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ON CONES ASSOCIATED WITH SCHAUDER FRAMES

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Abstract. In this paper, we define cone $C_{\mathcal{F}}$ associated with Schauder frame $\mathcal{F} = (\{x_n\}, \{f_n\})$ in Banach spaces. A necessary and sufficient condition for a cone $C_{\mathcal{F}}$ to be normal has been given. Also, we have obtained a sufficient condition for the cone $C_{\mathcal{F}}$ to be minihedral and show that the converse is not true in general. Moreover, we have given some necessary and sufficient conditions for the cone $C_{\mathcal{F}}$ to be generating and has a bounded base. Finally, a characterization of normalized Schauder frames of type P^*, al_+ and w_0 and a sufficient condition for the cone $C_{\mathcal{F}_1}$ associated with normalized shrinking Schauder frame $\mathcal{F}_1 = \{(e_n x_n, e_n f_n)\}$ has been given.

1. Introduction

Dennis Gabor [13] in 1946 gave a fundamental approach to signal decomposition in terms of elementary signals. Later, in 1952, Duffin and Schaeffer [10] abstracted Gabor's method to define frames for Hilbert spaces. These ideas did not generate much interest outside of non-harmonic Fourier series and signal processing until Daubechies, Grossmann and Meyer reintroduced frames in their land mark paper [8]. Frames are generalizations of orthonormal bases in Hilbert spaces. The main property of frames which makes them useful is their redundancy. Representation

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of signals using frames is advantageous over basis expansions in a variety of practical applications in science and engineering. In particular, frames are widely used in sampling theory [1, 11], wavelet theory [9], wireless communication [16, 28], signal processing [5], image processing [22], pseudo-differential operators [14], filter banks [3], geophysics [7], quantum computing [12], wireless sensor network [18], coding theory [29] and many more. The reason for such wide applications is that frames provide both great liberties in design of vector space.

Han and Larson [15] defined Schauder frames for Banach spaces. Casazza et al. [6] gave various definitions of frames for Banach spaces including that of Schauder frame for Banach spaces. Casazza et al. [4] in 2008 studied the coefficient quantization of Schauder frames in Banach spaces. Liu and Zheng [25] gave a characterization of Schauder frames which are near Schauder bases, which generalized some result due to Holub [17]. Beanland et al. [2] proved that the upper and lower estimates theorems for finite-dimensional decompositions of Banach spaces can be extended and modified to Schauder frames and gave a complete characterization on duality for Schauder frames. Liu [24] associated an operator with a Schauder frame and called it Hilbert-Schauder frame operator. In 2013, the concept of weak* Schauder frames for conjugate Banach spaces was introduced and studied by Kaushik et al. [21] and later on in 2014 Poumai [27] defined and studied various types of Schauder frames. Also he gave some theoretical application of theses types of Schauder frames. Marin and Sama [26] studied cones associated with Schauder basis in Banach spaces. shah et al. [20] have related the notation of retro Banach frame with the geometric notion of a support cone in Banach spaces. Shah et. al [19] relate Banach frames to another geometric notion called cone and associated it with Banach frames.

In the present paper, we relate Schauder frames to another geometric notion of Banach spaces. In fact, we study cones associated with various types of Schauder frames and gave an example of a cone $C_{\mathcal{F}}$ which is minihedral but not normal. Further, some necessary and sufficient conditions are given and some interesting and new results have been obtained in the context of Schauder frames.

2. Preliminaries

Throughout this paper $\mathcal{X}$ will denote an infinite dimensional real Banach space, $\mathcal{X}^*$ denote the conjugate space of $\mathcal{X}$. For a sequence $\{x_n\} \in \mathcal{X}$ and $\{f_n\} \in \mathcal{X}^*$, $[x_n]$ denotes the closure of linear span of $\{x_n\}$ in the norm topology of $\mathcal{X}$ and $[\tilde{f_n}]$ the closure of $\{f_n\}$ in the *weak*$^*-$topology of $\mathcal{X}^*$. A sequence space S is called a *BK-space* if it is a Banach space and the co-ordinate functionals are continuous on S, i.e., the relations $x_n = \{{\alpha_j}^{(n)}\}$, $x = \{\alpha_j\} \in S$ and $\lim\limits_{n\longrightarrow\infty} x_n = x$ imply $\lim\limits_{n\longrightarrow\infty} \alpha_j^{(n)} = \alpha_j$ $(j = 1, 2, 3, ...)$. The notion of Schauder frame was introduced and studied by Han and Larson [15] and they gave the following definition:

Definition 2.1. *Let $\mathcal{X}$ be a Banach space. A sequence $\{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ is called Schauder frame for $\mathcal{X}$ if*

$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \text{ for all } x \in \mathcal{X}.$$

Also, recall that a Schauder frame $\{(x_n, f_n)\}(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ is called normalized Schauder frame if $\|x_n\| = 1$ and $\|f_n\| = 1$.

