

HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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ABSTRACT. The aim of this paper is to give some results on the generalization of theorem the Guelfond .

1. INTRODUCTION AND RESULTS

In 1914 G.Polya show that the entire function f of a complex variable, with $f(\mathbb{N}) \subset \mathbb{Z}$ and $\lim_{r \rightarrow \infty} \sup(\log |f|_r / r) \leq \log 2$ is necessarily a polynomial.

F.Gramain [7] presente the situation in 1988 an attempt of the same kind: how to show by a method of transcendence, a result obtained in 1933 by A.O.Guelfond on entire functions taking integer values in all points of a geometric progression, like the previous multiplicative additive problem (see[6] theorem VIII). On the other hand, the same type of results for entire functions of several variables were obtained by P.Bundschuh1980 and J.P.Bézivin 1983 the first uses Newton interpolation series in several variables and the second linear récurrentes suites. J.P.Bézivin[1] in 1984 studied a multivariate generalization of a result the Guelfond. Tanguy Rivoal and

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Michael Welter [11] have found the following result: let F be a holomorphic function on $\{z \in \mathbb{C}^d : R(z_j) > 0, j = 1, \dots, d\}$, suppose that the following conditions are satisfied,

(i) $F(\mathbb{N}^d) \subset \mathbb{Z}$.

(ii) There are real $c > 0, \alpha \geq 0$ and β such that we have for all z satisfying $R(z_j) > 0$:

$$|F(z)| \leq c \prod_{j=1}^d \frac{e^{|z_j|\Psi(\theta_j)}}{R(z_j)^\alpha (1 + |z_j|)^\beta} \text{ or } \theta_j \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$\Psi(\theta_j) = \cos(\theta_j) \log(2 \cos(\theta_j)) + \theta_j \sin(\theta_j),$$

so if $\frac{1}{2} - \beta < \alpha < \frac{1}{2}$, the function F is a polynomial with rational coefficients.

B.Djebbar [3] says That a entire harmonic function h on $\mathbb{C}$ the exponential type $\tau < \pi$.

If $h(z) = 0$, for $z = 0, 1, 2, \dots$, and $h(z) = h(\bar{z})$, then $h \equiv 0$.

It was proven in [5] that if U an entire separately harmonic functions on $\mathbb{R}^{2N}$ such that:

$$\exists A > 0 \text{ such that } M_\Delta(U, r) \leq A \exp(\sigma r), \forall r > 0$$

and if $\sigma < \log 2$, then U is a polynomial.

Where for any complete bounded domain Δ of center 0 in $\mathbb{C}^N$, $M_\Delta(f, r) = \sup_{z \in r \cdot \Delta} |f(z)|$.

2. DEFINITIONS AND PROPERTIES

2.1. Order and type of an entire function. Let $\chi : [0, +\infty[\longrightarrow [0, +\infty[\cup \{+\infty\}$ be an increasing function. We define the growth order $\rho(\chi)$ of χ by the formula :

$$\rho(\chi) = \limsup_{r \rightarrow \infty} \frac{\log \chi(r)}{\log r},$$

if χ is of finite order, ($0 < \varrho(\chi) < \infty$) the growth type $\sigma(\chi)$ is defined by :

$$\sigma(\chi) = \limsup_{r \rightarrow \infty} \frac{\chi(r)}{r^\varrho} .$$

For any entire function f on $\mathbb{C}^N$, the N -growth of f is defined by the growth of the function $\chi(r) = \log^+ M_N(f, r)$ where N is a norm on $\mathbb{C}^N$ and $M_N(f, r) = \sup_{N(z) \leq r} |f(z)|$. For any complete bounded domain Δ of center 0 in $\mathbb{C}^N$, the Δ -growth of f is defined by the growth of the function $\chi(r) = \log^+ M_\Delta(f, r)$ where $M_\Delta(f, r) = \sup_{z \in r \cdot \Delta} |f(z)|$. (see [15] for more detail)

2.2. Holomorphic function. In complex analysis, a holomorphic function is a complex-valued function defined and differentiable at every point of an open subset of the complex plane $\mathbb{C}$.

