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QUASI-ZARISKI TOPOLOGY ON THE QUASI-PRIMARY SPECTRUM OF A MODULE

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ABSTRACT. Let R be a commutative ring with a nonzero identity and M be a unitary R-module. A submodule Q of M is called quasi-primary if $Q \neq M$ and, whenever $r \in R$, $x \in M$, and $rx \in Q$, we have $r \in \sqrt{(Q:M)}$ or $x \in \text{rad}Q$. A submodule N of M satisfies the primeful property if and only if M/N is a primeful R-module. We let q.Spec(M) denote the set of all quasi-primary submodules of M satisfying the primeful property. The aim of this paper is to introduce and study a topology on q.Spec(M) which is called quasi-Zariski topology of M. We investigate, in particular, the interplay between the properties of this space and the algebraic properties of the module under consideration. Modules whose quasi-Zariski topology is, respectively T_0, T_1 or irreducible, are studied, and several characterizations of such modules are given. Finally, we obtain conditions under which q.Spec(M) is a spectral space.

1. Introduction

Throughout this paper, R is a commutative ring with a nonzero identity and M is a unitary R-module. For any ideal I of R containing Ann(M) (the annihilator of M), $\overline{I}$ and $\overline{R}$ will denote $I/\text{Ann}(M)$ and $R/\text{Ann}(M)$, respectively.

Let M be an R-module and N a submodule of M. The colon ideal of M into

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N, denoted by $(N : M)$, is the annihilator of M/N as an R-module. P is a prime submodule or a p-prime submodule of M, where $p = (P : M)$, if $P \neq M$ and whenever $rx \in P$ for some $r \in R$ and $x \in M$, we have $r \in p$ or $x \in P$ ([14]). Spec(M), the prime spectrum of M, is the set of all prime submodules of M. Also the set of all maximal submodules of M is denoted by Max(M). It is easily seen that Max$(M) \subseteq$ Spec(M). If $p \in$ Spec(R), Spec$_p(M)$ denotes the set of all p-prime submodules of M ([15]). rad$N$ is the intersection of all prime submodules of M containing N and also rad$N = M$ when M has no prime submodule containing N. For an ideal I of R, the radical of I is denoted by $\sqrt{I}$.

Recall that a proper ideal q of R is quasi-primary if $rs \in q$ for $r, s \in R$ implies either $r \in \sqrt{q}$ or $s \in \sqrt{q}$ ([8]). Equivalently, q is a quasi-primary ideal of R if and only if $\sqrt{q}$ is a prime ideal of R [8, Definition 2, p. 176]. For an ideal I of R, the set of all quasi-primary ideals of R containing I is denoted by $V^{\mathfrak{q}}(I)$.

An R-module M is said to be primeful if either $M = 0$ or $M \neq 0$ and satisfies the following equivalent conditions (the equivalence is proved in [11, Theorem 2.1]):

(i) The natural map ψ : Spec$(M) \to$ Spec$(\overline{R})$, given by $\psi(P) = \overline{(P : M)}$, is surjective;

(ii) For every $p \in V(\mathrm{Ann}(M))$, there exists $P \in$ Spec(M) such that $(P : M) = p$;

(iii) $p_p M_p \neq M_p$ for every $p \in V(\mathrm{Ann}(M))$;

(iv) $S_p(pM)$, the contraction of $p_p M_p$ in M, is a p-prime submodule of M for every $p \in V(\mathrm{Ann}(M))$;

(v) Spec$_p(M) \neq \emptyset$ for every $p \in V(\mathrm{Ann}(M))$.

If N is a submodule of M and M/N is a primeful R-module, we say that N satisfies the primeful property.

