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COMPUTING INTERSECTIONS, DUAL AND DIVISORIAL CLOSURE OF IDEALS IN A CLASS OF RINGS

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ABSTRACT. Let D be an integral domain, X an indeterminate over D and let n be a positive integer. The set $\{a_0 + a_1X^n + a_2X^3 + \cdots a_nX^n \mid a_i \in D\}$ is a subrings of $D[X]$ denoted by $D + X^nD[X]$. This class of subrings is studied in [1] for $n = 2$. In this article we find explicit formulas to compute finite intersections, dual and divisorial closure of monomial ideals of $D + X^nD[X]$.

1. INTRODUCTION

Let D be an integral domain with quotient field K. A D-submodule J of K is called a fractional ideal of D if there exist $0 \neq a \in D$ such that $aJ \subseteq D$. For a nonzero fractional ideal J of D, the fractional ideal $D : J = \{x \in K \mid xJ \subseteq D\}$ is called *dual of J denoted by* J^{-1}, since it is isomorphic as a D-module to $Hom_D(J, D)$. The dual J^{-1} is not generally a subring of K (or we can say that J^{-1} is not generally an overrig of D). A natural question about the dual of an ideal has been studied in [3, Section 3.1], i.e., when is the dual of an ideal an overring? The fractional ideal $J_v = (J^{-1})^{-1} = (D : (D : J))$ is called v-closure or divisorial closure of J. If $J = J_v$ then J is called v-ideal or divisorial ideal. Many applications to multiplicative ideal theory can be derived from divisoriality. The map $J \mapsto J_v$ is a star operation called

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v-operation. A reader in need of a quick review of star operations may consult sections 32 and 34 of Gilmer's book [2].

Let X be an indeterminate over K. If $D \neq K$ then the ring $D[X]$ must have some non-principal ideals. However, if D is a PID, the ideals generated by non-constant monomials in $D[X]$ are principal, and hence their intersections, dual and divisorial closure is principal and can be easily computed. Such type of ideals are not always principal in a subring $D + X^nD[X]$ of $D[X]$ for $n > 1$. Note that for $n > 1$, $D + X^nD[X]$ is not integrally closed and hence it cannot be a GCD domain. Therefore, the finite intersection of principal ideals need not be principal for $n > 1$. For instance, $(X^3) \cap (X^4) = (X^4)$ in $D[X]$, $(X^3) \cap (X^4) = (X^6, X^7)$ in $D + X^2D[X]$ and $(X^3) \cap (X^4) = (X^7, X^8, X^9)$ in $D + X^3D[X]$. Moreover, in $D + X^2D[X]$, $(X^4, X^5)^{-1} = \frac{1}{X^4}(X^2, X^3)$ and $(X^4, X^5)_v = (X^4, X^5)$. Similarly, in $D + X^3D[X]$, $(X^4, X^5)^{-1} = \frac{1}{X^4}(X^3, X^4, X^5)$ and $(X^4, X^5)_v = (X^3, X^4, X^5)$. Note that the results varies by varying the values of n and it is not an easy job to compute the intersections, the dual and the divisorial closure for arbitrary large values of n. It appeals us to find some explicit formulas to compute the intersections, the dual and the divisorial closure of monomial ideals in $D + X^nD[X]$.

We obtain following results for the ring $D + X^nD[X]$. If $n \leq l < m$ are positive integers, then $\left(X^l\right) \cap \left(X^m\right) = \left(X^m\right)$ for $m - l \geq n$ and $\left(X^l\right) \cap \left(X^m\right) = \left(X^{m+n}, X^{m+(n+1)}, ..., X^{m+(2n-1)}\right)$ for $m - l < n$ (Theorem 2.1). If $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers, then $\left(X^{\lambda_1}\right) \cap \left(X^{\lambda_2}\right) \cap \cdots \cap \left(X^{\lambda_k}\right) = \left(X^{\lambda_k}\right)$ for $\lambda_k - \lambda_{k-1} \geq n$ and $\left(X^{\lambda_1}\right) \cap \left(X^{\lambda_2}\right) \cap \cdots \cap \left(X^{\lambda_k}\right) = X^{\lambda_k} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$ for $\lambda_k - \lambda_{k-1} < n$ (Theorem 2.2). If $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers, such that $\lambda_2 - \lambda_1 < n$, and $I = \left(X^{\lambda_1}, X^{\lambda_2}, ..., X^{\lambda_k}\right)$ be an ideal in $R = D + X^nD[X]$, then $I^{-1} = \frac{1}{X^{\lambda_1}} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$ and $I_v = \frac{X^{\lambda_1}}{X^n} \cdot$

