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A NEW CHARACTERIZATION OF $\mathbf{PSL(3,q)}$

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ABSTRACT. In this paper, we will show that the simple group $\mathrm{PSL}(3,q)$ can be uniquely characterized by order and one conjugacy class length, where q is a prime power. A main consequence of our result is the validity of Thompson's conjecture under a weak condition for the group under consideration.

1. Introduction

In group theory, as is well known, the elements of any group may be partitioned into conjugacy classes; members of the same conjugacy class share many group properties, and the study of conjugacy classes of non-abelian groups reveals many important features of their structures.

Let $\mathrm{SL}(n,q)$ denotes the group of n by n matrices of determinant 1 over the finite field $\mathrm{GF}(q)$ of q elements; $\mathrm{PSL}(n,q)$ denotes the projective special linear group which is equal to $\mathrm{SL}(n,q)$ modulo its center. In this paper, we prove that $\mathrm{PSL}(3,q)$ are uniquely determined by a conjugacy class length and order of $\mathrm{PSL}(3,q)$ when $p = (q^2+q+1)/(3,q-1)$ is a prime number. In fact, the main theorem of our paper is as follows:

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Main Theorem. Let G be a group and q be a prime power. Then $G \cong \mathrm{PSL}(3,q)$ if and only if $|G| = |\mathrm{PSL}(3,q)|$ and G has a conjugacy class length $|\mathrm{PSL}(3,q)|/p$, where $p = (q^2+q+1)/(3,q-1)$ is a prime number.

For related results, Chen et al. in [16] showed that $\mathrm{PSL}(2,p)$ is recognizable by its order and one conjugacy class length, where p is a prime number. As a consequence of their result, they showed that Thompson's conjecture is valid for $\mathrm{PSL}(2,p)$.

It is well known that the conjecture of J. G. Thompson which states that if G is a finite group with $Z(G) = 1$ and M is a non-abelian simple group satisfying $N(G) = N(M)$, where $N(G) = \{n : G$ has a conjugacy class of size $n\}$, then $G \cong M$. This conjecture is stated in [7, 8] in which the conjecture is verified for a few finite simple groups. We can give a positive answer to this conjecture by this characterization for our group under discussion.

In the [15], [1], [2], [3] and [9], it is proved that the groups: simple K_3-groups (a finite simple group is called a simple K_n -group if its order is divisible by exactly n distinct primes), ${}^2D_n(2)$, ${}^2D_{n+1}(2)$, alternating group of degree p, $p+1$, $p+2$, where p is a prime number, symmetric group of degree p, where p is a prime number, and sporadic simple groups, recognizable by their order and one conjugacy class length, respectively.

The *prime graph* of a finite group G that is denoted by $\Gamma(G)$ is defined as the simple undirected graph whose vertices are the prime divisors of the order, $|G|$, of G and two distinct vertices p, q are adjacent if and only if G contains an element of order pq.

We denote by $\pi(G)$ and $t(G)$, the set of prime divisors of $|G|$ and the number of connected components of $\Gamma(G)$, respectively. Let π_1, π_2 , ..., $\pi_{t(G)}$ be the connected components of $\Gamma(G)$. If the order of G is even, we set $2 \in \pi(G)$.

Now $|G|$ can be expressed as the product of integers m_1, m_2, ..., $m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. We call m_1, m_2, ..., $m_{t(G)}$ the order components of G. We write $\mathrm{OC}(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$, the set of order components of G. According to the classification theorem of finite simple groups and [14, 4, 11], we can list the order components of finite simple groups with disconnected prime graphs which are listed in Tables 1-4 in [8].

If n is a positive integer, then denote the r-part of n by $n_r = r^a$ which means $r^a \parallel n$, namely, $r^a \mid n$ and $r^{a+1} \nmid n$. If q is a prime, then we denote by $\mathrm{Syl}_q(G)$ a Sylow q-subgroup of G. The other notations and terminologies in this paper are standard, and the reader can refer to [10] if necessary.

2. Preliminary Results

Definition 2.1. [17] Let a and n be integers greater than 1. Then a Zsigmondy prime of $a^n - 1$ is a prime l such that $l \mid (a^n - 1)$ but $l \nmid (a^i - 1)$ for $1 \leq i < n$.

Lemma 2.1. [12] If a and n are positive integers greater than 1, then there exists a Zsigmondy prime of $a^n - 1$, unless $(a, n) = (2, 6)$ or $n = 2$ and $a = 2^s - 1$ for some natural number s.