A proper submodule Q of M is quasi-primary provided that $rx \in Q$, for $r \in R$ and $x \in M$, implies $r \in \sqrt{(Q : M)}$ or $x \in$ rad$Q$ (this notion has been introduced by the authors [6], [7]). If $\sqrt{(Q : M)} = p$ is a prime ideal, then Q is also called a p-quasi-primary submodule of M. If N is a proper submodule of an R-module M satisfying the primeful property, then, by definition, we have $\mathrm{rad}N \neq M$ and also, by [11, Proposition 5.3], we have $(\mathrm{rad}N : M) = \sqrt{(N : M)}$. Thus if Q is a quasi-primary submodule of M satisfying the primeful property, then $(Q : M)$ is a quasi-primary ideal of R. In this case, as we mentioned before, Q is called a p-quasi-primary submodule of M where $p = \sqrt{(Q : M)}$.

The quasi-primary spectrum q.Spec(M) is defined to be the set of all quasi-primary submodules of M satisfying the primeful property ([6], [7]). Also the set of all p-quasi-primary submodules of M satisfying the primeful property is denoted by q.Spec$_p(M)$. The authors studied the class of modules whose quasi-primary spectrums are empty ([5, section 2]). For example q.Spec$(\mathbb{Q}) = \emptyset$ while $Spec(\mathbb{Q}) = \{0\}$, where $\mathbb{Q}$ is the module of rational numbers over the ring of integers $\mathbb{Z}$. Throughout the rest of this paper, we assume that q.Spec(M) is non-empty.

An R-module M is called quasi-primaryful if either $M = (0)$ or $M \neq (0)$ and for every $q \in V^{\mathbf{q}}(\mathrm{Ann}(M))$, there exists $Q \in$ q.Spec(M) such that $\sqrt{(Q : M)} = \sqrt{q}$. This notion has been introduced and extensively studied by the authors in [5].

The Zariski topology on the spectrum of prime ideals of a ring is one of the main tools in algebraic geometry. In the literature, there are many different generalizations of the Zariski topology for modules over commutative rings. [13] defined a Zariski topology on Spec(M) whose closed sets are $V(N) = \{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\}$ for any submodule N of M. As a new generalization of the Zariski topology, we introduce the quasi-Zariski topology on q.Spec(M) for any R-module M in which closed sets are varieties $\nu(N) = \{Q \in \text{q.Spec}(M) : \sqrt{(Q : M)} \supseteq \sqrt{(N : M)}\}$ of all submodules N of M.

In section (2), when q.Spec$(M) \neq \emptyset$, we define a map $\psi^{\mathbf{q}} : \text{q.Spec}(M) \to \text{q.Spec}(\overline{R})$ by $\psi^{\mathbf{q}}(Q) = \overline{(Q : M)}$ for every $Q \in$ q.Spec(M). We show that, when q.Spec(M) is not empty, the injectivity and the surjectivity of the map $\psi^{\mathbf{q}}$ play a key role in our investigation and give some topological properties for q.Spec(M). We prove that q.Spec(M) is a T_0-space iff $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is injective iff q.Spec(M) has at most one p-quasi-primary submodule satisfying the primeful property for every $p \in$ Spec(R)(Theorem 2.1 and Proposition 3.2 (5)).

In section (3), and assuming suitable conditions for each result, we investigate when this space is connected (Theorem 3.1), T_0 or T_1 (Proposition 3.2 and Theorem 3.2) and irreducible (Corollary 3.2). Finally, we investigate this topological space q.Spec(M) of a module M from the point of view of spectral spaces, topological spaces each of which is homeomorphic to Spec(S) for some ring S. [10] has characterized spectral spaces as quasi-compact T_0-spaces W such that W has a quasi-compact open base closed under finite intersection and each irreducible closed subset of W has a generic point. We follow the Hochster's characterization closely in discussing whether q.Spec(M) of a module M is a spectral space.

We discover that when q.Spec$(M) \neq \emptyset$, the injectivity and the surjectivity of the map $\psi^{\mathbf{q}}$ of q.Spec(M) play, respectively, important roles for q.Spec(M) being spectral. We prove that if $\psi^{\mathbf{q}}$ is surjective, then q.Spec(M) is almost spectral in the sense that q.Spec(M) satisfies all the conditions to be a spectral space except for, possibly, that q.Spec(M) is a T_0-space (Proposition 3.3 (4) and Theorems 3.7, 3.4 (1)). We show that if $\psi^{\mathbf{q}}$ is surjective, then q.Spec(M) is a spectral space iff q.Spec(M) is a T_0-space iff $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is injective(Theorem 3.9).