$\left(X^n, X^{n+1}, ..., X^{2n-1}\right) = \left(X^{\lambda_1}, X^{\lambda_1+1}, ..., X^{\lambda_1+n-1}\right)$ (Theorem 2.3). If D is a GCD domain and $n \leq \lambda_1 < \lambda_2$ are positive integers, then $\left(aX^{\lambda_1}\right) \cap \left(bX^{\lambda_2}\right) = \left(lX^{\lambda_2}\right)$ for $\lambda_2 - \lambda_1 \geq n$ and $\left(aX^{\lambda_1}\right) \cap \left(bX^{\lambda_2}\right) = lX^{\lambda_2} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$ for $\lambda_2 - \lambda_1 < n$, where $a, b \in D$ and $l = lcm(a, b)$(Theorem 2.4). If D is a GCD domain and $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers, then $\left(a_1X^{\lambda_1}\right) \cap \left(a_2X^{\lambda_2}\right) \cap \cdots \cap \left(a_kX^{\lambda_k}\right) = \left(lX^{\lambda_k}\right)$ for $\lambda_k - \lambda_{k-1} \geq n$ and $\left(a_1X^{\lambda_1}\right) \cap \left(a_2X^{\lambda_2}\right) \cap \cdots \cap \left(a_kX^{\lambda_k}\right) = lX^{\lambda_k} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$ for $\lambda_k - \lambda_{k-1} < n$, where $a_1, a_2, ..., a_k \in D$ and $l = lcm(a_1, a_2, ..., a_k)$ (Theorem 2.5). If D is a GCD domain, $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers, such that $\lambda_k - \lambda_{k-1} < n$, and $I = \left(a_1X^{\lambda_1}, a_2X^{\lambda_2}, ..., a_kX^{\lambda_k}\right)$ is an ideal in $R = D + X^nD[X]$, then $I^{-1} = \frac{L}{a_1a_2\cdots a_kX^{\lambda_1}} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$ and $I_v = \frac{a_1a_2\cdots a_kX^{\lambda_1}}{LX^n} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right) = \frac{a_1a_2\cdots a_k}{L} \cdot \left(X^{\lambda_1}, X^{\lambda_1+1}, ..., X^{\lambda_1+n-1}\right)$, where $a_1, a_2, ..., a_k \in D$ and $L = lcm(a_2a_3\cdots a_k, a_1a_3a_4\cdots a_k, ..., a_1a_2\cdots a_{k-1})$ (Theorem 2.6). If D is PID, $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers, such that $\lambda_k - \lambda_{k-1} < n$, and $I = \left(a_1X^{\lambda_1}, a_2X^{\lambda_2}, ..., a_kX^{\lambda_k}\right)$ is an ideal in $R = D + X^nD[X]$, then $I^{-1} = \frac{1}{gcd(a_1,a_2,...,a_k)X^{\lambda_1}} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$ and $I_v = gcd(a_1, a_2, ..., a_k) \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$, where $a_1, a_2, ..., a_k \in D$ (Corollary 2.1).

Throughout this paper all rings are (commutative unitary) integral domains. Any unexplained material is standard as in [2] and [4].

