Jordan Journal of Mathematics and Statistics (JJMS) 11(1), 2018, pp 33 - 49

GENERALIZATION OF VALUE DISTRIBUTION AND UNIQUENESS OF CERTAIN TYPES OF DIFFERENCE POLYNOMIALS

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ABSTRACT. In this paper, we study the distribution of zeros and uniqueness of differential polynomials of the form $f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{s_j}$ and $f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ where $c_j (j = 1, 2, \cdots, d)$ are complex constants, $v_j (j = 1, 2, \cdots, d)$ are non-negative integers and $\sigma = \sum_{j=1}^{d} v_j$ sharing a small function with finite weight. The result obtained improves and generalizes the recent result.

1. INTRODUCTION

In this article, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory (see, for example, [9],[22]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of finite logarithmic measure. Then, the meromorphic function α is called a small function of $f(z)$, if $T(r, \alpha) = S(r, f)$. If $f(z) - \alpha$ and $g(z) - \alpha$ have the same zeros, counting multiplicity (ignoring multiplicity), then we say $f(z)$ and $g(z)$ share a small function α CM (IM). For a small function α related to $f(z)$, we define

$$\delta(\alpha, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-\alpha})}{T(r, f)},$$

1991 *Mathematics Subject Classification.* 30D35.

Key words and phrases. Value distribution, Difference Polynomials, Fixed points, Sharing value.

Received: June 27, 2017 Accepted: Sept. 24, 2017 .

$$\Theta(\alpha, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-\alpha})}{T(r, f)}.$$

In 2010, X. G. Qi, L. Z. Yang and K. Liu [14] proved the following uniqueness result.

Theorem 1.1. *Let f, g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that c is a non zero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = g$.*

In 2013, S. S. Bhoosnrmath and S. R. Kabbur [2] considered the zeros of difference polynomials of the form $f^n(z)(f^m(z)-1)f(z+c)$, where n, m are positive integers and c is a non zero complex constant and obtained the following theorems.

Theorem 1.2. *Let f be an entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c a non zero complex constant and n, m are positive integers. If $n \geq 2$, then $f^n(z)(f^m(z)-1)f(z+c) - \alpha(z)$ has infinitely many zeros.*

Theorem 1.3. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant and n, m are positive integers such that $n \geq m+6$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = tg$, where $t^m = 1$.*

Theorem 1.4. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant and n, m are positive integers such that $n \geq 4m+12$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ IM, then $f = tg$, where $t^m = 1$.*

Recently, P. Sahoo and B. Saha [21] studied the zeros and uniqueness of certain type of difference polynomial sharing a small function with finite weight and obtained the following results.

Theorem 1.5. *Let f be a transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a non zero complex constant, $n(\geq 1), m(\geq 1)$ and $k(\geq 0)$ are integers. If $n \geq k+2$, then $[f^n(z)(f^m(z)-1)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem 1.6. *Let f be a transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers. If $n \geq k+2$, when $m \leq k+1$ and $n \geq 2k-m+3$ when $m > k+1$, then $[f^n(z)(f(z)-1)^m f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem 1.7. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$. If $[f^n(z)(f^m(z)-1)f(z+c)]^{(k)}$ and $[g^n(z)(g^m(z)-1)g(z+c)]^{(k)}$ share $(\alpha, 2)$ then $f = tg$, where $t^m = 1$.*

Theorem 1.8. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$ when $m \leq k+1$ and $n \geq 4k-m+10$ when $m > k+1$. If $[f^n(z)(f(z)-1)^m f(z+c)]^{(k)}$ and $[g^n(z)(g(z)-1)^m g(z+c)]^{(k)}$ share $(\alpha, 2)$ then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by*

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \omega_1(z+c) - \omega_2^n(\omega_2 - 1)^m \omega_2(z+c).$$

Theorem 1.9. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero* *complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 5k + 4m + 12$. If $[f^n(z)(f^m(z) - 1)f(z + c)]^{(k)}$ and $[g^n(z)(g^m(z) - 1)g(z + c)]^{(k)}$ share $\alpha(z)$ IM, then $f = tg$, where $t^m = 1$.*

Theorem 1.10. *Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 5k + 4m + 12$ when $m \leq k + 1$ and $n \geq 10k - m + 19$ when $m > k + 1$.If $[f^n(z)(f(z) - 1)^m f(z + c)]^{(k)}$ and $[g^n(z)(g(z) - 1)^m g(z + c)]^{(k)}$ share $\alpha(z)$, then conclusion of Theorem H hold.*

Regarding Theorems 1.5-1.10, the following question is inevitable which is motivation of the present paper.