2. Surjectivity and injectivity of spectral maps

In this section, we introduce a commutative square of spectral maps that the surjectivity of two of its sides determine the class of quasi-primaryful modules. In fact every non-zero quasi-primaryful modules possess the non-empty quasi-primary spectrum with a surjective natural map.

The saturation of a submodule N of M with respect to a prime ideal p of R is the contraction of N_p in M and designated by $S_p(N)$. It is known that $S_p(N) = \{m \in M \mid cm \in N$ for some $c \in R - p\}$ ([12]).

Lemma 2.1. *Let M be an R-module and $Q \in \text{q.Spec}_p(M)$. Then $S_p(pM)$ is a p-prime submodule of M. In particular, the map $\phi^{\mathbf{M}} : \text{q.Spec}(M) \to \text{Spec}(M)$ defined by $\phi^{\mathbf{M}}(Q) = S_p(pM)$, is well-defined.*

Proof. By [12, Corollary 3.7], it suffices to show that $p_pM_p \neq M_p$ where $p = \sqrt{(Q : M)}$. It is clear that $\sqrt{(Q : M)}M = (\text{rad}Q : M)M \subseteq \text{rad}Q$ and so $(\text{rad}Q : M)_pM_p \subseteq (\text{rad}Q)_p$. By [6, Theorem 2.15], $(\text{rad}Q)_p = \text{rad}Q_p$ is a prime submodule of M_p and hence $p_pM_p \subseteq \text{rad}Q_p \neq M_p$. It follows that $S_p(pM)$ is a p-prime submodule of M. □

To prepare our way for this section, it is convenient to introduce the following spectral maps:

$$\begin{array}{ccc} \text{q.Spec}(M) & \xrightarrow{\psi^{\mathbf{q}}} & \text{q.Spec}(\overline{R}) \\ \phi^{\mathbf{M}}\big\downarrow & & \phi^{\mathbf{R}}\big\downarrow \\ \text{Spec}(M) & \xrightarrow{\psi} & \text{Spec}(\overline{R}) \end{array}$$

where $\psi^{\mathbf{q}}(Q) = \overline{(Q : M)}$, $\psi(N) = \overline{(N : M)}$, $\phi^{\mathbf{R}}(\overline{q}) = \sqrt{\overline{q}}$ and $\phi^{\mathbf{M}}(Q) = S_p(pM)$ with $p = \sqrt{(Q : M)}$.

It is clear that for a non-zero R-module M, the above diagram is commutative; i.e., $\phi^{\mathbf{R}} o \psi^{\mathbf{q}} = \psi o \phi^{\mathbf{M}}$. Indeed, suppose $Q \in \text{q.Spec}(M)$ and $p = \sqrt{(Q : M)}$. It follows from Lemma 2.1 that $(S_p(pM) : M) = p$, i.e., $\psi o \phi^{\mathbf{M}}(Q) = \overline{p}$. On the other hand, by definition, $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}(Q) = \overline{p}$, as required.

It is easy to see that the surjectivity of $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is naturally equivalent to M being a quasi-primaryful module.

Proposition 2.1. (1) *Let p be a prime ideal of a ring R and let M be an R-module. If the map $\psi^{\mathbf{q}}$ is injective, then every p-prime submodule of M satisfying the primeful property is of the form $S_p(pM)$.*

(2) *If every prime submodule of M satisfies the primeful property then the map $\phi^{\mathbf{M}}$ is surjective.*