2. Main Results

Theorem 2.1. *Suppose that $n \leq l < m$ are positive integers. Then the intersection of the ideals $\left(X^l\right)$ and $\left(X^m\right)$ in the ring $D + X^nD[X]$ is given by:*

$$\left(X^l\right) \cap \left(X^m\right) = \begin{cases} \left(X^m\right), & \text{if } m - l \geq n; \\ \left(X^{m+n}, X^{m+(n+1)}, ..., X^{m+(2n-1)}\right), & \text{if } m - l < n \end{cases}$$

Proof. Case 1. If $m-l \geq n$: Since $X^{m-l} \in D+X^nD[X]$, we can write $X^m = X^lX^{m-l}$. Therefore $\left(X^m\right) \subset \left(X^l\right)$ and hence $\left(X^l\right) \cap \left(X^m\right) = \left(X^m\right)$.

Case 2. If $m-l < n$: Since $m-l+n+t > n$ for every integer $t \geq 0$, we can write $X^{m+(n+t)} = X^lX^{m-l+n+t}$ for every integer $t \geq 0$. Also $X^{m+(n+t)} = X^mX^{n+t}$ for every integer $t \geq 0$. Therefore, $X^{m+n}, X^{m+(n+1)}, ..., X^{m+(2n-1)} \in \left(X^l\right) \cap \left(X^m\right)$ and hence $\left(X^{m+n}, X^{m+(n+1)}, ..., X^{m+(2n-1)}\right) \subseteq \left(X^l\right) \cap \left(X^m\right)$.
Let $f \in \left(X^l\right) \cap \left(X^m\right)$. Then $f \in \left(X^l\right)$ and $f \in \left(X^m\right)$. Therefore, for $m-l = \lambda < n$, and $a, a_i, b, b_i \in D$, we have

$$
\begin{aligned}
f =& X^l(a + a_0X^n + a_1X^{n+1} + \cdots + a_\lambda X^{n+\lambda} + a_{\lambda+1}X^{n+(\lambda+1)} + \cdots + \\
& a_nX^{n+n} + a_{n+1}X^{n+(n+1)} + \cdots + a_kX^k). \\
=& X^m(b + b_0X^n + b_1X^{n+1} + \cdots + b_\lambda X^{n+\lambda} + b_{\lambda+1}X^{n+(\lambda+1)} + \cdots + \\
& b_nX^{n+n} + b_{n+1}X^{n+(n+1)} + \cdots + b_qX^q).
\end{aligned} \tag{2.1}
$$

This implies that

$$
\begin{aligned}
f =& aX^l + a_0X^{l+n} + a_1X^{l+(n+1)} + \cdots + a_\lambda X^{l+n+\lambda} + a_{\lambda+1}X^{l+n+(\lambda+1)} + \cdots + \\
& a_{n+1}X^{l+2n+1} + \cdots + a_kX^{l+k}. \\
=& bX^m + b_0X^{m+n} + b_1X^{m+n+1} + \cdots + b_\lambda X^{m+n+\lambda} + b_{\lambda+1}X^{m+n+(\lambda+1)} + \cdots \\
& + b_nX^{m+2n} + b_{n+1}X^{(m+2n+1)} + \cdots + b_qX^{m+q}.
\end{aligned} \tag{2.2}
$$

Since $m-l < n \Rightarrow m < n+l$, so $a = b = 0$. Hence equation 2.2 becomes

$$
\begin{aligned}
f =& a_0X^{l+n} + a_1X^{l+(n+1)} + \cdots + a_\lambda X^{l+n+\lambda} + a_{\lambda+1}X^{l+n+(\lambda+1)} + \cdots + \\
& a_{n+1}X^{l+2n+1} + \cdots + a_kX^{l+k}. \\
=& b_0X^{m+n} + b_1X^{m+n+1} + \cdots + b_\lambda X^{m+n+\lambda} + b_{\lambda+1}X^{m+n+(\lambda+1)} + \cdots + \\
& b_nX^{m+2n} + b_{n+1}X^{(m+2n+1)} + \cdots + b_qX^{m+q}.
\end{aligned} \tag{2.3}
$$