2.3. Arithmetic function. Let f be an entire function on $\mathbb{C}$. f is said to be arithmetic if for each non negative integer n , $f(n)$ is an integer.

2.4. Residue. In complex analysis, the residue is a complex number which describes the behavior of the line integral of a holomorphic function in the neighborhood of a singularity. The residuals are calculated quite easily and, once known, can calculate more complicated curvilinear integrals with the residue theorem.

2.5. Theorem of residue. Let U an open simply connected subset of the complex plane $\mathbb{C}$, $\{z_1, \dots, z_n\}$ a set of points U , separate and isolated, and f a function defined and holomorphic on $U \setminus \{z_1, \dots, z_n\}$.

If ∇ is a rectifiable curve in U which meets no singular points z_k and whose starting point to the end point then:

$$\int_{\nabla} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k) \text{Ind}_{\nabla}(z_k)$$

where

$$\text{Ind}_{\nabla}(z_k) = \frac{1}{2\pi i} \int_{\nabla} \frac{dz}{z - z_k}.$$

3. THEOREM OF GUELFOND

Theorem 3.1. [6] *Let $g(z)$ an entire function, and if the $g(\beta^n)$ are integers, $n = 1, 2, 3, \dots$, ($\beta > 1$, integer), if $g(z)$ verifies :*

$$\ln |g(z)| < \frac{\ln^2 r}{4 \ln \beta} - \frac{1}{2} \ln r - \omega(r), \quad r = |z|, \quad \lim_{r \rightarrow +\infty} \omega(r) = \infty,$$

then $g(z)$ is a polynomial.

To prove this theorem we must show that the quantity:

$$P_k^n(t) = \frac{(1-t^n) \dots (1-t^{n-k})}{(1-t) \dots (1-t^{k+1})} \text{ avec: } n \geq k$$

is a polynomial in t with integer rational coefficients.

Proof. We put :

$$I_{k+1} = (1-t) \dots (1-t^{k+1}),$$

a) $t = 1$ is a root of I_{k+1} with the order of multiplicity $k+1$.

b) $t = \exp(2\pi i \frac{p}{q})$ is a root of $(1-t^q), (1-t^{2q}), \dots, (1-t^{\alpha q})$,

$$\text{when: } \left\{ \begin{array}{l} p \text{ and } q \text{ have no common divisor} \\ 0 \leq p < q \leq k+1, \end{array} \right\}$$

with

$$1 \leq \alpha q \leq k+1 \Rightarrow \frac{1}{q} \leq \alpha \leq \frac{k+1}{q}.$$

Or the order of the plurality of root $\exp(2\pi i \frac{p}{q})$ is $\alpha = \left[\frac{k+1}{q} \right]$, then the root of the polynomial I_{k+1} are of type $\exp(2\pi i \frac{p}{q})$ with multiplicity equal to the order $\left[\frac{k+1}{q} \right]$.

We denote the polynomial: $(1-t^n) \dots (1-t^{n-k})$ by I_{n-k} with $n \geq k$,

a) $t = 1$ is a root of I_{n-k} with the order of multiplicity $k+1$.

b) $t = \exp(2\pi i \frac{p}{q})$ is a root of $(1 - t^q), (1 - t^{2q}), \dots, (1 - t^{\beta q})$,

when $\left\{ \begin{array}{l} p \text{ and } q \text{ have no common divisor} \\ 0 \leq p < q \leq k+1, \end{array} \right\}$

with

$$n - k \leq \beta q \leq n \Rightarrow \frac{n - k}{q} \leq \beta \leq \frac{n}{q} \leq \frac{n + 1}{q}; (n \geq k).$$

Or $\beta = \left\lfloor \frac{n+1}{q} \right\rfloor$ the order of the multiplicity of root $\exp(2\pi i \frac{p}{q})$.