Question. What would happen if one replaces the difference polynomials $[f^n(z)(f^m(z)-1)f(z + c)]^{(k)}$ by $f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j}$ in Theorems 1.5-1.10, where k is any positive integer?

In this paper, we study the zero and uniqueness of difference polynomial of the form $f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j}$ and $f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f(z + c_j)^{v_j}$ where $c_j(j = 1, 2, \cdots, d)$ are complex constants, $v_j(j = 1, 2, \cdots, d)$ are non-negative integers and $\sigma = v_1 + v_2 + \cdots + v_d = \sum_{j=1}^{d} v_j$ and hence obtain the following results.

Theorem 1.11. *Let f be a transcendental entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that $c_j(j = 1, 2, \cdots, d)$ are non zero complex constants, $v_j(j = 1, 2, \cdots, d)$ are non-negative integers, $n, m \geq 1$ and $k(\geq 0)$ are integers. If $n \geq k + 2$, then $\left[f^n(z)(f^m(z) - 1) \prod_{j=1}^{d} f(z + c_j)^{v_j}\right]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem 1.12. *Let f be a transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that $c_j(j = 1, 2, \cdots, d)$ is a non zero complex constants, $v_j(j = 1, 2, \cdots, d)$ are non-negative integers, $n, m \geq 1$ and $k(\geq 0)$*

are integers. If $n \geq k+2$ *when* $m \leq k+1$ *and* $n \geq 2k-m+3$ *when* $m > k+1$, *then* $\left[f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)} - \alpha(z)$ *has infinitely many zeros.*

Theorem 1.13. *Let* f *and* g *be two transcendental entire functions of finite order and* $\alpha(z)(\not\equiv 0)$ *be a small function with respect to* f *and* g. *Suppose that* $c_j(j = 1, 2, \cdots, d)$ *are non zero complex constants,* $v_j(j = 1, 2, \cdots, d)$ *are non-negative integers,* $n, m \geq 1$ *and* $k(\geq 0)$ *are integers satisfying* $n \geq 2k + m + \sigma + 5$. *If* $\left[f^n(z)(f^m(z)-1) \prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ *and* $\left[g^n(z)(g^m(z)-1) \prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$ *share* $(\alpha, 2)$, *then* $f = tg$ *where* $t^m = 1$.

Theorem 1.14. *Let* f *and* g *be two transcendental entire functions of finite order and* $\alpha(z)(\not\equiv 0)$ *be a small function with respect to* f *and* g. *Suppose that* $c_j(j = 1, 2, \cdots, d)$ *are non zero complex constants,* $v_j(j = 1, 2, \cdots, d)$ *are non-negative integers,* $n, m \geq 1$ *and* $k(\geq 0)$ *are integers satisfying* $n \geq 2k + m + \sigma + 5$ *when* $m \leq k + 1$ *and* $n \geq 4k - m + \sigma + 9$ *when* $m > k + 1$. *If* $\left[f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ *and* $\left[g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$ *share* $(\alpha, 2)$, *then either* $f = g$ *or* f *and* g *satisfy the algebraic equation* $R(f, g) = 0$ *where* $R(f, g)$ *is given by*
$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \prod_{j=1}^{d} \omega_1(z+c_j)^{v_j} - \omega_2^n(\omega_2 - 1)^m \prod_{j=1}^{d} \omega_2(z+c_j)^{v_j}$.

Theorem 1.15. *Let* f *and* g *be two transcendental entire functions of finite order and* $\alpha(z)(\not\equiv 0)$ *be a small function with respect to* f *and* g. *Suppose that* $c_j(j = 1, 2, \cdots, d)$ *are non-negative complex constants,* $v_j(j = 1, 2, \cdots, d)$ *are non-negative integers,* $n, m \geq 1$ *and* $k(\geq 0)$ *are integers satisfying* $n \geq 5k + 4m + 4\sigma + 8$. *If* $\left[f^n(z)(f^m(z)-1) \prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ *and* $\left[g^n(z)(g^m(z)-1) \prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$ *share* $\alpha(z)$ *IM, then* $f = tg$ *where* $t^m = 1$.