Since $m+n=l+n+\lambda, m+(n+1)=l+\lambda+n+1$ and so on, therefore $a_0=a_1=\cdots=a_{\lambda-1}=0$ and $a_\lambda=b_0, a_{\lambda+1}=b_1, a_{\lambda+2}=b_2$, and so on. Hence from equation 2.3 we get,

$$\begin{aligned} f =& b_0X^{m+n}+b_1X^{m+(n+1)}+\cdots+b_\lambda X^{m+n+\lambda}+b_{\lambda+1}X^{m+n+(\lambda+1)}+\cdots+ \\ & b_nX^{m+2n}+b_{n+1}X^{m+2n+1}+\cdots+b_qX^{m+q}. \end{aligned} \tag{2.4}$$

Since $X^i \notin D+X^nD[X]$ if $i<n$, therefore equation 2.4 can be written as

$$\begin{aligned} f =& X^{m+n}(b_0+b_nX^n+b_{n+1}X^{n+1}+\cdots+b_qX^{q-n})+b_1X^{m+(n+1)}+\cdots+ \\ & b_\lambda X^{m+n+\lambda}+b_{\lambda+1}X^{m+n+(\lambda+1)}+\cdots+b_{n-1}X^{m+(2n-1)}. \end{aligned} \tag{2.5}$$

Hence, $f \in \left(X^{m+n}, X^{m+(n+1)}, ..., X^{m+(2n-1)}\right)$. □

Theorem 2.2. *Assume that $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers. Then the intersection of the ideals $\left(X^{\lambda_1}\right), \left(X^{\lambda_2}\right), ..., \left(X^{\lambda_k}\right)$ in the ring $D+X^nD[X]$ is given by:*

$$\left(X^{\lambda_1}\right)\cap\left(X^{\lambda_2}\right)\cap\cdots\cap\left(X^{\lambda_k}\right)=\begin{cases}\left(X^{\lambda_k}\right), & \text{if } \lambda_k-\lambda_{k-1}\geq n; \\ \\ X^{\lambda_k}\cdot\left(X^n, X^{n+1}, ..., X^{2n-1}\right), & \text{if } \lambda_k-\lambda_{k-1}<n.\end{cases}$$

Proof. Case 1. If $\lambda_k-\lambda_{k-1}\geq n$: Since, $\lambda_k-\lambda_{k-1}\geq n$ we have $\lambda_k-\lambda_i\geq n; \forall i=1,2,...,k-1$. So, we can write $X^{\lambda_k}=X^{\lambda_i}X^{\lambda_k-\lambda_i}$, which gives $X^{\lambda_k}\in\left(X^{\lambda_i}\right); \forall i=1,2,...,k-1$.

Therefore, $\left(X^{\lambda_1}\right)\cap\left(X^{\lambda_2}\right)\cap\cdots\cap\left(X^{\lambda_k}\right)=\left(X^{\lambda_k}\right)$.

Case 2. If $\lambda_k-\lambda_{k-1}<n$: Using Theorem 2.1 we have,

$$\begin{aligned} & \left(X^{\lambda_1}\right)\cap\left(X^{\lambda_2}\right)\cap\cdots\cap\left(X^{\lambda_k}\right) \\ & =\left(X^{\lambda_1}\right)\cap\left(X^{\lambda_2}\right)\cap\cdots\cap\left(X^{\lambda_{k-2}}\right)\cap\left(X^{\lambda_k+n}, X^{\lambda_k+(n+1)}, ..., X^{\lambda_k+(2n-1)}\right). \end{aligned} \tag{2.6}$$

Since, $\lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers, we have,

$\lambda_k - \lambda_i + n + t > n$, for every $t \geq 0$ and $\forall i = 1, 2, ..., k-2$.