3. Main Result

Let $\mathcal{X}$ be a real Banach space and let $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ be a Schauder frame for $\mathcal{X}$. Define $\mathcal{C}_{\mathcal{F}} = \{x \in \mathcal{X} : f_n(x) \geq 0, \text{ for all } n \in \mathbb{N}\}$. Then $\mathcal{C}_{\mathcal{F}}$ is a *cone associated with the Schauder frame* $\mathcal{F}$ and satisfies the following properties:

(i) $\mathcal{C}_{\mathcal{F}}$ is a closed set satisfying

$$\mathcal{C}_{\mathcal{F}} + \mathcal{C}_{\mathcal{F}} \subset \mathcal{C}_{\mathcal{F}} \text{ and } \lambda\mathcal{C}_{\mathcal{F}} \subset \mathcal{C}_{\mathcal{F}} \ (\lambda \geq 0),$$

(ii) $\mathcal{C}_{\mathcal{F}} \cap (-\mathcal{C}_{\mathcal{F}}) = \{0\}$.

Definition 3.1. *The cone $\mathcal{C}_{\mathcal{F}}$ associated with a Schauder frame $\mathcal{F}$ is called*

(a) generating if

$$\mathcal{X} = \{y - z : y, z \in \mathcal{C}_{\mathcal{F}}\}$$

(b) normal if there exists a constant $L > 0$ such that

$$0 \leq x \leq y \Rightarrow \|x\| \leq L\|y\|; \ x, y \ \in \mathcal{X},$$

(c) minihedral if for every $x, \ y \in C_{\mathcal{F}}$ there exists $z_0 = sup(x, y)$ and if $z \geq x, \ y$ then $z \geq z_0$.

Recall that the cone $\mathcal{C}_{\mathcal{F}}$ induces a natural partial order relation on $\mathcal{X}$, namely $x \geq y$ if and only if $x - y \in \mathcal{C}_{\mathcal{F}}$.
A subset $\mathcal{B}$ of $\mathcal{C}_{\mathcal{F}}$ is called a *base* of $\mathcal{C}_{\mathcal{F}}$ if it is closed and convex and if for every $x \in \mathcal{C}_{\mathcal{F}} \setminus \{0\}$ has a unique representation of the form $x = \lambda y, \ \lambda > 0, \ y \in \mathcal{B}$.
A set $\mathcal{E}$ contained in $\mathcal{C}_{\mathcal{F}}$ is called an *extremal subset* of $\mathcal{C}_{\mathcal{F}}$ if $x, \ y \in \mathcal{C}_{\mathcal{F}}$ with $\lambda x + (1 - \lambda)y \in \mathcal{E}$ and $0 \leq \lambda \leq 1$ imply $x, \ y \in \mathcal{E}$.
Define a *hyper plane* L to be a set of the form $L = \{x \in \mathcal{X} : f(x) = 0\}$.

In the following result, we give a necessary and sufficient condition for a cone $C_{\mathcal{F}}$ associated with a Schauder frame to be a normal cone.

Theorem 3.1. *Let $\mathcal{X}$ be a real Banach space and let $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ be a Schauder frame for $C_{\mathcal{F}}$, then, $\mathcal{F} = \{(x_n, f_n)\}$ is an unconditional Schauder frame for $C_{\mathcal{F}}$ if and only if $C_{\mathcal{F}}$ is normal.*

Proof. Suppose first that $\mathcal{F} = \{(x_n, f_n)\}$ is an unconditional Schauder frame for $C_{\mathcal{F}}$. Then, for every $x \in C_{\mathcal{F}}$, we have