Then I_{k+1} and I_{n-k} are polynomials with roots type $\exp(2\pi i \frac{p}{q})$ was the order of multiplicity equal $\left\lfloor \frac{k+1}{q} \right\rfloor$ of I_{k+1} , and $\left\lfloor \frac{n+1}{q} \right\rfloor$ of I_{n-k} .

I_{k+1} divided I_{n-k} , with $\left\lfloor \frac{k+1}{q} \right\rfloor$ (the order of the multiplicity of roots of polynomial I_{k+1}) $\leq \left\lfloor \frac{n+1}{q} \right\rfloor$ (the order of the multiplicity of roots of polynomial I_{n-k}).

Then

$$p_k^n(t) = \frac{(1 - t^n) \dots (1 - t^{n-k})}{(1 - t) \dots (1 - t^{k+1})}; (n \geq k),$$

is a polynomial, with coefficients are integers because the coefficient of the highest degree of the denominator $((1 - t) \dots (1 - t^{k+1}))$ equal to unity. $\square$

4. HOLOMORPHIC FUNCTIONS WITH COMPLEX VARIABLES

Theorem 4.1. Let g an holomorphic function on $\mathbb{C}^2$, with $g(\beta^n, \beta^m)$ are integers, $\forall n, m \in \mathbb{N}$ (β , integer greater than unity) and $g(z_1, z_2)$ verifies the inequality:

$$\ln |g(z_1, z_2)| < \frac{\ln^2 r_1 r_2}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 - \omega(r_1, r_2),$$

or

$$r_1 = |z_1|, r_2 = |z_2|, \lim_{\min(r_1, r_2) \rightarrow +\infty} \omega(r_1, r_2) = \infty.$$

Then g is a polynomial.

Proof. since $g(z_1, z_2)$ verified the condition:

$$\ln |g(z_1, z_2)| < \frac{\ln^2 r_1 r_2}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 - \omega(r_1, r_2),$$

therefore

$$g(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{n,m} (z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^n)(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^m),$$

Or

$$A_{n,m} = \frac{1}{(2\pi i)^2} \int_{c_n} \int_{c_m} \frac{g(z_1, z_2)}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n+1})(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{m+1})} dz_1 dz_2.$$

Take as an integration contour : $|z_1| = \beta^{2n}, |z_2| = \beta^{2m}$ after inequality the Cauchy with two variables :

$$|A_{n,m}| \leq \frac{\beta^{2n+2m} \max_{0 \leq \theta_1, \theta_2 \leq 2\pi} |g(\beta^{2n} \exp i\theta_1, \beta^{2m} \exp i\theta_2)|}{\prod_{k=1}^{n+1} (\beta^{2n} - \beta^k) \prod_{j=1}^{m+1} (\beta^{2m} - \beta^j)}$$

$$|A_{n,m}| \leq k \beta^{-n(n+3)} \beta^{-m(m+3)} \exp(-w(\beta^{2n}, \beta^{2m})),$$

but

$$\begin{aligned} A_{n,m} &= \frac{1}{(2\pi i)^2} \int_{c_n} \int_{c_m} \frac{g(z_1, z_2)}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n+1})(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{m+1})} dz_1 dz_2 \\ &= \frac{1}{2\pi i} \int_{c_n} \left(\frac{1}{2\pi i} \int_{c_m} \frac{g(z_1, z_2)}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{m+1})} dz_2 \right) \frac{dz_1}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n+1})} \end{aligned}$$

$$(4.1) \quad A_{n,m} = \frac{1}{2\pi i} \int_{c_n} F(z_1) \frac{dz_1}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n+1})},$$

with

$$(4.2) \quad F(z_1) = \frac{1}{2\pi i} \int_{c_m} \frac{g(z_1, z_2)}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{m+1})} dz_2.$$