So, we can write $X^{\lambda_k+n+t} = X^{\lambda_k-\lambda_i+n+t}X^{\lambda_i}$, which gives $X^{\lambda_k+n+t} \in \left(X^{\lambda_i}\right)$ for every $t \geq 0$ and $\forall i = 1, 2, ..., k-2$. Therefore,

$X^{\lambda_k+n}, X^{\lambda_k+(n+1)}, ..., X^{\lambda_k+(2n-1)} \in \left(X^{\lambda_i}\right); \forall i = 1, 2, ..., k-2$. This implies $\left(X^{\lambda_{k+n}}, X^{\lambda_{k+(n+1)}}, ..., X^{\lambda_{k+(2n-1)}}\right) \subseteq \left(X^{\lambda_1}\right) \cap \left(X^{\lambda_2}\right) \cap \cdots \cap \left(X^{\lambda_{k-2}}\right)$. Hence, from equation 2.6, we get

$$\begin{aligned} \left(X^{\lambda_1}\right) \cap \left(X^{\lambda_2}\right) \cap \cdots \cap \left(X^{\lambda_k}\right) &= \left(X^{\lambda_k+n}, X^{\lambda_k+(n+1)}, ..., X^{\lambda_k+(2n-1)}\right) \\ &= X^{\lambda_k} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right). \end{aligned} \tag{2.7}$$

□

Theorem 2.3. *Suppose that $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers, such that $\lambda_2 - \lambda_1 < n$, and $I = \left(X^{\lambda_1}, X^{\lambda_2}, ..., X^{\lambda_k}\right)$ be an ideal in $R = D + X^nD[X]$. Then*

(i) $I^{-1} = \frac{1}{X^{\lambda_1}} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$.

(ii) $I_v = \frac{X^{\lambda_1}}{X^n} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right) = \left(X^{\lambda_1}, X^{\lambda_1+1}, ..., X^{\lambda_1+n-1}\right)$.

Proof. (i):

$$\begin{aligned} I^{-1} &= \left(X^{\lambda_1}, X^{\lambda_2}, ..., X^{\lambda_k}\right)^{-1} \\ &= \frac{1}{X^{\lambda_1}}R \cap \frac{1}{X^{\lambda_2}}R \cap \cdots \cap \frac{1}{X^{\lambda_k}}R \\ &= \frac{1}{X^{\lambda_1+\lambda_2+\cdots+\lambda_k}}[\left(X^{\lambda_2+\lambda_3+\cdots+\lambda_k}\right) \cap \left(X^{\lambda_1+\lambda_3+\cdots+\lambda_k}\right) \cap \cdots \cap \left(X^{\lambda_1+\lambda_2+\cdots+\lambda_{k-1}}\right)] \\ &= \frac{1}{X^{\lambda_1+\lambda_2+\cdots+\lambda_k}}[X^{\lambda_2+\lambda_3+\cdots+\lambda_k} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)]; \text{(by Theorem 2.2)} \\ &= \frac{1}{X^{\lambda_1}} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right). \end{aligned}$$

(ii):

$$\begin{aligned} I_v = \left(I^{-1}\right)^{-1} &= X^{\lambda_1} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)^{-1} \\ &= \frac{X^{\lambda_1}}{X^n} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right); \text{using case (i)} \\ &= \left(X^{\lambda_1}, X^{\lambda_1+1}, ..., X^{\lambda_1+n-1}\right). \end{aligned}$$

□

Example 2.1. *Let* $R = \mathbb{Z} + X^{10}\mathbb{Z}[X]$ *and* $I = \left(X^{12}, X^{13}, X^{15}, X^{16}, X^{17}\right)$*. Then* $I^{-1} = \frac{1}{X^{12}} \cdot \left(X^{10}, X^{11}, X^{12}, ..., X^{19}\right)$ *and* $I_v = \left(X^{12}, X^{13}, X^{14}, ..., X^{21}\right)$*.*

Remark 2.1. Assume that $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n$ are consecutive positive integers. Then $\left(X^{\lambda_1}, X^{\lambda_2}, ..., X^{\lambda_n}\right)$ is a divisorial ideal in the ring $D + X^n D[X]$.