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)x_k, \tag{3.1}$$

where the series in (3.1) converges unconditionally in $C_{\mathcal{F}}$. Let $x, y \in \mathcal{X}$ be such that $0 \leq x \leq y$. Then, for each $i \in \mathbb{N}$, we have $0 \leq f_i(x) \leq f_i(y)$. Therefore, for each $i \in \mathbb{N}$, there exist a real no $\lambda_i (0 \leq \lambda_i \leq 1)$ such that

$$f_i(x) = \lambda_i f_i(y),\ i \in \mathbb{N}. \tag{3.2}$$

Therefore using (3.1), one can find $L > 0$ such that $\|x\| \leq L\|y\|$.
Conversely, let $x \in C_{\mathcal{F}}$ and let $\epsilon > 0$ be arbitrary. Since $\mathcal{F} = \{(x_n, f_n)\}$ is a Schauder frame for $C_{\mathcal{F}}$ and $C_{\mathcal{F}}$ is normal, there exists a positive integer M such that

$$\|\sum_{k=M}^{\infty} f_k(x)x_k\| < \frac{\epsilon}{L}. \tag{3.3}$$

Let $\sum_{l=1}^{\infty} f_{k_l}(x)x_{k_l}$ be any subseries of $\sum_{k=1}^{\infty} f_k(x)x_k$. Choose N such that $k_l \geq M$, whenever $k \geq N$. Then for any $m_1, m_2 \geq N$, we have $0 \leq \sum_{l=m_1}^{m_2} f_{k_l}(x)x_{k_l} \leq \sum_{k=N}^{\infty} f_k(x)x_k$. Since $C_{\mathcal{F}}$ is normal, by condition (3.3), we get

$$\|\sum_{l=m_1}^{m_2} f_{k_l}(x)x_{k_l}\| \leq L\|\sum_{k=N}^{\infty} f_k(x)x_k\| < \epsilon.$$

Therefore, $\sum_{k=1}^{\infty} f_k(x)x_k$ is unconditionally convergent. Hence $\mathcal{F} = \{(x_n, f_n)\}$ is unconditional Schauder frame for $C_{\mathcal{F}}$. □

Next, in the following result we have given a sufficient condition for the cone $C_{\mathcal{F}}$ associated with Schauder frame to be minihedral cone.

Theorem 3.2. *Let $\mathcal{X}$ be a real Banach space and let $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ be a unconditional Schauder frame for $C_{\mathcal{F}}$, then $C_{\mathcal{F}}$ is minihedral.*

Proof. Suppose that $C_{\mathcal{F}}$ has unconditional Schauder frame, then for any $x,\ y \in C_{\mathcal{F}}$ we have

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)x_k \text{ and } y = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(y)x_k.$$

Therefore, the series

$$\lim_{n\to\infty} \sum_{k=1}^{n} [f_k(x), f_k(y)]x_k$$

is unconditionally convergent and so, the series

$$z_0 = \lim_{n\to\infty} \sum_{k=1}^{n} [sup(f_k(x), f_k(y))]x_k \tag{3.4}$$

is also unconditionally convergent. Thus, the sum of the series in (3.4) is $Sup(x, y) = z_0$. Hence $C_{\mathcal{F}}$ is minihedral. □

The converse of Theorem 3.2 is not true. In this context we have the following example.

Example 3.1. *Let $\mathcal{X}$ be the closed hyperplane in l^1 defined by $\mathcal{X} = \{x = \{\eta_n\} \in l^1 : f_n(x) \geq 0\}$. Define $x_n = \xi_n - \xi_{n-1}$ $(n = 1, 2, 3...)$, where $\xi_n = \{\delta_{ij}\}_{j=1}^{\infty}$. Then, $\{x_n\}$ is a Schauder frame for $\mathcal{X}$ with associate sequence of coefficient functionals $\{f_n\} \in \mathcal{X}^*$. Define*

$$C_{\mathcal{F}} = \{x = \{\eta_i\} \in l^1 : f_n(\eta_i) \geq 0 \textit{ for all } i = 1, 2, 3...\}.$$