On the other hand the integral giving $A_{n,m}$ equals some residues of the function $F(z_1)$.

$$(4.3) \quad A_{n,m} = \sum_{s=1}^{n+1} \frac{F(\beta^s)}{\prod_{\substack{j=1 \\ s \neq j}}^{n+1} (\beta^s - \beta^j)}$$

from (4.2)

$$F(\beta^s) = \frac{1}{2\pi i} \int_{c_m} \frac{g(\beta^s, z_2)}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{m+1})} dz_2, 1 \leq s \leq n+1$$

for fixed s : $F(\beta^s) =$

$$\frac{1}{2\pi i} \int_{c_m} \frac{G(z_2)}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{m+1})} dz_2,$$

$$\text{with : } g(\beta^s, z_2) = G(z_2)$$

$$G(z_2) = \sum_{m=0}^{\infty} b_m (z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^m)$$

$$\text{or : } b_m = \frac{1}{2\pi i} \int_{c_m} \frac{G(z_2)}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{m+1})} dz_2$$

then

$$F(\beta^s) = b_m .$$

The integral giving b_m equals Some residues of the function $G(z_2)$

$$b_m = \sum_{k=1}^{m+1} \frac{G(\beta^k)}{\prod_{\substack{r=1 \\ r \neq k}}^{m+1} (\beta^k - \beta^r)}$$

$$b_m = \sum_{k=1}^{m+1} \frac{g(\beta^s, \beta^k)}{\prod_{\substack{r=1 \\ r \neq k}}^{m+1} (\beta^k - \beta^r)}$$

$$(4.4) \quad F(\beta^s) = \sum_{k=1}^{m+1} \frac{g(\beta^s, \beta^k)}{\prod_{\substack{r=1 \\ r \neq k}}^{m+1} (\beta^k - \beta^r)}, (1 \leq s \leq n+1)$$

replaces (4.4) in (4.3)

$$\begin{aligned} A_{n,m} &= \sum_{s=1}^{n+1} \frac{F(\beta^s)}{\prod_{\substack{j=1 \\ s \neq j}}^{n+1} (\beta^s - \beta^j)} \\ A_{n,m} &= \sum_{s=1}^{n+1} \frac{\sum_{k=1}^{m+1} \frac{g(\beta^s, \beta^k)}{\prod_{\substack{r=1 \\ r \neq k}}^{m+1} (\beta^k - \beta^r)}}{\prod_{\substack{j=1 \\ s \neq j}}^{n+1} (\beta^s - \beta^j)} \\ A_{n,m} &= \sum_{s=1}^{n+1} \sum_{k=1}^{m+1} \frac{g(\beta^s, \beta^k)}{\prod_{\substack{j=1 \\ s \neq j}}^{n+1} (\beta^s - \beta^j) \prod_{\substack{r=1 \\ r \neq k}}^{m+1} (\beta^k - \beta^r)}. \end{aligned}$$

Then $A_{n,m}$ is an integer, *which implies that all $A_{n,m}$ vanish for large rank.*

Therefore g is a polynom. □

In general this can résulat for holomrphes functions has several complex variables.

Theorem 4.2. *Let g an holomorphic function on $\mathbb{C}^d$,*

with: $g(\beta^{n_1}, \dots, \beta^{n_d})$ are integers, $\forall n_1, \dots, n_d \in \mathbb{N}^d(\beta, \text{integer greater than unity})$,

and $g(z_1, z_2, \dots, z_d)$ verifies the inequality:

$$\ln |g(z_1, z_2, \dots, z_d)| < \frac{\ln^2 r_1 r_2 \dots r_d}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 \dots r_d - \omega(r_1, r_2, \dots, r_d)$$

or

$$r_j = |z_j|, j = 1, \dots, d, \quad \lim_{\min(r_1, r_2, \dots, r_d) \rightarrow +\infty} \omega(r_1, r_2, \dots, r_d) = \infty.$$

Then g is a polynom.