Lemma 2.1. *Let* $d \in D - \{0\}$*,* $n \geq 1$ *be an integer and* $R = D + X^n D[X]$*. Then* $dR \cap X^k R = dX^k R$ *for any* $k \geq n$*.*

Proof. Clearly, $dR \cap X^k R \supseteq dX^k R$. Let $f \in dR \cap X^k R$. Then for $\lambda = k - n$,

$$\begin{aligned} f =& d(a + a_0 X^n + a_1 X^{n+1} + \cdots + a_\lambda X^k + \cdots + a_m X^m). \\ =& X^k(b + b_0 X^n + b_1 X^{n+1} + \cdots + b_q X^q). \end{aligned} \tag{2.8}$$

This implies that

$$\begin{aligned} f =& da + da_0 X^n + da_1 X^{n+1} + \cdots + da_\lambda X^k + \cdots + da_m X^m. \\ =& bX^k + b_0 X^{k+n} + b_1 X^{k+n+1} + \cdots + b_q X^q. \end{aligned} \tag{2.9}$$

This implies $a = a_0 = a_1 = \cdots = a_{\lambda-1} = 0$ and $d \mid b, b_0, b_1, ..., b_q$.

$$f = X^k(b + b_0X^n + b_1X^{n+1} + \cdots + b_qX^{q-k}). \tag{2.10}$$

Hence $f \in dX^kR$. □

Theorem 2.4. *Let D be a GCD domain and $n \leq \lambda_1 < \lambda_2$ be positive integers. Then the intersection of the ideals $\big(aX^{\lambda_1}\big)$ and $\big(bX^{\lambda_2}\big)$ in the ring $D + X^nD[X]$ is given by:*

$$\big(aX^{\lambda_1}\big) \cap \big(bX^{\lambda_2}\big) = \begin{cases} \big(lX^{\lambda_2}\big), & \text{if } \lambda_2 - \lambda_1 \geq n; \\ \\ lX^{\lambda_2} \cdot \big(X^n, X^{n+1}, ..., X^{2n-1}\big), & \text{if } \lambda_2 - \lambda_1 < n. \end{cases}$$

where $a, b \in D$ and $l = lcm(a, b)$.

Proof. By using Lemma 2.1, we get that $\big(aX^{\lambda_1}\big) \cap \big(bX^{\lambda_2}\big) = \big(a\big) \cap \big(b\big) \cap \big(X^{\lambda_1}\big) \cap \big(X^{\lambda_2}\big)$. Now apply Theorem 2.1. □

Theorem 2.5. *Let D be a GCD domain and $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ be positive integers. Then the intersection of the monomial ideals $\big(a_1X^{\lambda_1}\big), \big(a_2X^{\lambda_2}\big), ..., \big(a_kX^{\lambda_k}\big)$ in the ring $D + X^nD[X]$ is given by:*

$$\big(a_1X^{\lambda_1}\big) \cap \big(a_2X^{\lambda_2}\big) \cap \cdots \cap \big(a_kX^{\lambda_k}\big) = \begin{cases} \big(lX^{\lambda_k}\big), & \text{if } \lambda_k - \lambda_{k-1} \geq n; \\ \\ lX^{\lambda_k} \cdot \big(X^n, X^{n+1}, ..., X^{2n-1}\big), & \text{if } \lambda_k - \lambda_{k-1} < n. \end{cases}$$

where $a_1, a_2, ..., a_k \in D$ and $l = lcm(a_1, a_2, ..., a_k)$.

Proof. By using Lemma 2.1, we get that

$$\big(a_1X^{\lambda_1}\big) \cap \big(a_2X^{\lambda_2}\big) \cap \cdots \cap \big(a_kX^{\lambda_k}\big) = \big(a_1\big) \cap \big(a_2\big) \cap \cdots \cap \big(a_k\big) \cap \big(X^{\lambda_1}\big) \cap \big(X^{\lambda_2}\big) \cap \cdots \cap \big(X^{\lambda_k}\big).$$

Now apply Theorem 2.2.

□

Theorem 2.6. *Let D be a GCD domain, $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ be positive integers, such that $\lambda_k - \lambda_{k-1} < n$, and $I = \big(a_1X^{\lambda_1}, a_2X^{\lambda_2}, ..., a_kX^{\lambda_k}\big)$ be an ideal in $R = D + X^nD[X]$. Then*

(i) $I^{-1} = \frac{L}{a_1a_2\cdots a_kX^{\lambda_1}} \cdot \big(X^n, X^{n+1}, ..., X^{2n-1}\big)$.