Let $\{\eta_i\} \in l^1$ be arbitrary element, write $\eta_i = \phi_i - \psi_i$, where $\phi_i \geq 0$ and $\psi_i \geq 0$. Then $y = \{\phi_i\} \in C_{\mathcal{F}}$ and $z = \{\psi_i\} \in C_{\mathcal{F}}$ are such that every x can be expressed as $x = y - z$. So, $C_{\mathcal{F}}$ is generating. Further, if $\sum_{n=1}^{\infty} f_n(x)x_n \in \mathcal{X}$, then we have

$$\begin{aligned}\sum_{n=1}^{\infty} f_n(x)x_n &= \sum_{n=1}^{\infty} f_n(x)(\xi_n - \xi_{n-1}) \\ &= f_1(x)\xi_1 + \sum_{n=2}^{\infty}(f_n - f_{n-1})(x)\xi_n,\end{aligned}$$

where ξ_n is the unit vector of l^1. That is, $|f_1| + \sum_{n=2}^{\infty} |f_n - f_{n-1}| < \infty$. Since $||f_n| - |f_{n-1}|| \leq |f_n - f_{n-1}|$ $(n = 2, 3...)$. Therefore, it follows that $\sum_{n=1}^{\infty} |f_n|(x)x_n$ converges, whenever $\sum_{n=1}^{\infty} f_n(x)x_n$ converges. Thus, for each $x \in \mathcal{X}$, there exists an element $|x| \in \mathcal{X}$ such that $x_+ = sup(x, 0)$ and $x_- = sup(-x, 0)$. Hence $C_{\mathcal{F}}$ is minihedral. Finally, we will show that $C_{\mathcal{F}}$ is not normal. Suppose on the contrary that $C_{\mathcal{F}}$ is normal. For this, let $x = \{x_n\}$ and $y = \{y_n\}$ be two sequences in $\mathcal{X}$ defined by

$$x_n = \frac{1}{n}[\xi_1 + \xi_2+, ..., \xi_{2n-1}] - \frac{1}{n}[\xi_2 + \xi_4 + ... + \xi_{2n}]$$

and

$$y_n = \frac{1}{n}[\xi_1 - \xi_{2n}], \text{ where } (n = 1, 2, ...).$$

Then, we have $0 \leq \|x\| \leq \|y\|$, for all $x,\ y \in \mathcal{X}$, but $\|x\| = 2$ and $\|y\| = \frac{2}{n}$, a contradiction. Hence $C_{\mathcal{F}}$ is not normal.

In [27], Poumai in 2014 introduced Schauder frame of types P, P^*, Q and Q^* and has given some theoretical applications of these types of Schauder frames. Here we have defined normalized Schauder frame of type l_+ and has given some necessary and sufficient conditions for the associated cone $C_{\mathcal{F}}$. Next, we give the definition of normalized Schauder frame of the type l_+.

Definition 3.2. *A normalized Schauder frame $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ for a Banach space $\mathcal{X}$ is called a Schauder frame of type l_+ if there exist a constant $A > 0$ such that*

$$\| \sum_{n=1}^{\infty} f_n(x)x_n \| \geq A \sum_{n=1}^{\infty} f_n(x) \tag{3.5}$$

and there exist $f \in \mathcal{X}^*$ such that $f(x_n) \geq 1,$ for all n.

In the following result, we have given the necessary and sufficient condition for the cone $C_{\mathcal{F}}$ associated with normalized Schauder frame to have a bounded base.

Theorem 3.3. *Let $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ be a normalized Schauder frame for a Banach space $\mathcal{X}$. Then, $\mathcal{F} = \{(x_n, f_n)\}$ is a Schauder frame of type l_+ if and only if the associated cone $C_{\mathcal{F}}$ has a bounded base.*

Proof. Suppose first that $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame of type l_+. Define $\mathcal{B} = \{x \in C_{\mathcal{F}} : f(x) = 1\}$, where $f \in \mathcal{X}^*$ such that $f(x_n) \geq 1$. Clearly $\mathcal{B}$ is closed and convex set. Let $0 \neq x \in C_{\mathcal{F}}$. Take $y = \frac{x}{f(x)}$. Then $f(y) = 1$, so $y \in \mathcal{B}$. Now $x = \lambda y$, where $\lambda = f(x) > 0$. If $x = \lambda_1 y_1 = \lambda_2 y_2$, where $\lambda_1,\ \lambda_2 > 0$ and $y_1, y_2 \in \mathcal{B}$ then $f(x) = \lambda_1 = \lambda_2$. Thus the representation $x = \lambda y$ is unique. Therefore $\mathcal{B}$ is a base for $C_{\mathcal{F}}$. Since $\{x_n\}$ is a normalized Schauder frame, so for every $x = \sum_{n=1}^{\infty} f_n(x)x_n \in \mathcal{B}$, we have

$$\|x\| \leq \sum_{n=1}^{\infty} f_n(x)x_n \leq \sum_{n=1}^{\infty} f_n(x)f(x_n) = f(\sum_{n=1}^{\infty} f_n(x)x_n) = 1.$$