Theorem 1.16. *Let* f *and* g *be two transcendental entire functions of finite order and* $\alpha(z)(\not\equiv 0)$ *be a small function with respect to* f *and* g. *Suppose that* $c_j(j = 1, 2, \cdots, d)$ *are non zero complex constants,* $v_j(j = 1, 2, \cdots, d)$ *are non-negative integers,* $n, m \geq 1$

and $k(\geq 0)$ *are integers satisfying* $n \geq 5k + 4m + 4\sigma + 8$ *when* $m \leq k + 1$ *and* $n \geq 10k - m + 4\sigma + 15$ *when* $m > k + 1$. *If* $\left[f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{v_j}\right]^{(k)}$ *and* $\left[g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{v_j}\right]^{(k)}$ *share* $\alpha(z)$ *IM, then the conclusion of theorem 4 hold.*

Remark. *For* $\sigma = 1$ *in Theorems 1.11 to 1.16, we get Theorems 1.5 to 1.10. Hence Theorems 1.11 to 1.16 generalizes Theorems 1.5 to 1.10.*

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in the complex plane $\mathbb{C}$. We denote by H the following function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right)$$

Lemma 2.1. [13] *Let* f *be a meromorphic function of finite order* ρ *and let* $c(\neq 0)$ *be a fixed non zero complex constant. Then*

$$\overline{N}(r, \infty; f(z + c)) \leq \overline{N}(r, \infty; f) + S(r, f)$$

outside a possible exceptional set of finite logarithmic measure.

Lemma 2.2. [3] *Let* f *be an entire function of finite order and* $F = f^n(z)(f^m(z) - 1)f(z + c)$. *Then* $T(r, F) = (n + m + 1)T(r, f) + S(r, f)$.

Arguing in a similar manner as in Lemma 2.6[3] *we obtain the following Lemma.*

Lemma 2.3. *Let* f *be an entire function of finite order and* $F = f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)$. *Then* $T(r, F) = (n + m + \sigma)T(r, f) + S(r, f)$.

Lemma 2.4. [20] *Let* f *be a non-constant meromorphic functions and* p, k *be two positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f) \tag{2.1}$$

$$N_p(r,0;f^{(k)}) \le k\overline{N}(r,\infty,f) + N_{p+k}(r,0;f) + S(r,f) \tag{2.2}$$

Lemma 2.5. [10] *Let f and g be two non-constant meromorphic functions sharing (1,2). Then one of the following cases holds.*

(i) $T(r) \le N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r)$,

(ii) $f = g$,

(iii) $fg = 1$, *when* $T(r) = max\{T(r,f), T(r,g)\}$ *and* $S(r) = o\{T(r)\}$.

Lemma 2.6. [1] *Let F and G be two non-constant meromorphic functions sharing the value 1 IM and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \le & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) \\ & + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,F) + S(r,G) \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.7. *Let f and g be two entire functions, suppose that $c_j (j = 1, 2, \cdots, d)$ are non zero complex constants, $v_j (j = 1, 2, \cdots, d)$ are non-negative integers, $n, m \ge 1$ and $k(\ge 0)$ are integers and let $F = \left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ and $G = \left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$. If there exists non zero constants c_1 and c_2 such that $\overline{N}(r,c_1;F) = \overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G) = \overline{N}(r,0;F)$, then $n \le 2k+m+\sigma+2$.*

Proof. We put $F_1 = f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ and $G_1 = g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}$, by the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,0;F) + \overline{N}(r,c_1;F) + S(r,F) \\ &\le \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) \end{aligned} \tag{2.3}$$

Using equation (2.3), in Lemmas 2.2 and 2.4, we obtain

$$
\begin{aligned}
(n+m+\sigma)T(r,f) &\le T(r,F) - \overline{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f)\\
&\le \overline{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f)\\
&\le N_{k+1}(r,0;F_1) + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g)\\
&\le (k+m+\sigma+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g).
\end{aligned} \tag{2.4}
$$

Similarly,

$$(n+m+\sigma)T(r,g) \le (k+m+\sigma+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$(n-2k-m-\sigma-2)(T(r,f)+T(r,g)) \le S(r,f)+S(r,g).$$

Which gives $n \le 2k+m+\sigma+2$.
This proves the lemma.