(ii) $I_v = \frac{a_1a_2\cdots a_kX^{\lambda_1}}{LX^n} \cdot \big(X^n, X^{n+1}, ..., X^{2n-1}\big) = \frac{a_1a_2\cdots a_k}{L} \cdot \big(X^{\lambda_1}, X^{\lambda_1+1}, ..., X^{\lambda_1+n-1}\big)$.

where $a_1, a_2, ..., a_k \in D$ and $L = lcm(a_2a_3\cdots a_k, a_1a_3a_4\cdots a_k, ..., a_1a_2\cdots a_{k-1})$.

Proof. <u>(i):</u>

$$\begin{aligned} I^{-1} &= \big(a_1X^{\lambda_1}, a_2X^{\lambda_2}, ..., a_kX^{\lambda_k}\big)^{-1} \\ &= \frac{1}{a_1X^{\lambda_1}}R \cap \frac{1}{a_2X^{\lambda_2}}R \cap \cdots \cap \frac{1}{a_kX^{\lambda_k}}R. \end{aligned}$$

Let $A = a_1a_2a_3\cdots a_k, A_1 = a_2a_3a_4\cdots a_k, A_2 = a_1a_3a_4\cdots a_k, ...,$ *and* $A_k = a_1a_2\cdots a_{k-1}$. Then we have

$$\begin{aligned} I^{-1} &= \frac{1}{AX^{\lambda_1+\lambda_2+\cdots+\lambda_k}}\big[\big(A_1X^{\lambda_2+\lambda_3+\cdots+\lambda_k}\big) \cap \big(A_2X^{\lambda_1+\lambda_3+\cdots+\lambda_k}\big) \cap \cdots \cap \big(A_kX^{\lambda_1+\lambda_2+\cdots+\lambda_{k-1}}\big)\big] \\ &= \frac{LX^{\lambda_2+\lambda_3+\cdots+\lambda_k}}{AX^{\lambda_1+\lambda_2+\cdots+\lambda_k}}\big(X^n, X^{n+1}, ..., X^{2n-1}\big); \text{(by Theorem 2.5)} \\ &= \frac{L}{AX^{\lambda_1}}\big(X^n, X^{n+1}, ..., X^{2n-1}\big). \end{aligned}$$

<u>(ii):</u>

$$\begin{aligned} I_v = \big(I^{-1}\big)^{-1} &= \frac{AX^{\lambda_1}}{L}\big(X^n, X^{n+1}, ..., X^{2n-1}\big)^{-1} \\ &= \frac{AX^{\lambda_1}}{LX^n} \cdot \big(X^n, X^{n+1}, ..., X^{2n-1}\big); \text{(by using Theorem 2.3)} \\ &= \frac{A}{L}\big(X^{\lambda_1}, X^{\lambda_1+1}, ..., X^{\lambda_1+n-1}\big). \end{aligned}$$

□

Corollary 2.1. *If D is a PID, $n \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{k-1} < \lambda_k$ are positive integers, such that $\lambda_k - \lambda_{k-1} < n$, and $I = \left(a_1 X^{\lambda_1}, a_2 X^{\lambda_2}, ..., a_k X^{\lambda_k}\right)$ is an ideal of $R = D + X^n D[X]$, then*

(i) $I^{-1} = \frac{1}{gcd(a_1,a_2,...,a_k)X^{\lambda_1}} \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$.

(ii) $I_v = gcd(a_1, a_2, ..., a_k) \cdot \left(X^n, X^{n+1}, ..., X^{2n-1}\right)$.

where $a_1, a_2, ..., a_k \in D$.

Example 2.2. *Let $R = \mathbb{Z}[i] + X^5\mathbb{Z}[i][X]$ and $I = \left(4X^6, (1+i)X^8, 2(1-i)X^9\right)$. Then $I^{-1} = \frac{1}{(1+i)X^6} \cdot \left(X^5, X^6, X^7, X^8, X^9\right)$ and $I_v = (1+i).\left(X^6, X^7, X^8, X^9, X^{10}\right)$.*

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