Conversely, suppose that the cone $C_{\mathcal{F}}$ associated with normalized Schauder frame has a bounded base $\mathcal{B}$. Then, $0 \notin \mathcal{B}$. Since $\mathcal{B}$ is closed and convex, so by Hahn-Banach theorem there exist a functional $f \in \mathcal{X}^*$ such that

$$\inf_{x \in \mathcal{B}} f(x) = \epsilon > 0 \tag{3.6}$$

and let $\{x_n\}$ be a sequence in $C_{\mathcal{F}}$ such that $x_n \neq 0,\ n \in \mathbb{N}$. Then for each n, there exist $\lambda_n > 0$ and $z_n \in \mathcal{B}$ such that $x_n = \lambda_n x_n,\ n \in \mathbb{N}$. Also, the representation of each x_n is unique. So, $\frac{x_n}{\lambda_n} \in \mathcal{B}$. Therefore, $\frac{1}{\lambda_n} = \|\frac{x_n}{\lambda_n}\| \leq \sup\limits_{z \in \mathcal{B}} \|z\| = A < \infty$ and so,

$$\lambda_n \geq \frac{1}{A},\ n \in \mathbb{N}. \tag{3.7}$$

Thus, from equations (3.6) and (3.7), we get $f(x_n) = \lambda_n f(\frac{x_n}{\lambda_n}) \geq \frac{\epsilon}{A}$.
Hence $\mathcal{F} = \{(x_n, f_n)\}$ is a Schauder frame of type l_+. □

Next, a sufficient condition for a cone $C_{\mathcal{F}}$ to be normal is given.

Corollary 3.1. *If* $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ *is a normalized Schauder frame of type* l_+ *for a Banach space* $\mathcal{X}$, *then the associated cone* $C_{\mathcal{F}}$ *is normal.*

Proof. By hypothesis, $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame of type l_+ and $0 \leq x \leq y$. Then,

$$\begin{aligned} \|x\| &= \|\sum_{n=1}^{\infty} f_n(x)x_n\| \\ &\leq \|\sum_{n=1}^{\infty} f_n(x)\| \\ &\leq \|\sum_{n=1}^{\infty} g_n(x)\| \\ &\leq \frac{1}{A}\|\sum_{n=1}^{\infty} g_n(x)x_n\| \\ &= L\|y\|. \end{aligned}$$

This implies that $\|x\| \leq L\|y\|$, where $L = \frac{1}{A}$. Hence $C_{\mathcal{F}}$ is normal. ☐

Next, a characterization of Schauder frame of type l_+ has been given.

Proposition 3.1. *Let* $\mathcal{X}$ *be a Banach space, a normalized Schauder frame* $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ *of* $\mathcal{X}$ *is of type* l_+ *if and only if there exists a constant* $B > 0$ *such that for every* $x \in C_{\mathcal{F}}$, *the series* $\sum_{n=1}^{\infty} f_n(x)x_n$ *is absolutely convergent and*

$$\sum_{n=1}^{\infty} \|f_n(x)x_n\| \leq B\|x\|, \text{ for all } x \in C_{\mathcal{F}}. \tag{3.8}$$

Proof. Let $\mathcal{F} = \{(x_n, f_n)\}$ be a normalized Schauder frame of type l_+, then for every $x \in C_{\mathcal{F}}$ and $n \in \mathbb{N}$, we have $\|\sum_{n=1}^{\infty} f_n(x)x_n\| \geq A\sum_{n=1}^{\infty} f_n(x)$ and there exist $f \in \mathcal{X}^*$ such that $f(x_n) \geq 1$. Therefore,

$$\sum_{n=1}^{\infty} \|f_n(x)x_n\| = \sum_{n=1}^{\infty} f_n(x) \leq \frac{1}{A}\sum_{n=1}^{\infty} \|f_n(x)x_n\|.$$

