
\geq & \left[A - \left(\sup_{1\leq n<\infty} \alpha_n\right)\|U\|\right] \|\Lambda\|
\end{aligned}$$

Hence, for each $\Lambda \in B(X,Y)$ we obtain

$$\left[A-\left(\sup_{1\le n<\infty}\right)\|U\|\right]\|\Lambda\| \le \|\{\Lambda(x_n+\alpha_n y_n)\}\|_{\mathcal{B}_d} \le \left[B+\left(\sup_{1\le n<\infty}(\alpha_n)\right)\|U\|\right]\|\Lambda\|.$$

Put $S_0 = SQ$. Then $S_0 : \mathcal{B}_d \to B(X,Y)$ is a bounded linear operator such that $S_0(\{\Lambda(x_n+\alpha_n y_n)\}) = \Lambda$, $\Lambda \in B(X,Y)$. Hence, $(\{x_n+\alpha_n y_n\}, \Lambda, S_0)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$ and having required frame bounds. □

6. Application

Let us now deal with an application of Λ-Banach frame in eigenvalue problem, which is inspired by Theorem 5.2 in [10]. In this section, we assume that X and Y be Banach spaces over the same field $\mathbb{K} = \mathbb{R}$, endowed with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and recall that $\mathcal{B}_d = \{\{\Lambda(x_n)\} : \Lambda \in B(X,Y)\}$ $(\{x_n\} \subset X)$ be the Banach space associated with Y.

Theorem 6.1. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$. Let $\{z_k\}_{k=1}^m$ be linearly independent set of vectors in X and let for each k $(1 \le k \le m)$ there exists an operator $\Theta_k \in B(X,Y)$ such that $\|\Theta_k(x_n)\|_Y = c_k^{(n)}$, for all $n \in \mathbb{N}$. If $(\{x_n + \frac{1}{\lambda}\sum_{k=1}^m c_k^{(n)} z_k\}, \Lambda, S)$ is a Λ-Banach frame for $B(X,Y)$ with respect to $\mathcal{B}_d$, where λ is any nonzero real number and $S : \mathcal{B}_d \to B(X,Y)$ is a reconstruction operator, then $-\lambda$ is not an eigenvalue of the matrix*

$$\begin{pmatrix} \|\Theta_1(z_1)\|_Y & \|\Theta_2(z_1)\|_Y & \dots & \|\Theta_m(z_1)\|_Y \\ \|\Theta_1(z_2)\|_Y & \|\Theta_2(z_2)\|_Y & \dots & \|\Theta_m(z_2)\|_Y \\ \vdots & \vdots & \vdots & \vdots \\ \|\Theta_1(z_m)\|_Y & \|\Theta_2(z_m)\|_Y & \dots & \|\Theta_m(z_m)\|_Y \end{pmatrix}.$$

Proof. Assume that $-\lambda$ is an eigenvalue of the given matrix. Then we have,

$$\begin{vmatrix} \|\Theta_1(z_1)\|_Y + \lambda & \|\Theta_2(z_1)\|_Y & \dots & \|\Theta_m(z_1)\|_Y \\ \|\Theta_1(z_2)\|_Y & \|\Theta_2(z_2)\|_Y + \lambda & \dots & \|\Theta_m(z_2)\|_Y \\ \vdots & \vdots & \vdots & \vdots \\ \|\Theta_1(z_m)\|_Y & \|\Theta_2(z_m)\|_Y & \dots & \|\Theta_m(z_m)\|_Y + \lambda \end{vmatrix} = 0.$$

So, there exist scalars $\alpha_i \in \mathbb{R}$ $(i = 1, 2, ..., m)$ such that

$$\alpha_1\|\Theta_1(z_1)\|_Y + \alpha_2\|\Theta_2(z_1)\|_Y + \ldots + \alpha_m\|\Theta_m(z_1)\|_Y = -\lambda\alpha_1$$

$$\alpha_1\|\Theta_1(z_2)\|_Y + \alpha_2\|\Theta_2(z_2)\|_Y + \ldots + \alpha_m\|\Theta_m(z_2)\|_Y = -\lambda\alpha_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\alpha_1\|\Theta_1(z_m)\|_Y + \alpha_2\|\Theta_2(z_m)\|_Y + \ldots + \alpha_m\|\Theta_m(z_m)\|_Y = -\lambda\alpha_m.$$

Now, we put $\Lambda = -\lambda \sum_{k=1}^{m} \alpha_k\Theta_k$. Then, Λ is a nonzero operator in $B(X, Y)$ such that

$$\Lambda(x_n) = -\lambda \sum_{k=1}^{m} \alpha_k\Theta_k(x_n).$$

Thus we obtain

$$\|\Lambda(x_n)\|_Y \leq \lambda \sum_{k=1}^{m} \alpha_k c_k^{(n)}, \tag{6.1}$$

where $\|\Theta_k(x_n)\|_Y = c_k^{(n)}$, for $k = 1, 2, ..., m$.

Further, $\Lambda(z_k) = -\lambda \sum_{i=1}^{m} \alpha_i\Theta_i(z_k)$, $(k = 1, 2, ..., m)$, gives

$$\|\Lambda(z_k)\|_Y \leq \lambda \sum_{i=1}^{m} \alpha_i\|\Theta_i(z_k)\|_Y = -\lambda^2\alpha_k. \tag{6.2}$$

Therefore, using (6.1) and (6.2), we obtain

$$\begin{aligned} \left\|\Lambda\left(x_n + \frac{1}{\lambda}\sum_{k=1}^{m} c_k^{(n)} z_k\right)\right\|_Y &\leq \|\Lambda(x_n)\|_Y + \frac{1}{\lambda}\sum_{k=1}^{m} c_k^{(n)}\|\Lambda(z_k)\|_Y \\ &\leq \lambda\sum_{k=1}^{m} \alpha_k c_k^{(n)} + \frac{1}{\lambda}\sum_{k=1}^{m} c_k^{(n)}(-\lambda^2\alpha_k) = 0. \end{aligned}$$

Hence, $\Lambda(x_n+\frac{1}{\lambda}\sum_{k=1}^{m} c_k^{(n)} z_k) = 0$, $(k = 1, 2, ..., m)$. Since, $(\{x_n+\frac{1}{\lambda}\sum_{k=1}^{m} c_k^{(n)} z_k\}, \Lambda, S)$ is a Λ-Banach frame for $B(X, Y)$, it follows by the Λ-frame inequality that $\Lambda = 0$, a contradiction. □

As a spacial case of Theorem 6.1, for $\lambda = 1$, we have the following result:

Corollary 6.1. *Let $(\{x_n\}, \Lambda, S)$ be a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$. Let $\{z_k\}_{k=1}^{m}$ be linearly independent set of vectors in X and let for each k $(1 \leq k \leq m)$ there exists an operator $\Theta_k \in B(X, Y)$ such that $\|\Theta_k(x_n)\|_Y = c_k^{(n)}$, for all $n \in \mathbb{N}$. If $(\{x_n+\sum_{k=1}^{m} c_k^{(n)} z_k\}, \Lambda, S)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$, where $S : \mathcal{B}_d \to B(X, Y)$ is a reconstruction operator, then -1 is not an eigenvalue of the matrix given in Theorem 6.1.*

Proof. Straight forward. □

Remark 3. If $Y = \mathbb{K}$ is the field of scalars, then Λ-Banach frame reduces to the retro Banach frame (see remark 1), and hence, Theorem 5.2 in [10] can be retrieve by taking Y as the field of scalars in Corollary 6.1.

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