Lemma 2.8. *Let f and g be two entire functions, suppose that $c_j (j = 1, 2, \cdots, d)$ are non zero complex constants, $v_j (j = 1, 2, \cdots, d)$ are non-negative integers, $n, m \ge 1$ and $k (\ge 0)$ are integers and let $F = \left[f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ and $G = \left[g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$. If there exists non zero constants c_1 and c_2 such that $\overline{N}(r, c_1; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, c_2; G) = \overline{N}(r, 0; F)$, then $n \le 2k+m+\sigma+2$ when $m \le k+1$ and $n \le 4k-m+\sigma+4$ when $m > k+1$.*
Proof. By the same reasoning as in proof of Lemma 2.7, we can easily deduce the result. Hence we omit the details.
Arguing in a similar manner as in lemma 2.5([2]), we obtain the following lemma.

Lemma 2.9. *Suppose that f and g are two transcendental entire function of finite order. Suppose that $c_j (j = 1, 2, \cdots, d)$ are non zero complex constants, $v_j (j = 1, 2, \cdots, d)$ are non-negative integers, $n, m \ge 1$ and $k (\ge 0)$ are integers. If $n \ge m+5$*

and $\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)} = \left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$ *then* $f = tg$ *where* $t^m = 1$.

3. Proofs of Main Theorems

Proof of Theorem 1.11. Let $F_1 = f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$. Then F_1 is a transcendental entire function.
If possible, we assume $F_1^{(k)} - \alpha(z)$ has only finitely many zeros. Then, we have

$$N(r,\alpha,F_1^{(k)}) = o\{\log r\} = S(r,f). \tag{3.1}$$

Using (2.1), (3.1) and Nevanlinna's three small function theorem, we obtain

$$\begin{aligned} T(r,F_1^{(k)}) &\le \overline{N}(r,0,F_1^{(k)}) + \overline{N}(r,\alpha;F_1^{(k)}) + S(r,f) \\ &\le T(r,F_1^{(k)}) - T(r,F_1) + N_{k+1}(r,0;F_1) + S(r,f) \end{aligned} \tag{3.2}$$

Applying lemma 2.2, we obtain from (3.2),

$$\begin{aligned} (n+m+\sigma)T(r,f) &\le N_{k+1}(r,0;F_1) + S(r,f) \\ &\le (k+m+\sigma+1)T(r,f) + S(r,f) \end{aligned}$$

This gives

$$(n-k-1)T(r,f) \le S(r,f),$$

a contradiction with the assumption that $n \ge k+2$. This proves the theorem. □

Proof of Theorem 1.12. Let $F_2 = f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{v_j}$. Then F_2 is a transcendental entire function.
If possible, suppose that $F_2^{(k)} - \alpha(z)$ has only finitely many zeros. Then, we have

$$N(r,\alpha,F_2^{(k)}) = o\{\log r\} = S(r,f) \tag{3.3}$$

Now, using (2.1), (3.3) and Nevanlinna's three small function theorem, we obtain

$$(3.4)\quad \begin{aligned} T(r,F_2^{(k)}) &\le \overline{N}(r,0,F_2^{(k)}) + \overline{N}(r,\alpha;F_2^{(k)}) + S(r,f) \\ &\le T(r,F_2^{(k)}) - T(r,F_2) + N_{k+1}(r,0;F_2) + S(r,f) \end{aligned}$$

Applying Lemma 2.3, we obtain from (3.4)

$$(3.5)\quad \begin{aligned} (n+m+\sigma)T(r,f) &\le N_{k+1}(r,0;F_2) + S(r,f) \\ &\le (k+m+\sigma+1)T(r,f) + S(r,f) \end{aligned}$$

If $m \le k+1$, we deduce from (3.5) that

$$(n-k-1)T(r,f) \le S(r,f),$$

a contradiction to the assumption that $n \ge k+2$.