Thus, taking $n \to \infty$, we get

$$\sum_{n=1}^{\infty} \|f_n(x)x_n\| \leq \frac{1}{A}\|x\|.$$

Hence

$$\sum_{n=1}^{\infty} \|f_n(x)x_n\| \leq B\|x\|,$$

where $B = \frac{1}{A}$. Conversely, if $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame satisfying (3.8). Then, for any $x \in \mathcal{X}$ with $x = \sum_{n=1}^{\infty} f_n(x)x_n$, we have

$$\frac{1}{B}\sum_{n=1}^{\infty} f_n(x) = \frac{1}{B}\sum_{n=1}^{\infty} \|f_n(x)x_n\| \leq \|\sum_{n=1}^{\infty} f_n(x)x_n\|.$$

This implies that

$$\|\sum_{n=1}^{\infty} f_n(x)x_n\| \geq A\sum_{n=1}^{\infty} f_n(x),$$

where $A = \frac{1}{B}$. Hence $\mathcal{F} = \{(x_n, f_n)\}$ is a Schauder frame of type l_+. □

In the following result, we have given a necessary and sufficient condition for the cone $C_{\mathcal{F}}$ associated with a unit vector basis of l^1 to be generating and has a bounded base.

Theorem 3.4. *Let $\mathcal{X}$ be a Banach space and let $\mathcal{F} = \{(x_n, f_n)\}$ ($\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*$) be a normalized Schauder frame for $\mathcal{X}$. Then $\{x_n\}$ is equivalent to the unit vector basis of l^1 if and only if the associated cone $C_{\mathcal{F}}$ is generating and has a bounded base.*

Proof. First, suppose that $\mathcal{F} = \{(x_n, f_n)\}$ ($\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*$) is a normalized Schauder frame for $\mathcal{X}$ and $\{x_n\}$ is equivalent to the unit vector basis of l^1. Then, by Theorem 3.3, $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame of type l_+. Let $x \in \mathcal{X}$ be arbitrary. Then, since $C_{\mathcal{F}}$ is generating, we have $x = y - z$, for all $y, z \in C_{\mathcal{F}}$.

Therefore by proposition 3.1, we have

$$\begin{aligned}\sum_{n=1}^{\infty}|f_n(x)| &= \sum_{n=1}^{\infty}|f_n(y)-f_n(z)| \\ &\leq \sum_{n=1}^{\infty}|f_n(y)|+\sum_{n=1}^{\infty}|f_n(z)| \\ &= \sum_{n=1}^{\infty}|f_n(y)x_n|+\sum_{n=1}^{\infty}|f_n(z)x_n| < \infty.\end{aligned}$$

Conversely, suppose that the associated cone $C_{\mathcal{F}}$ is generating and has a bounded base and since $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame and $\mathcal{X}$ is a complete space. Therefore by Theorem3.3 $\mathcal{F} = \{(x_n, f_n)\}$ is a Schauder frame of type l_+. Thus $\{x_n\}$ is equivalent to the unit vector basis of l^1. □

Next, we will show that if $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame for the Banach space $\mathcal{X}$ and $e_n = \pm 1$, then $\mathcal{F}_1 = \{(e_n x_n, e_n f_n)\}$ is also a normalized Schauder frame for $\mathcal{X}$.

Theorem 3.5. *If $\mathcal{F} = \{(x_n, f_n)\}(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ be a normalized Schauder frame for the Banach space $\mathcal{X}$ and $\{e_n\}$ $(n = 1, 2, ...)$ be any sequence with $(e_n) = \pm 1$. Then, $\mathcal{F}_1 = \{(e_n x_n, e_n f_n)\}$ is also a normalized Schauder frame for $\mathcal{X}$.*

Proof. Since $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame for $\mathcal{X}$, we have

$$x = \sum_{n=1}^{\infty} f_n(x) x_n. \tag{3.9}$$

Let $\{e_n\}, n = 1, 2, ...$ be any sequence with $e_n = \pm 1$. Then by (3.9), we have

$$x = \sum_{n=1}^{\infty} f_n(x) x_n = \sum_{n=1}^{\infty} {e_n}^2 f_n(x) x_n = \sum_{n=1}^{\infty} e_n f_n(x) x_n e_n.$$