Proof. Since

$$\ln |g(z_1, z_2, \dots, z_d)| < \frac{\ln^2 r_1 r_2 \dots r_d}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 \dots r_d - \omega(r_1, r_2, \dots, r_d),$$

therefore

$$\begin{aligned} g(z_1, z_2, \dots, z_d) = & \sum_{n_1=0}^{\infty} \dots \sum_{n_d=0}^{\infty} A_{n_1, \dots, n_d} (z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n_1}) \dots \\ & \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d}), \end{aligned}$$

or

$$A_{n_1, \dots, n_d} = \frac{1}{(2\pi i)^d} \int_{c_{n_1}} \dots \int_{c_{n_d}} \frac{g(z_1, z_2, \dots, z_d) dz_1 \dots dz_d}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n_1+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})}$$

$$\begin{aligned} A_{n_1, \dots, n_d} = & \frac{1}{(2\pi i)} \int_{c_{n_1}} \left(\frac{1}{(2\pi i)^{d-1}} \int_{c_{n_2}} \dots \right. \\ & \left. \dots \int_{c_{n_d}} \frac{g(z_1, z_2, \dots, z_d) dz_2 \dots dz_d}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n_1+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})} \right) dz_1 \end{aligned}$$

$$A_{n_1, \dots, n_d} = \frac{1}{(2\pi i)} \int_{c_{n_1}} \frac{F(z_1)}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n_1+1})} dz_1, \text{ with :}$$

$$\begin{aligned} F(z_1) = & \frac{1}{(2\pi i)^{d-1}} \\ & \int_{c_{n_2}} \dots \int_{c_{n_d}} \frac{g(z_1, z_2, \dots, z_d) dz_2 \dots dz_d}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{n_2+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})} \end{aligned}$$

$A_{n_1, \dots, n_d}$ equals some residues of the function $F(z_1)$

$$(4.5) \quad A_{n_1, \dots, n_d} = \sum_{s=1}^{n_1+1} \frac{F(\beta^s)}{\prod_{\substack{j=1 \\ s \neq j}}^{n_1+1} (\beta^s - \beta^j)}, \quad 1 \leq s \leq n_1 + 1,$$

or

$$F(\beta^s) = \frac{1}{(2\pi i)^{d-1}} \int_{c_{n_2}} \dots \int_{c_{n_d}} \frac{g(\beta^s, z_2, \dots, z_d) dz_2 \dots dz_d}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{n_2+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})},$$

$$(1 \leq s \leq n + 1)$$

for fixed s :

$$F(\beta^s) = \frac{1}{(2\pi i)^{d-1}} \int_{c_m} \frac{G(z_2, \dots, z_d) dz_2 \dots dz_d}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{n_2+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})}$$

with

$$g(\beta^s, z_2, \dots, z_d) = G(z_2, \dots, z_d)$$

$$G(z_2, \dots, z_d) = \sum_{n_2=0}^{\infty} \dots \sum_{n_d=0}^{\infty} b_{n_2, \dots, n_d} (z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{n_2+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1}),$$

or

$$b_{n_2, \dots, n_d} = \frac{1}{(2\pi i)^{d-1}} \int_{c_m} \frac{G(z_2, \dots, z_d) dz_2 \dots dz_d}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{n_2+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})},$$

then

$$F(\beta^s) = b_{n_2, \dots, n_d}.$$

The integral giving $b_{n_2, \dots, n_d}$ equals some residues of the function $G(z_2, \dots, z_d)$

$$b_{n_2, \dots, n_d} = \sum_{k=1}^{n_2+1} \frac{G(\beta^k, z_3, \dots, z_d)}{\prod_{\substack{r=1 \\ r \neq k}}^{n_2+1} (\beta^k - \beta^r)}$$

$$F(\beta^s) = \sum_{k=1}^{n_2+1} \frac{g(\beta^s, \beta^k, z_3, \dots, z_d)}{\prod_{\substack{r=1 \\ r \neq k}}^{n_2+1} (\beta^k - \beta^r)},$$