Remark 2.1. If l is a Zsigmondy prime of $a^n - 1$, then Fermat's little theorem shows that $n \mid (l - 1)$. Put $Z_n(a) = \{l : l \text{ is a Zsigmondy prime of } a^n - 1\}$. If $r \in Z_n(a)$ and $r \mid a^m - 1$, then $n \mid m$.

Definition 2.2. A Frobenius group is a transitive permutation group in which the stabilizer of any two points is trivial. A subgroup H of a Frobenius group G fixing a point of the set X is called the Frobenius complement. The identity element together with all elements not in any conjugate of H form a normal subgroup called the Frobenius kernel K.

Lemma 2.2. [5, Theorem 2] Let G be a 2-Frobenius group of even order, i.e., G is a finite group and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:
(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;
(b) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$, $(|G/K|, |K/H|) = 1$ and $G/K \lesssim \mathrm{Aut}(K/H)$.

Lemma 2.3. [11] If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:
(a) G is a Frobenius group or a 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \lesssim \mathrm{Aut}(K/H)$ and the odd order components of G are the odd order components of K/H.

Lemma 2.4 [4] If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $(n + 1)/2 < p_i < n$. Here

$$s(n) = 1,\ for\ 6 \leq n \leq 13;$$

$$s(n) = 2,\ for\ 14 \leq n \leq 17;$$

$$s(n) = 3,\ for\ 18 \leq n \leq 37;$$

$$s(n) = 4,\ for\ 38 \leq n \leq 41;$$

$$s(n) = 5,\ for\ 42 \leq n \leq 47;$$

$$s(n) = 6,\ for\ n \geq 48.$$

3. Proof of the Main Theorem

By [13, Corollary 2.11], PSL$(3, q)$ has one conjugacy class of length $\frac{|\mathrm{GL}(3,q)|}{(q^3-1)}$. Since the necessity of the theorem can be checked easily, we only need to prove the sufficiency.

By hypothesis, there exists an element x of order p in G such that $C_G(x) =< x >$ and $C_G(x)$ is a Sylow p-subgroup of G. By the Sylow's theorem, we have that $C_G(y) =< y >$ for any element y in G of order p . So, $\{p\}$ is a prime graph component of G and $t(G) \geq 2$. In addition, p is the maximal prime divisor of $|G|$ and an odd order component of G.

We are going to prove the main theorem in the following steps:

Step 1. G is neither a Frobenius group nor a 2-Frobenius group.

Proof. Let G be a Frobenius group with Frobenius kernel K and Frobenius complement H. Thus $\pi(K) = \{p\}$ or $\pi(H) = \{p\}$.

First, let $\pi(K) = \{p\}$. Since $K \trianglelefteq G$, $p \mid |G|$, and $p^2 \nmid |G|$, we have $\mathrm{Syl}_p(G) = K$. Thus $|G/C_G(K)| = |N_G(K)/C_G(K)| \mid (p-1)$. It follows that $|G| \leq p(p-1)$, which is a contradiction.

Now let $\pi(H) = \{p\}$. Since H is a Frobenius complement G, we have $N_G(H) = H$. Therefore, $|K| = q^3(q^2-1)(q-1)$. We consider the following cases:

(a) Let there exists a prime number r in $Z_2(q)$. Then $\mathrm{Syl}_r(K)$ is a normal subgroup of G. Hence, the semidirect product $\mathrm{Syl}_r(K) \rtimes H$ is a Frobenius subgroup of G. Then $|H| \mid |\mathrm{Syl}_r(K)| - 1$, and so $p < |\mathrm{Syl}_r(K)| \leq |K|_r$. However, $|K|_r \leq (q+1)_r$, which is a contradiction.

(b) Assume there is no prime number in $Z_2(q)$. Then $q + 1 = 2^k$, for some natural number k. Hence, $q = 3$ or there exists a prime t such that $t \mid (q-1)$ and $t \neq 2$. If $q = 3$, then $p = 13$ and $|\mathrm{Syl}_2(K)| = 16$. Similar to (a) we have $p \mid |\mathrm{Syl}_2(K)| - 1$, which is a contradiction. If there exists a prime t such that $t \mid (q-1)$ and $t \neq 2$, then similar to (a), $p < |\mathrm{Syl}_t(K)| \leq |K|_t$. However, $|K|_t = (\frac{(q-1)^2}{(3,q-1)})_t = (\frac{q^2-2q+1}{(3,q-1)})_t$, which is a contradiction.