If $m < k+1$, by (3.5) we get,

$$(n+m-2k-2)T(r,f) \le S(r,f)$$

a contradiction with the assumption that $n \ge 2k-m+3$. This proves the theorem.

□

Proof of Theorem 1.13. Let $F_1 = f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$
and $G_1 = g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}$,
$F = \dfrac{F_1^{(k)}}{\alpha(z)}$ and $G = \dfrac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share (1,2) except the zeros and poles of $\alpha(z)$. Using (2.1) and Lemma 2.2, we get

$$\begin{aligned} N_2(r,0;F) &\le N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\le T(r,F_1^{(k)}) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f) \\ &\le T(r,F) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f) \end{aligned}$$

From this we get,

$$(n+m+\sigma)T(r,f) \leq T(r,F) + N_{k+2}(r,0;F_1) - N_2(r,0;F) + S(r,f) \tag{3.6}$$

Again by (2.2), we have

$$N_2(r,0;F) \leq N_2(r,0;F_1^{(k)}) + S(r,f) \leq N_{k+2}(r,0;F_1) + S(r,f) \tag{3.7}$$

Suppose, if possible that (i) of Lemma 2.5 holds, Then, using (3.7), we obtain from (3.6)

$$\begin{aligned}(n+m+\sigma)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F)\\ &\quad + N_2(r,\infty;G) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g)\\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g)\\ &\leq (k+2+m+\sigma)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g) \qquad (3.8)\end{aligned}$$

In a similar manner we obtain,

$$(n+m+\sigma)T(r,g) \leq (k+2+m+\sigma)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{3.9}$$

From (3.8) and (3.9) together gives,

$$(n-2k-m-\sigma-4)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

contradicting with the fact that $n \geq 2k+m+\sigma+5$. Therefore, by Lemma 2.5 we have either $FG = 1$ or $F = G$.

Let $FG = 1$. Then,

$$\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)} \cdot \left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)} = \alpha^2$$

$\left[f^n(z)(f(z)-1)(f^{m-1}(z)+f^{m-2}(z)+\cdots+1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}\cdot\left[g^n(z)(g(z)-1)(g^{m-1}(z)+g^{m-2}(z)+\cdots+1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\right]^{(k)}=\alpha^2$

It can be easily viewed from above that

$$N(r,0;f)=S(r,f) \text{ and } N(r,1;f)=S(r,f)$$

Thus,

$$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3,$$

Which is not possible. Therefore, we must have $F=G$, and then

$$\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}=\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\right]^{(k)}$$

Integrating above, we get,

$$\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k-1)}=\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\right]^{(k-1)}+C_{k-1}$$

Where C_{k-1} is a constant. If $C_{k-1}\neq 0$, using Lemma 2.7, it follows that $n\leq 2k+m+\sigma$ a contradiction. Hence $C_{k-1}=0$, repeating k times, we deduce that,

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}=g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}$$

which by Lemma 2.9, gives $f=tg$ where t is a constant satisfying $t^m=1$. This proves Theorem 1.13. □

Proof of Theorem 1.14. Let $F_1=f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{v_j}$ and $G_1=g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{v_j}$. $F=\dfrac{F_1^{(k)}}{\alpha(z)}$ and $G=\dfrac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share (1,2) except possibly the zeros and poles of $\alpha(z)$.

Arguing in a manner similar to the proof of Theorem 3.13, we obtain either $FG=1$ or $F=G$.

If $F = G$, then applying the same techniques as in the proof of Theorem 3.13 and using Lemma 2.8, we obtain.

$$f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j} = g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j} \tag{3.10}$$

Set $h = \dfrac{f}{g}$. If h is a constant, then substituting $f = gh$ in equation (3.10), we duduce that

$$g^n \prod_{j=1}^{d} g(z+c_j)^{v_j} \Big[g^m(h^{n+m+\sigma}-1) - {}^m c_1 g^{m-1}(h^{n+m+\sigma-1}-1) + \cdots + (-1)^m(h^{n+\sigma}-1) \Big] = 0$$