from (4.5)

$$A_{n_1, \dots, n_d} = \sum_{s=1}^{n_1+1} \sum_{k=1}^{n_2+1} \frac{g(\beta^s, \beta^k, z_3, \dots, z_d)}{\prod_{\substack{r=1 \\ r \neq k}}^{n_2+1} (\beta^k - \beta^r) \prod_{\substack{j=1 \\ s \neq j}}^{n_1+1} (\beta^s - \beta^j)}, 1 \leq k \leq n_2 + 1$$

for fixed s and k

$$A_{n_1, \dots, n_d} = \frac{1}{(2\pi i)^2} \int_{c_{n_1}} \int_{c_{n_2}} \left(\frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \dots \right. \\ \left. \dots \int_{c_{n_d}} \frac{g(z_1, z_2, \dots, z_d) dz_3 \dots dz_d}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n_1+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})} \right) dz_1 dz_2$$

$$F(z_1, z_2) = \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \dots \\ \dots \int_{c_{n_d}} \frac{g(z_1, z_2, \dots, z_d) dz_3 \dots dz_d}{(z_3 - \beta)(z_3 - \beta^2) \dots (z_3 - \beta^{n_3+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})}$$

$$A_{n_1, \dots, n_d} = \frac{1}{(2\pi i)^2} \int_{c_{n_1}} \dots \\ \dots \int_{c_{n_2}} \frac{F(z_1, z_2)}{(z_1 - \beta)(z_1 - \beta^2) \dots (z_1 - \beta^{n_1+1}) (z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{n_2+1})} dz_1 dz_2$$

$A_{n_1, \dots, n_d}$ equals some residues of the function $F(z_1, z_2)$,

$$(4.6) \quad A_{n_1, \dots, n_d} = \sum_{s=1}^{n_1+1} \sum_{k=1}^{n_2+1} \frac{F(\beta^s, \beta^k)}{\prod_{\substack{j=1 \\ s \neq j}}^{n_1+1} (\beta^s - \beta^j) \prod_{\substack{l=1 \\ k \neq j}}^{n_2+1} (\beta^k - \beta^l)}, \left\{ \begin{array}{l} 1 \leq s \leq n_1 + 1 \\ 1 \leq k \leq n_2 + 1 \end{array} \right\},$$

or

$$F(\beta^s, \beta^k) = \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \dots \int_{c_{n_d}} \frac{g(\beta^s, \beta^k, \dots, z_d) dz_3 \dots dz_d}{(z_3 - \beta)(z_3 - \beta^2) \dots (z_3 - \beta^{n_3+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})},$$

with

$$\left\{ \begin{array}{l} 1 \leq s \leq n_1 + 1 \\ 1 \leq k \leq n_2 + 1 \end{array} \right\}.$$

For s and k

$$F(\beta^s, \beta^k) = \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \dots \int_{c_{n_d}} \frac{G(z_3, \dots, z_d) dz_3 \dots dz_d}{(z_3 - \beta)(z_3 - \beta^2) \dots (z_3 - \beta^{n_3+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})}$$

with

$$g(\beta^s, \beta^k, z_3, \dots, z_d) = G(z_3, \dots, z_d)$$

$$G(z_3, \dots, z_d) = \sum_{n_3=0}^{\infty} \dots \sum_{n_d=0}^{\infty} b_{n_3, \dots, n_d} (z_3 - \beta)(z_3 - \beta^2) \dots (z_3 - \beta^{n_3+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1}),$$

or

$$b_{n_3, \dots, n_d} = \frac{1}{(2\pi i)^{d-2}} \int_{c_{n_3}} \int_{c_{n_d}} \frac{G(z_2, \dots, z_d) dz_2 \dots dz_d}{(z_2 - \beta)(z_2 - \beta^2) \dots (z_2 - \beta^{n_2+1}) \dots (z_d - \beta)(z_d - \beta^2) \dots (z_d - \beta^{n_d+1})}$$