Let G be a 2-Frobenius group. Then G is a finite group with a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with Frobenius kernels H and K/H, respectively. Since $K \trianglelefteq G$, $p \mid |K|$ and $p \nmid |G/K|$, we have $\mathrm{Syl}_p(G) \leq K$, and $N_G(\mathrm{Syl}_p(G)) = N_K(\mathrm{Syl}_p(G))$. But K is a Frobenius group with a Frobenius complement $\mathrm{Syl}_p(G)$, so $N_K(\mathrm{Syl}_p(G)) = \mathrm{Syl}_p(G)$. By similar discussion as above to K, we can get a contradiction.

By Lemma 2.3, and Step 1, G has normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a non-abelian simple group and p is an odd order component of K/H.

According to the classification theorem of finite simple groups and the results in Tables 1–4 in [8], K/H is either an alternating group, sporadic group or simple group of Lie type. We will show that K/H is isomorphic to $\mathrm{PSL}(3, q)$.

Here we consider two cases: $q \equiv 1 \pmod 3$ and $q \not\equiv 1 \pmod 3$.

First, let $q \equiv 1 \pmod 3$. Then $q \geq 4$, and so $p = (q^2 + q + 1)/3$ is one of the odd order components of K/H.

Step 2. K/H cannot be an alternating group Alt_m, where $m \geq 5$.

Proof. Suppose that $K/H \cong \mathrm{Alt}_m$. Since $(q^2 + q + 1)/3 = p \in \pi(K/H)$, $p \leq m$. Because $q \geq 2$ is a prime power, we have $p \geq 7$. By Lemma 2.4, there exists a prime number $u \in \pi(\mathrm{Alt}_m) \subseteq \pi(G)$ such that $(p + 1)/2 < u < p$. It is easy to see that $u \nmid q$, $u \nmid q - 1$ and $u \nmid p - 1$. Thus $u \in Z_2(q)$. It follows that $u = p - 2$, where $p = 7$ and $q = 4$. So $|G| = |\mathrm{PSL}(3, 4)| = 2^6.3^2.5.7$. Since $|\mathrm{Alt}_m|$ divides $|G|$, we have $m \in \{7, 8\}$. Since $|H| \mid |G|/|K/H|$ and $\mathrm{Syl}_7(G)$ acts fixed point freely on H, we can see that the semidirect product $H \rtimes \mathrm{Syl}_7(G)$ is a Frobenius subgroup of G. Thus $7 \mid (|H| - 1)$. Therefore, considering the orders of G and K/H it follows that either $|H| = 8$ and $m = 7$ or $|H| = 1$. If $|H| = 8$ and $m = 7$, then we can assume that H is a 2-elementary abelian group. Thus $\mathrm{Alt}_7 \lesssim \mathrm{GL}(3, 2)$ and so, $|\mathrm{Alt}_7| \mid |\mathrm{GL}(3, 2)|$, which is a contradiction. If $|H| = 1$, then $G \cong \mathrm{Alt}_8$, we get a contradiction.

Step 3. K/H is not a sporadic simple group.

Proof. Suppose that K/H is a sporadic simple group. Thus $p \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71\}$. If $p \in \{5, 11\}$, then since q is a prime power, we get a contradiction. Assume that $(q^2+q+1)/3 = 7$, then $q = 4$ and $|\mathrm{PSL}(3, 4)| = 2^6.3^2.5.7$. We have $K/H \in \{M_{22}, J_1, J_2, HS\}$. If $K/H = M_{22}$ (Mathieu group of degree 22), then $11 \mid |K/H|$, a contradiction.

If $K/H = J_1$ (Janko group), then $19 \mid |K/H|$, which is impossible.

If $K/H = J_2$ (Janko group), then $5^2 \mid |K/H|$, which is a contradiction.

Also, if $K/H = HS$ (Higman–Sims group), then $11 \mid |K/H|$, which is impossible.

Similarly, we can rules out the other possibilities of p.

Step 4. $K/H = \mathrm{PSL}(3, q)$.