□

By theorem 3.5, $\mathcal{F}_1 = \{(e_n x_n, e_n f_n)\}$ is a normalized Schauder frame for $\mathcal{X}$. Define the cone $C_{\mathcal{F}_1}$ associated with the normalized Schauder frame $\mathcal{F}_1 = \{(e_n x_n, e_n f_n)\}$ as follows :

$$C_{\mathcal{F}_1} = \{x \in \mathcal{X} : e_n f_n(x) \geq 0, \text{ for all } n \in \mathbb{N}\}.$$

Recall that a subset $\mathcal{B}_1$ of the cone $C_{\mathcal{F}_1}$ associated with the Schauder frame $\mathcal{F}_1 = \{(e_n x_n, e_n f_n)\}$ in a Banach space $\mathcal{X}$ is called hyper base of the cone $C_{\mathcal{F}_1}$ if there exists $f \in \mathcal{X}^*(f(x) > 0$ for all $x \in C_{\mathcal{F}_1} \setminus 0)$ such that $\mathcal{B}_1 = \{z \in C_{\mathcal{F}_1} : f(z) = 1\}$. Define $\mathcal{B}n_1 = \mathcal{B}_1 \bigcap [x_n, x_{n+1}, x_{n+2}, ...]$, where $n = 1, 2, 3...$ From this we have characterize geometrically some other classes of normalized Schauder frames in Banach spaces. Next, we have given the definition of normalized Schauder frame of type P^*, al_+ and w_0.

Definition 3.3. *A normalized Schauder frame* $\mathcal{F} = \{(x_n, f_n)\}(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ *of a Banach space* $\mathcal{X}$ *is called*

(a) normalized Schauder frame of type P^* *if* $\sup\limits_n \|\sum_{i=1}^n f_i\| < \infty$ *and there exists* $f \in \mathcal{X}^*$ *such that* $f(x_n) = 1,$ *for all* $n \in \mathbb{N}$.

(b) normalized Schauder frame of type al^+*, if there exists a sequence* $\{e_n\}$*, where* $e_n = \pm 1,\ n = 1, 2, ...$ *such that* $\{e_n x_n\}$*, is a normalized Schauder frame of type* l_+.

(c) normalized Schauder frame of type w_0 *if* $f(x_n) \longrightarrow 0,$ *for all* $f \in \mathcal{X}^*$.

In the following result, we have given the characterization of normalized Schauder frame of type P^*, al^+ and w_0.

Theorem 3.6. *Let* $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ *be a normalized Schauder frame for the Banach space* $\mathcal{X}$*. Then*

(a) $\mathcal{F} = \{(x_n, f_n)\}$ *is of type* P^* *if and only if there exists a hyper base* $\mathcal{B}$ *of* $C_{\mathcal{F}}$ *containing* $x_n,\ (n = 1, 2, ...)$.

(b) $\mathcal{F} = \{(x_n, f_n)\}$ *is not of type* al_+ *if and only if for every sequence* $\{e_n\}$*,with* $e_n = \pm 1$ *and for every hyper base* $\mathcal{B}_1$ *of the cone* $C_{\mathcal{F}_1}$ *the unique number* $\lambda_n > 0$ *for which* $\mathcal{B}_1 \supset \{\lambda_n e_n x_n\}$ *satisfying* $\sup_n \lambda_n = \infty$.

(c) $\mathcal{F} = \{(x_n, f_n)\}$ *is of type* w_0 *if and only if for every sequence* $\{e_n\}$*,with* $e_n = \pm 1$ *and for every hyper base* $\mathcal{B}_1$ *of the cone* $C_{\mathcal{F}_1}$ *the unique number* $\lambda_n > 0$ *for which* $\mathcal{B}_1 \supset \{\lambda_n e_n x_n\}$ *satisfying* $\lim_{n\to\infty} \lambda_n = \infty$.

Proof. (a) If $\mathcal{F} = \{(x_n, f_n)\}$ is of type P^*, then by definition, there exists $f \in \mathcal{X}^*$ such that $f(x_n) = 1$. Define $\mathcal{B} = \{z \in C_{\mathcal{F}} : f(z) = 1\}$. Then, $\mathcal{B}$ is a hyper base of $C_{\mathcal{F}}$ containing x_n. Conversely, if $\mathcal{B}$ is a hyper base of $C_{\mathcal{F}}$ such that $x_n \in \mathcal{B}$. Then, there exists $f \in \mathcal{X}^*$ such that $\mathcal{B} = \{z \in C_{\mathcal{F}} : f(z) = 1\}$. Then, $f(x_n) = 1$ and therefore, $\mathcal{F} = \{(x_n, f_n)\}$ is a Schauder frame of type P^*.