Proof. (1). Suppose $\psi^{\mathbf{q}}$ is injective. Let P be a p-prime submodule of M satisfying the primeful property. Then $S_p(pM) \subseteq S_p(P) = P \neq M$. It follows from [12, Proposition 2.4] that $S_p(pM)$ is a p-prime submodule of M. Since P satisfies the primeful property, clearly $S_p(pM)$ also does. Thus, we have $\psi^{\mathbf{q}}(S_p(pM)) = \psi^{\mathbf{q}}(P)$ and hence $S_p(pM) = P$, since $\psi^{\mathbf{q}}$ is injective.
(2) is trivial. Indeed, if $P \in \mathrm{Spec}_p(M)$, then $P \in \mathrm{q.Spec}(M)$ and hence $\phi^{\mathbf{M}}(P) = S_p(pM)$. □

Recall that for any submodule N of M,

$$\nu(N) = \{Q \in \mathrm{q.Spec}(M) : \sqrt{(Q:M)} \supseteq \sqrt{(N:M)}\}.$$

Theorem 2.1. *The following statements are equivalent for any R-module M.*

(1) *$\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is injective;*
(2) *If $\nu(N) = \nu(K)$, then $N = K$, for any $N, K \in \mathrm{q.Spec}(M)$;*
(3) *$\mid \mathrm{q.Spec}_p(M) \mid \leq 1$ for any $p \in \mathrm{Spec}(R)$;*
(4) *$\phi^{\mathbf{M}}$ is injective.*

Moreover, if every prime submodule of M satisfies the primeful property, then the above statements are equivalent to:

(5) *$\phi^{\mathbf{M}}$ is bijective.*

Proof. (1) ⇒ (2) Suppose that $\nu(N) = \nu(K)$ for $N, K \in \mathrm{q.Spec}(M)$. By definition, we have then $\sqrt{(N:M)} = \sqrt{(K:M)}$; i.e., $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}(N) = \phi^{\mathbf{R}} o \psi^{\mathbf{q}}(K)$. Now the injectivity of $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ implies that $N = K$, so we have proved (2).
(2) ⇒ (3). Let $N, K \in \mathrm{q.Spec}_p(M)$. Then $\sqrt{(N:M)} = \sqrt{(K:M)}$ implies that $\nu(N) = \nu(K)$. Thus, $N = K$ by (2).
(3) ⇒ (4). Suppose $Q, Q' \in \mathrm{q.Spec}(M)$ such that $p = \sqrt{(Q:M)}$, $p' = \sqrt{(Q':M)}$

and $\phi^{\mathbf{M}}(Q) = \phi^{\mathbf{M}}(Q')$. Then $S_p(pM) = S_{p'}(p'M)$ and Lemma 2.1 show that $S_p(pM)$ and $S_{p'}(p'M)$ are p-prime submodules of M. Thus $Q, Q' \in \text{q.Spec}_p(M)$ and hence (3) implies that $Q = Q'$.

(4) $\Rightarrow$ (1). Suppose $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}(Q) = \phi^{\mathbf{R}} o \psi^{\mathbf{q}}(Q')$ for some $Q \in \text{q.Spec}_p(M)$ and $Q' \in \text{q.Spec}_{p'}(M)$. Thus $p = p'$ and so $\phi^{\mathbf{M}}(Q) = \phi^{\mathbf{M}}(Q')$. This implies that $Q = Q'$.

(4) $\Rightarrow$ (5) is clear where every prime submodule of M satisfies the primeful property. □

An R-module M is said to be multiplication if for every submodule N of M, there exists an ideal I of R such that $N = IM$ ([4]). In this case, we can take $I = (N : M)$. An R-module M is called content if for every family $\{I_\lambda \mid \lambda \in \Lambda\}$ of ideals of R, $(\underset{\lambda\in\Lambda}{\cap} I_\lambda)M = \underset{\lambda\in\Lambda}{\cap}(I_\lambda M)$ ([16]). For example faithful multiplication modules and projective modules are content modules [4, Theorem 1.6] and [1, Theorem 2.1 and Theorem 3.1].

Let M be a finitely generated module over a ring R. Then M is called Laskerian if every submodule of M is the intersection of a finite number of primary submodules ([9]). It is well-known that every finitely generated module over a Noetherian ring is Laskerian. However the converse is not true in general [9, Example 4.2].