Proof. By Steps 2 and 3, K/H is isomorphic to a simple group of Lie type. Let $t(K/H) = 2$. Then $\mathrm{OC}_2(K/H) = p = \frac{q^2+q+1}{3}$. Thus we have:

1. Let $K/H \cong C_n(q')$, where $n = 2^u \geq 2$, and $q' = p'^{\alpha}$. Then $\frac{q'^n+1}{(2,q'-1)} = \frac{q^2+q+1}{3}$. If $q'^{\alpha} \neq 2^{\alpha}$, then $\frac{q'^n+1}{2} = p$, so $q'^n = \frac{2(q^2+q-1)}{3}$ and so, $(q', p) = (q', q) = 1$. Furthermore, since p' is odd, we have $r \mid q-1$ or $p' \mid q+1$ ($q = r^l$, where l is a natural number). Hence, $(p'^{\alpha})^{n^2} = |K/H|_{p'} \leq |G|_{p'} \leq \frac{1}{(3,p')}(q^2-1)_{p'}(q-1)_{p'}(\frac{q^2+q-2}{3})_{p'} < (\frac{2(q^2+q-1)}{3})^2 < (p'^{\alpha})^{2n}$. Therefore, $n < 2$, which is a contradiction.

If $q'^{\alpha} = 2^{\alpha}$, then $q'^2 + 1 = p$. Thus $2^{2\alpha} + 1 = \frac{q^2+q+1}{3}$ and so, $2^{2\alpha} = \frac{(q-1)(q+2)}{3}$. Since $3 \mid q-1$, we have $3 \mid q+2$. Hence, $3 \mid 2^{2\alpha}$, a contradiction.

Similarly, we can rules out the case $K/H \cong B_n(q')$, or $K/H \cong {}^2D_n(q')$, where $n = 2^u \geq 4$.

2. Let $K/H \cong C_t(3)$ or $B_t(3)$, where t is prime. Then $\frac{3^t-1}{2} = \frac{q^2+q+1}{3}$. So $3^t = \frac{2}{3}(q^2+q+\frac{5}{2})$ and so, either $(3, q) = 1$ and $3^{t+1} > q^2+q+\frac{5}{2}$ or $q = t = 3$. If $(3, q) = 1$, then $3^{t^2} = |K/H|_3 \leq |G|_3 = \frac{1}{3}(q^2-1)_3(q-1)_3(\frac{q^2+q-2}{3})_3 < (\frac{q^2+q-2}{3})^3 < 3^{3(t+1)}$, so

$t^2 < 3(t+1)$. It follows that $t = 3$. Hence, $q(q+1) \in \{12, 38\}$. Clearly, $q(q+1) \neq 38$. Suppose that $q(q+1) = 12$, then $q = 3$. Since $q \equiv 1 \pmod 3$ leads to contradiction.

If $q = t = 3$, then since $q \equiv 1 \pmod 3$ we get a contradiction.

3. Let $K/H \cong C_t(2)$, where t is prime. Then $2^t - 1 = \frac{q^2+q+1}{3}$ and so, $2^t = \frac{q^2+q+4}{3}$. If $q = 4$, then $\frac{q^2+q+1}{3} = 7$ and hence, $t = 3$. In this case $|\mathrm{PSL}(3,4)| = |G| = 2^6.3^2.5.7$. So $2^9 \nmid |G|$. But, $2^9 \mid |C_3(2)|$, which is a contradiction. If $q \neq 4$, then $(2, q) = 1$ and hence, $2^{t^2} = |K/H|_2 \leq |G|_2 \leq (\frac{q^2-1}{3})_2(q-1)_2(\frac{q^2+q-2}{3})_2 < (\frac{q^2+q+4}{3})^3 = 2^{3t}$. It follows that $t^2 < 3t$, and so $t = 2$. Therefore, $\frac{q^2+q+1}{3} = 3$, a contradiction.