Since g is a transcendental entire function, we have $g^n \prod_{j=1}^{d} g(z+c_j)^{v_j} \neq 0$. So from above we obtain,

$$g^m(h^{n+m+\sigma}-1) - {}^m c_1 g^{m-1}(h^{n+m+\sigma-1}-1) + \cdots + (-1)^m(h^{n+\sigma}-1) = 0$$

which implies $h = 1$

Hence $f = g$. If h is not a constant, then it follows from equation (3.10) that f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \prod_{j=1}^{d} \omega_1(z+c_j)^{v_j} - \omega_2^n(\omega_2 - 1)^m \prod_{j=1}^{d} \omega_2(z+c_j)^{v_j}$$

If $FG = 1$, proceeding in a like manner as in the proof of Theorem 1.13 we arrive at a contradiction. This completes the proof of Theorem 1.14. □

Proof of Theorem 1.15. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 3.13. Then, F and G are transcendental meromorphic functions that share the value 1 IM except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H \not\equiv 0$. Using

Lemma 2.6 and (3.7), we obtain from (3.6).

(3.11)

$$\begin{aligned}(n+m+\sigma)T(r,f) \leq & N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)\\ & +N_{k+2}(r,0;F_1)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\ \leq & N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+2N_{k+1}(r,0;F_1)\\ & +2N_{k+1}(r,0;G_1)+S(r,f)+S(r,g)\\ \leq & (3k+4+3m+3\sigma)T(r,f)+(2k+3+2m+2\sigma)T(r,g)+\\ & S(r,f)+S(r,g)\\ \leq & (5k+5m+5\sigma+7)T(r)+S(r)\end{aligned}$$

Similarly,

$$(n+m+\sigma)T(r,f) \leq (5k+5m+12)T(r)+S(r). \tag{3.12}$$

From equations (3.11) and (3.12), together yields

$$(n-4m-4\sigma-5k-7)T(r) \leq S(r),$$

which is a contradiction with the assumption that $n \geq 5k+4m+4\sigma+8$. We now assume that $H \equiv 0$. Then

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0$$

Integrating both sides of above equality twice, we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B. \tag{3.13}$$

Where $A(\neq 0)$ and B are constants.

From (3.13) it is obvious that F, G share value 1 CM and hence they share (1,2). Therefore $n \geq 2k+m+\sigma+5$.

We now discuss the following three cases separately.

case i. Suppose that $B \neq 0$ and $A = B$, then from (3.13) we obtain.

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.14}$$

If $B = -1$, then from (3.14), we obtain $FG = 1$, which is a contradiction as in the proof of Theorem 1.13.

If $B \neq -1$, from (3.14), we have,

$$\frac{1}{F} = \frac{BG}{(1+B)G-1}$$

and so $\overline{N}\left(r, \frac{1}{1+B}; G\right) = \overline{N}(r, 0; F)$.
Using (2.1), (2.2) and Second Fundamental Theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1) - (n+m+\sigma)T(r,g) + S(r,g) \end{aligned}$$

This gives,

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+1)\{T(r,f) + T(r,g)\} + S(r,g)$$

Thus we obtain

$$(n-2k-m-\sigma-2)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction as $n \geq 2k+m+\sigma+5$.

case ii. Let $B \neq 0$ and $A \neq B$. Then From (3.13), we get

$$F = \frac{(B+1)G - (B-A+1)}{BG + (A-B)},$$

and so $\overline{N}\left(r, \frac{B-A+1}{B+1}; G\right) = \overline{N}(r, 0; F)$,
Proceeding in a manner similar to case i we can arrive at a contradiction.

case iii. Let $B = 0$ and $A \neq 0$. Then from (3.13) we get

$$F = \frac{G + A - 1}{A} \text{ and } G = AF - (A - 1)$$

If $A \neq 1$, it follows that $\overline{N}\left(r, \frac{A-1}{A}; F\right) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1 - A; G) = \overline{N}(r, 0; F)$ Now applying Lemma 2.7, it can be shown that $n \leq 2k + m + \sigma + 2$, which is a contradiction.

Thus, $A = 1$ and then $F = G$. Now the result follows from the proof of Theorem 1.13. This completes the proof of Theorem 1.15. □

Proof of Theorem 1.16. Arguing in a like manner as in proof of Theorem 1.15, the conclusion of Theorem 1.16 follows. Here we omit the details. □

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