then

$$F(\beta^s, \beta^k) = b_{n_3, \dots, n_d}.$$

The integral giving $b_{n_3, \dots, n_d}$ equals some residues of the function $G(z_3, \dots, z_d)$

$$b_{n_3, \dots, n_d} = \sum_{c=1}^{n_3+1} \frac{G(\beta^c, z_4, \dots, z_d)}{\prod_{\substack{c=1 \\ c \neq m}}^{n_3+1} (\beta^c - \beta^m)}$$

$$F(\beta^s, \beta^k) = \sum_{c=1}^{n_3+1} \frac{g(\beta^s, \beta^k, \beta^c, z_4, \dots, z_d)}{\prod_{\substack{c=1 \\ c \neq m}}^{n_3+1} (\beta^c - \beta^m)}.$$

From (4.6)

$$A_{n_1, \dots, n_d} = \sum_{s=1}^{n_1+1} \sum_{k=1}^{n_2+1} \sum_{c=1}^{n_3+1} \frac{g(\beta^s, \beta^k, \beta^c, z_4, \dots, z_d)}{\prod_{\substack{j=1 \\ s \neq j}}^{n_1+1} (\beta^s - \beta^j) \prod_{\substack{l=1 \\ k \neq l}}^{n_2+1} (\beta^k - \beta^l) \prod_{\substack{c=1 \\ c \neq m}}^{n_3+1} (\beta^c - \beta^m)}, \left\{ \begin{array}{l} 1 \leq s \leq n_1 + 1 \\ 1 \leq k \leq n_2 + 1 \end{array} \right\}$$

then for fixed all the s_i ($1 \leq i \leq d-1, 1 \leq s_i \leq n_i + 1$)

$$A_{n_1, \dots, n_d} = \sum_{s_1=1}^{n_1+1} \sum_{s_2=1}^{n_2+1} \dots \sum_{s_d=1}^{n_d+1} \frac{g(\beta^{s_1}, \beta^{s_2}, \dots, \beta^{s_d})}{\prod_{\substack{j=1 \\ s_1 \neq j}}^{n_1+1} (\beta^{s_1} - \beta^j) \prod_{\substack{l=1 \\ s_2 \neq l}}^{n_2+1} (\beta^{s_2} - \beta^l) \dots \prod_{\substack{s_d=1 \\ s_d \neq m}}^{n_d+1} (\beta^{s_d} - \beta^m)}$$

$A_{n_1, \dots, n_d}$ is an integer, which implies that all $A_{n_1, \dots, n_d}$ vanish for large rank.

Therefore g is a polynomial. □

Corollary 4.1. Let F the holomorphic function on $\mathbb{C}^d$, such that $F(\beta^{n_1}, \dots, \beta^{n_d})$ are integers, $\forall n_1, \dots, n_d \in N^d(\beta, \text{integer greater than unity})$,

and $F(z_1, z_2, \dots, z_d)$ satisfies the inequality:

$$\ln |F(z_1, z_2, \dots, z_d)| < \frac{\ln^2 r_1 r_2 \dots r_d}{4 \ln \beta} - \frac{1}{2} \ln r_1 r_2 \dots r_d - \omega(r_1, r_2, \dots, r_d)$$

or

$$r_j = |z_j|, j = 1, \dots, d, \lim_{\min(r_1, r_2, \dots, r_d) \rightarrow +\infty} \omega(r_1, r_2, \dots, r_d) = \infty.$$

If $F(\mathbb{N}^d) = 0$, then $F \equiv 0$ on $\mathbb{C}^d$.

Proof. For Theorem (4.2), F is a polynom and $F(\mathbb{N}^d) = 0$, therefore all the coefficients of polynom are zero.

Then $F \equiv 0$ on $\mathbb{C}^d$. □

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