4. Let $K/H \cong \mathrm{PSL}(t, q')$, where $(t, q') \neq (3, 2)$, $(3, 4)$, and t is an odd prime. Then $p = \frac{q'^t-1}{(t,q'-1)(q'-1)}$, and $q'^{\frac{t(t-1)}{2}} \prod_{i=1}^{t-1}(q'^i-1) \mid q^3(q^2-1)(q^3-1)/3$. On the other hand, $p^5 = \frac{(q'^t-1)^5}{(t,q'-1)^5(q'-1)^5} < q'^{5t}$ and $q'^{t(t-1)-t} < q'^{\frac{t(t-1)}{2}} \prod_{i=1}^{t-1}(q'^i-1) \leq q^3(q^2-1)(q^3-1)/3 < p^5$. It follows that $t(t-1) - t < 5t$, and so $t = 3, 5$. If $t = 5$, then $\frac{q^2+q+1}{3} = \frac{q'^4+q'^3+q'^2+q'+1}{(5,q'-1)}$. Then $(q', q) = 1$. We have $q'^{10}(q'-1)(q'^2-1)(q'^3-1)(q'^4-1) \mid q^3(q^2-1)(q^3-1)/3$. But $q'^{10} \nmid q^3(q^2-1)(q^3-1)/3$, which is a contradiction. If $t = 3$, then $\frac{q'^3-1}{(3,q'-1)(q'-1)} = \frac{q^2+q+1}{3}$ and so, $\frac{q'^2+q'+1}{(3,q'-1)} = \frac{q^2+q+1}{3}$. We consider the following subcases:

(a) Assume $(q', q) \neq 1$ such that $q \mid q'$.

(a.1) If $(3, q'-1) = 3$, then $\frac{q'^2+q'+1}{3} = \frac{q^2+q+1}{3}$. Therefore, $q' = q$ and so $K/H \cong \mathrm{PSL}(3, q)$.

(a.2) If $(3, q'-1) = 1$, then $q'^2 + q' + 1 = \frac{q^2+q+1}{3}$. Hence, $q(q+1) = 3q'^2 + 3q' + 2$. It follows that $q \mid 3q'^2 + 3q' + 2$. Since $q \mid q'$ and $q \mid 2$, we have $q = 2$. Therefore, $q'(q'+1) = \frac{4}{3}$, which is a contradiction.

Similarly, if $q' \mid q$, then we can complete the proof.

(b) Suppose that $(q', q) = 1$.

(b.1) If $(3, q'-1) = 3$, then $\frac{q'^2+q'+1}{3} = \frac{q^2+q+1}{3}$. Thus $q'^2 + q' + 1 = q^2 + q + 1$ and so, $q(q+1) = q'(q'+1)$. But $(q', q) = 1$, a contradiction.

(b.2) If $(3, q'-1) = 1$, then $q'^2 + q' + 1 = \frac{q^2+q+1}{3}$. Thus $q'(q'+1) = \frac{(q-1)(q+2)}{3}$. It follows that $3k(k+1) = q'(q'+1)$. Considering the different possibilities of q' leads us to get a contradiction.

5. Let $K/H \cong \mathrm{PSL}(t+1,q')$, where $(q'-1) \mid (t+1)$ and t is an odd prime. Then $p = \frac{q'^t-1}{q'-1}$ and $q'^{\frac{t(t+1)}{2}}(q'^{t+1}-1)\prod_{i=1}^{t-1}(q'^i-1) \mid q^3(q^2-1)(q^3-1)/3$. On the other hand, $p^5 = \frac{(q'^t-1)^5}{(q'-1)^5} < q'^{5t}$ and $q'^{t(t+1)-t} < q'^{\frac{t(t+1)}{2}}(q'^{t+1}-1)\prod_{i=1}^{t-1}(q'^i-1) \le q^3(q^2-1)(q^3-1)/3 < p^5 < q'^{5t}$. It follows that $t+1 < 6$. Hence, $t=3$, and so $q'^2+q'+1 = \frac{q^2+q+1}{3}$. Since $(q'-1) \mid (t+1)$, $q' \in \{2,3,5\}$. So $K/H \cong \mathrm{PSL}(4,2)$, $\mathrm{PSL}(4,3)$, or $\mathrm{PSL}(4,5)$.

If $K/H \cong \mathrm{PSL}(4,2) \cong \mathrm{Alt}_8$, then by Step 2, we get a contradiction.

If $K/H \cong \mathrm{PSL}(4,3)$, then $\frac{q^2+q+1}{3} = 13$, which is a contradiction.

Similarly, we can rules out the case when $K/H \cong \mathrm{PSL}(4,5)$.

The other cases are very similar, and we omit them.

If $t(K/H)=3$, then $p \in \{\mathrm{OC}_2(K/H),\ \mathrm{OC}_3(K/H)\}$, and if $t(K/H) \in \{4,5\}$, then $p \in \{\mathrm{OC}_2(K/H), \mathrm{OC}_3(K/H), \mathrm{OC}_4(K/H), \mathrm{OC}_5(K/H)\}$.