Theorem 2.2. *Let M be an R-module and the map $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ be injective.*

1. *Let M be a Laskerian module and every primary submodule of M satisfies the primeful property. Then every quasi-primary submodule of M satisfying the primeful property is primary.*
2. *Let M be a flat content R-module. Then $Q = (Q : M)M$ for every $Q \in \text{q.Spec}(M)$.*
3. *If M is free, then $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is bijective.*

Proof. Let $Q \in \text{q.Spec}(M)$ and $\overset{t}{\underset{i=1}{\cap}} N_i$ be a primary decomposition for Q. Since $\sqrt{(Q : M)}$ is a prime ideal of R,

$$\sqrt{(N_j : M)} \subseteq \sqrt{(Q : M)} = \bigcap_{i=1}^{t} \sqrt{(N_i : M)} \subseteq \sqrt{(N_j : M)}$$

for some $1 \leq j \leq t$. Since N_j satisfies the primeful property, we have $N_j \in \text{q.Spec}(M)$ and so the injectivity of $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ implies that $Q = N_j$.

(2). Suppose $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is injective and $Q \in \text{q.Spec}_p(M)$. By Theorem 2.1, it suffices to show that $(Q : M)M \in \text{q.Spec}_p(M)$. It is easy to see directly that $\sqrt{((Q : M)M : M)} = \sqrt{(Q : M)} = p$ and $(Q : M)M$ satisfies the primeful property. It remains to show that $(Q : M)M$ is quasi-primary. Let $rx \in (Q : M)M$ for $r \in R$ and $x \notin \text{rad}((Q : M)M)$. Since M is flat content, $\text{rad}((Q : M)M) = \bigcap_{p \supseteq (Q:M)} (pM) = (\bigcap_{p \supseteq (Q:M)} p)M = \sqrt{(Q : M)}M = pM$ and hence $rx \in pM$ and $x \notin pM$. On the other hand, rad$Q$ is a proper submodule of M, because Q satisfies the primeful property. Thus $pM \neq M$ is a p-prime submodule of M, by [14, Theorem 3], and so $r \in p$, i.e. $(Q : M)M$ is a p-quasi-primary submodule of M.

(3). By [5, Theorem 4.3(1)], free modules are quasi-primaryful and hence the proof is easy. □

3. Some topological properties of q.Spec(M)

Recall that for any submodule N of an R-module M, $\nu(N)$ is the set of all quasi-primary submodules Q of M satisfying the primeful property, namely $\sqrt{(Q : M)} \supseteq \sqrt{(N : M)}$. We begin this section by showing that if $\eta(M)$ denotes the collection of all subsets $\nu(N)$ of q.Spec(M), then $\eta(M)$ satisfies the axioms for the closed subsets of a topological space on q.Spec(M), called quasi-Zariski topology.

Lemma 3.1. *Let M be an R-module. Then for submodules N, N' and $\{N_i \mid i \in I\}$ of M we have*

(1) $\nu(0) = \text{q.Spec}(M)$ *and* $\nu(M) = \emptyset$.
(2) $\bigcap_{i \in I} \nu(N_i) = \nu((\sum_{i \in I} (N_i : M))M)$.
(3) $\nu(N) \cup \nu(N') = \nu(N \cap N')$.

Proof. (1) and (3) are trivial. (2) follows from the following implications:

$$\begin{aligned}
Q \in \cap_{i\in I}\nu(N_i) &\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{(N_i:M)}\ \forall i \in I \\
&\Rightarrow \sqrt{(Q:M)} \supseteq (N_i:M)\ \forall i \in I \\
&\Rightarrow \sqrt{(Q:M)} \supseteq \sum_{i\in I}(N_i:M) \\
&\Rightarrow \sqrt{(Q:M)}M \supseteq (\sum_{i\in I}(N_i:M))M \\
&\Rightarrow (\sqrt{(Q:M)}M:M) \supseteq ((\sum_{i\in I}(N_i:M))M:M) \\
&\Rightarrow ((\mathrm{rad}Q:M)M:M) \supseteq ((\sum_{i\in I}(N_i:M))M:M) \\
&\Rightarrow (\mathrm{rad}Q:M) \supseteq ((\sum_{i\in I}(N_i:M))M:M) \\
&\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{((\sum_{i\in I}(N_i:M))M:M)} \\
&\Rightarrow Q \in \nu((\sum_{i\in I}(N_i:M))M).
\end{aligned}$$