(b) If $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame not of type al_+, then by definition, we have $\inf_n |f(x_n)| = 0$. Let $e_n = \pm 1$ and let $\mathcal{B}_1$ be arbitrary base of the cone $C_{\mathcal{F}_1}$. Then, there exists $f \in \mathcal{X}^*$ such that $\mathcal{B}_1 = \{z \in C_{\mathcal{F}_1} : f(z) = 1\}$. Therefore, $\frac{1}{f(e_n x_n)} e_n x_n \in \mathcal{B}_1$ and thus $\lambda_n = \frac{1}{f(e_n x_n)}$, $n = 1, 2, ...$ Hence $\sup_n \lambda_n = \infty$. Conversely, suppose that the condition holds, and we have to show that $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame not of type al_+, suppose if $\mathcal{F} = \{(x_n, f_n)\}$ is a normalized Schauder frame of type al_+, then by definition there exists $f \in \mathcal{X}^*$ such that $|f(x_n)| \geq 1; n = 1, 2, ...$ Define $e_n = Sign f(x_n)$. Then, $f(e_n x_n) \geq 1$. Therefore, the set $\mathcal{B}_1 = \{z \in C_{\mathcal{F}_1} : f(z) = 1\}$ is a hyper base of the cone $C_{\mathcal{F}_1}$ and $\frac{1}{f(e_n x_n)} e_n x_n \in \mathcal{B}_1$, $n = 1, 2, ...$ Hence $\lambda_n = \frac{1}{f(e_n x_n)} e_n x_n \leq 1;\ n = 1, 2, ...$

(c) Proof is similar to the proof of part (b) with slightly modification.

□

Liu [23] in 2010 introduced the concept of Shrinking and boundedly complete frames in Banach spaces and had proven some elementary facts. Recall that if $\mathcal{F} =$

$\{(x_n, f_n)\}$ is a normalized Schauder frame for the Banach space $\mathcal{X}$, let $T_n : \mathcal{X} \longrightarrow \mathcal{X}$ be the operator such that $T_n(x) = \sum_{n=1}^{\infty} f_n(x)x_n$. The frame $\mathcal{F} = \{(x_n, f_n)\}$ is *called shrinking* if $\|f^* o T_n\| \to 0$, for all $f^* \in \mathcal{X}^*$.

In the following result, a sufficient condition for the cone $C_{\mathcal{F}_1}$ associated with normalized shrinking Schauder frame $\mathcal{F}_1 = \{(e_n x_n, e_n f_n)\}$ has been given.

Theorem 3.7. *Let* $\mathcal{F} = \{(x_n, f_n)\}$ $(\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*)$ *be a normalized shrinking Schauder frame for the Banach space* $\mathcal{X}$. *If for every* $\{e_n\}$,*with* $e_n = \pm 1$ *and every hyper base* $\mathcal{B}_1$ *of the cone* $C_{\mathcal{F}_1}$, *we have* $dist(0, \mathcal{B}n_1) \to \infty$ *as* $n \to \infty$.

Proof. If $\mathcal{F} = \{(x_n, f_n)\}$ is a shrinking normalized Schauder frame for the Banach space $\mathcal{X}$. Let $e_n = \pm 1,\ n = 1, 2, ...$ and let $\mathcal{B}_1$ be an arbitrary hyper base of the cone $C_{\mathcal{F}_1}$. Then, there exists $f \in \mathcal{X}^*$ such that $\mathcal{B}_1 = \{z \in C_{\mathcal{F}_1} : f(z) = 1\}$. Thus, for any $z \in \mathcal{B}n_1$, we have $z = \sum_{n=1}^{\infty} f_n(z)x_n$. Let $\epsilon > 0$ and m be any natural number. Then, for all $n > m$, we have

$$\frac{1}{\| \sum_{n=1}^{\infty} f_n(z)x_n\|} = f\left(\frac{\sum_{n=1}^{\infty} f_n(z)x_n}{\| \sum_{n=1}^{\infty} f_n(z)x_n\|}\right) < \epsilon.$$

This implies that $\|z\| > \frac{1}{\epsilon}$, for all $z \in \mathcal{B}n_1$, whenever $n > m$. □

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