Then by Tables 1-4 in [8], all of possibilities for K/H are $\mathrm{PSL}(2,q')$, where $4 \mid q'$, $\mathrm{PSL}(2,q')$, where $4 \mid q'-1$, $\mathrm{PSU}(6,2)$, $\mathrm{PSL}(3,2)$, ${}^2D_t(3)$, where $t = 2^u+1 \ge 5$, ${}^2D_{t+1}(2)$, where $t = 2^n-1$ and $n \ge 2$, $G_2(q')$, where $q' \equiv 0 \pmod 3$, ${}^2G_2(q')$, where $q'^{(2t+1)} > 3$, $F_4(q')$, where q' is even, ${}^2F_4(q')$, where $q'^{(2t+1)} \ge 2$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, $\mathrm{PSL}(3,4)$, ${}^2B_2(q')$, where q'^{2t+1} and $t \ge 1$, or $E_8(q')$. The cases $K/H \cong \mathrm{PSL}(3,2)$ and $\mathrm{PSL}(3,4)$ are our desired, but for the other cases, we can get a contradiction. For example, we consider once K/H with $t(K/H)=3$, and once again with $t(K/H) > 3$.

For the case $t(K/H)=3$, we consider $F_4(q')$, where q' is even. Then $q'^4+1 = \frac{q^2+q+1}{3}$, or $q'^4-q'^2+1 = \frac{q^2+q+1}{3}$. So, $p^6 = (q'^4+1)^6 < (q'^5)^6 = q'^{30}$ and $q'^{24}(q'^{12}-1)(q'^8-1)(q'^6-1)(q'^2-1) \mid q^3(q^2-1)(q^3-1)/3$. Therefore, $q'^{36} \le q^3(q^2-1)(q^3-1)/3 < p^6 < q'^{30}$, which is a contradiction.

For the case $t(K/H) > 3$, we consider ${}^2B_2(q')$, where $q'^{(2t+1)}$ and $t \ge 1$. We have $p \in \{q'-1, q' \pm \sqrt{2q'}+1\}$.

If $q' - 1 = p$, then we can see that $2^2(3.2^{2t-1} - 1) = q(q+1)$. If $|q|_2 = 2^2$, then $q + 1 = 5$ and $t = 1$, Therefore, $13 \nmid |G|$ and $13 \mid |K/H|$, a contradiction. Thus $|q+1|_2 = 2^2$ and so, $|q-1|_2 = 2$. Furthermore, $|p-1|_2 = 2$. So $2^{2(2t+1)} \leq |K/H|_2 \leq |G|_2 \leq 2^5$. Since $t \geq 1$, we get a contradiction.

If $q' + \sqrt{2q'} + 1 = p$, then $\frac{q^2+q-2}{3} = 2^{t+1}(2^t+1)$. Hence, $(q-1)(q+2) = 3.2^{t+1}(2^t+1)$. Since $3 \mid q-1$, we have $q - 1 = 3k$ for some positive integer k. Thus $3k(k+1) = 2^{t+1}(2^t+1)$ and so, $k(k+1) = 2^{t+1}(\frac{2^t+1}{3})$.

Now, if $2^{t+1} \mid k$, then $k+1 \leq \frac{2^t+1}{3}$ and if $2^{t+1} \mid k+1$, then $k \leq \frac{2^t+1}{3}$, which are impossible.

Similarly, we can rules out the case when $q' - \sqrt{2q'} + 1 = p$.

If $q \not\equiv 1 \pmod 3$, then $q \geq 3$. The proof is similar to that of $q \equiv 1 \pmod 3$, and we omit the proof.

The above steps show that $K/H \cong \mathrm{PSL}(3,q)$. Since $|G| = |\mathrm{PSL}(3,q)|$, $H = 1$ and $K = G \cong \mathrm{PSL}(3,q)$. This completes the proof of our main theorem.

Corollary 3.1 Let q be prime power. Then Thompson's conjecture holds for the simple group $\mathrm{PSL}(3,q)$, where $(q^2+q+1)/(3,q-1)$ is a prime number.

Proof. Let G be a group with trivial center and $N(G) = N(\mathrm{PSL}(3,q))$. By [6, Lemma 1.4], we have $|G| = |\mathrm{PSL}(3,q)|$. Hence, the corollary follows from the main theorem.

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