For the reverse inclusion we have

$$\begin{aligned}
Q \in \nu(\sum_{i\in I}(N_i:M)M) &\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{((\sum_{i\in I}(N_i:M))M:M)} \\
&\Rightarrow \sqrt{(Q:M)} \supseteq ((\sum_{i\in I}(N_i:M))M:M) \\
&\Rightarrow \sqrt{(Q:M)} \supseteq ((N_i:M)M:M)\ \ \forall i \in I \\
&\Rightarrow \sqrt{(Q:M)} \supseteq (N_i:M)\ \ \forall i \in I \\
&\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{(N_i:M)}\ \ \forall i \in I \\
&\Rightarrow Q \in \cap_{i\in I}\nu(N_i)
\end{aligned}$$

□

Let Y be a subset of q.Spec(M) for an R-module M. We will denote the intersection of all elements in Y by $\xi(Y)$ and the closure of Y in q.Spec(M) with respect to the quasi-Zariski topology by $cl(Y)$. In the following Lemma, we gather some basic facts about the varieties.

Lemma 3.2. *Let M be an R-module. Let N, N' and $\{N_i \mid i \in I\}$ be submodules of M. Then the following hold.*

(1) If $N \subseteq N'$, then $\nu(N') \subseteq \nu(N)$.
(2) $\nu(\text{rad}N) \subseteq \nu(N)$ and equality holds if M is multiplication.
(3) $\nu(N) = \nu(\sqrt{(N:M)}M)$.
(4) If $\sqrt{(N:M)} = \sqrt{(N':M)}$, then $\nu(N) = \nu(N')$. The converse is also true if both $N, N' \in$ q.Spec(M).
(5) $\nu(N) = \bigcup_{(N:M)\subseteq p \in \text{Spec}(R)} \text{q.Spec}_p(M)$.
(6) Let Y be a subset of q.Spec(M). Then $Y \subseteq \nu(N)$ if and only if $\sqrt{(N:M)} \subseteq \sqrt{(\xi(Y):M)}$.

Proof. (1) is clear.
(2). $\nu(\text{rad}N) \subseteq \nu(N)$ is clearly true by (1). The equality can be deduced from the fact rad$N = \sqrt{(N:M)}$, where N is a submodule of a multiplication module M([4, Theorem 2.12].
(3). Let N be a proper submodule of M. Then

$$\begin{aligned} Q \in \nu(N) &\Rightarrow \sqrt{(Q:M)}M \supseteq \sqrt{(N:M)}M \\ &\Rightarrow \text{rad}Q \supseteq \sqrt{(N:M)}M \\ &\Rightarrow \sqrt{(Q:M)} \supseteq (\sqrt{(N:M)}M : M) \\ &\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{(\sqrt{(N:M)}M : M)} \\ &\Rightarrow Q \in \nu(\sqrt{(N:M)}M). \end{aligned}$$

Thus $\nu(N) \subseteq \nu(\sqrt{(N:M)}M)$. For the reverse inclusion, we have

$$\begin{aligned} Q \in \nu(\sqrt{(N:M)}M) &\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{(\sqrt{(N:M)}M:M)} \\ &\Rightarrow \sqrt{(Q:M)} \supseteq (\sqrt{(N:M)}M:M) \\ &\Rightarrow \sqrt{(Q:M)} \supseteq \sqrt{(N:M)} \\ &\Rightarrow Q \in \nu(N) \end{aligned}$$

Finally, $(4), (5)$ and (6) are clearly true by definitions. □

Proposition 3.1. *Let M be an R-module.*

(1) $(\phi^{\mathbf{R}})^{-1}(V(\overline{I})) = \nu(\overline{I})$ *for every ideal I of R containing* $\mathrm{Ann}(M)$. *In particular,* $(\phi^{\mathbf{R}} o \psi^{\mathbf{q}})^{-1}(V(\overline{I})) = (\psi^{\mathbf{q}})^{-1}(\nu(\overline{I}))$.

(2) $\phi^{\mathbf{R}}(\nu(\overline{I})) = V(\overline{I})$ *and* $\phi^{\mathbf{R}}(\mathrm{q.Spec}(\overline{R}) - \nu(I)) = \mathrm{Spec}(\overline{R}) - V(\overline{I})$ *i.e. $\phi^{\mathbf{R}}$ is both closed and open.*

(3) $(\phi^{\mathbf{M}})^{-1}(V(N)) = \nu(N)$, *for every submodule N of M; i.e. the map $\phi^{\mathbf{M}}$ is continuous.*

(4) *The natural maps $\psi^{\mathbf{q}}$ and $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ are continuous with respect to the quasi-Zariski topology; more precisely for every ideal I of R containing* $\mathrm{Ann}(M)$,

$$(\phi^{\mathbf{R}} o \psi^{\mathbf{q}})^{-1}(V(\overline{I})) = (\psi^{\mathbf{q}})^{-1}(\nu(\overline{I})) = \nu(IM).$$

(5) *Let M be a quasi-primaryful R-module. If $\varphi = \phi^{\mathbf{R}} o \psi^{\mathbf{q}}$, then* $\varphi(\nu(N)) = V(\overline{\sqrt{(N:M)}})$ *and* $\varphi(\mathrm{q.Spec}(M) - \nu(N)) = \mathrm{Spec}(\overline{R}) - V(\overline{\sqrt{(N:M)}})$ *i.e. φ is both closed and open.*

(6) $\varphi = \phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ *is bijective if and only if it is a homeomorphism.*

Proof. (1). Let I be an ideal of R containing $\mathrm{Ann}(M)$. Then

$$\begin{aligned} \overline{q} \in (\phi^{\mathbf{R}})^{-1}(V(\overline{I})) &\Leftrightarrow \phi^{\mathbf{R}}(\overline{q}) \in V(\overline{I}) \\ &\Leftrightarrow \sqrt{\overline{q}} \supseteq \overline{I} \\ &\Leftrightarrow \sqrt{q} \supseteq I \\ &\Leftrightarrow q \in \nu(\overline{I}). \end{aligned}$$

(2). As we have seen in (1), $\phi^{\mathbf{R}}$ is a continuous map such that $(\phi^{\mathbf{R}})^{-1}(V(\overline{I})) = \nu(\overline{I})$ for every ideal I of R containing $\mathrm{Ann}(M)$. It follows that $\phi^{\mathbf{R}}(\nu(\overline{I})) = \phi^{\mathbf{R}}((\phi^{\mathbf{R}})^{-1}(V(\overline{I}))) = V(\overline{I})$ as $\phi^{\mathbf{R}}$ is surjective. Similarly,

$$\begin{aligned} \phi^{\mathbf{R}}(\text{q.Spec}(\overline{R}) - \nu(\overline{I})) &= \phi^{\mathbf{R}}((\phi^{\mathbf{R}})^{-1}(\mathrm{Spec}(\overline{R})) - (\phi^{\mathbf{R}})^{-1}(V(\overline{I}))) \\ &= \phi^{\mathbf{R}}((\phi^{\mathbf{R}})^{-1}(\mathrm{Spec}(\overline{R}) - V(\overline{I})) \\ &= \phi^{\mathbf{R}} o (\phi^{\mathbf{R}})^{-1}(\mathrm{Spec}(\overline{R}) - V(\overline{I}))) \\ &= \mathrm{Spec}(\overline{R}) - V(\overline{I}). \end{aligned}$$

(3). Suppose $Q \in (\phi^{\mathbf{M}})^{-1}(V(N))$. Then $\phi^{\mathbf{M}}(Q) \in V(N)$ and so $p = (S_p(pM) : M) \supseteq (N : M)$, in which $p = \sqrt{(Q : M)}$. Hence $\sqrt{(Q : M)} \supseteq \sqrt{(N : M)}$ and so $Q \in \nu(N)$. The argument is reversible and so $\phi^{\mathbf{M}}$ is continuous.

(4). It follows from [13, Proposition 3.1] that ψ is a continuous map with $\psi^{-1}(V(\overline{I})) = V(IM)$ for every ideal I of R containing $\mathrm{Ann}(M)$. Also, we showed that $\phi^{\mathbf{R}} o \psi^{\mathbf{q}} = \psi o \phi^{\mathbf{M}}$. This implies that $\psi^{\mathbf{q}}$ and $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ are also continuous and $(\phi^{\mathbf{R}} o \psi^{\mathbf{q}})^{-1}(V(\overline{I})) = (\psi^{\mathbf{q}})^{-1}(\nu(\overline{I})) = \nu(IM)$ for every ideal I of R containing $\mathrm{Ann}(M)$, by (1) and (3).

(5). Take $\varphi = \phi^{\mathbf{R}} o \psi^{\mathbf{q}}$. Since M is quasi-primaryful, φ is surjective. Also by (4), φ is a continuous map such that $\varphi^{-1}(V(\overline{I})) = \nu(IM)$ for every ideal I of R containing $\mathrm{Ann}(M)$. Hence, by Lemma 3.2(3), for every submodule N of M, $\varphi^{-1}(V(\overline{\sqrt{(N : M)}})) = \nu(\sqrt{(N : M)}M) = \nu(N)$. Since the map φ is surjective, we have $\varphi(\nu(N)) = \varphi o \varphi^{-1}(V(\overline{\sqrt{(N : M)}})) = V(\overline{\sqrt{(N : M)}})$. Similarly, we conclude that

$$\begin{aligned} \varphi(\text{q.Spec}(M) - \nu(N)) &= \varphi(\varphi^{-1}(\text{Spec}(\overline{R})) - (\varphi)^{-1}(V(\overline{\sqrt{(N:M)}}))) \\ &= \varphi((\varphi)^{-1}(\text{Spec}(\overline{R}) - V(\overline{\sqrt{(N:M)}}))) \\ &= \varphi o \varphi^{-1}(\text{Spec}(\overline{R}) - V(\overline{\sqrt{(N:M)}})) \\ &= \text{Spec}(\overline{R}) - V(\overline{\sqrt{(N:M)}}). \end{aligned}$$

(6). This follows from (5). □

Lemma 3.3. *For any ring R,* q.Spec$(\overline{R})$ *is connected if and only if* Spec$(\overline{R})$ *is connected.*

Proof. Suppose that q.Spec$(\overline{R})$ is a connected space. By Proposition 3.1, the map $\phi^{\mathbf{R}}$ is surjective and continuous and so Spec$(\overline{R})$ is also a connected space. Conversely, suppose on the contrary that q.Spec$(\overline{R})$ is disconnected. Then there exists a non-empty proper subset W of q.Spec$(\overline{R})$ that is both open and closed. By Proposition 3.1, $\phi^{\mathbf{R}}(W)$ is a non-empty subset of Spec$(\overline{R})$ that is both open and closed. To complete the proof, it suffices to show that $\phi^{\mathbf{R}}(W)$ is a proper subset of Spec$(\overline{R})$ that in this case Spec$(\overline{R})$ is disconnected, a contradiction.

Since W is open, $W = \text{q.Spec}(\overline{R}) - \nu(\overline{I})$ for some ideal I of R containing Ann(M). Thus $\phi^{\mathbf{R}}(W) = \text{Spec}(\overline{R}) - V(\overline{I})$ by Proposition 3.1. Therefore, if $\phi^{\mathbf{R}}(W) = \text{Spec}(\overline{R})$, then $V(\overline{I}) = \emptyset$, and so $\overline{I} = \overline{R}$, i.e., $I = R$. It follows that $W = \text{q.Spec}(\overline{R}) - \nu(\overline{R}) = \text{q.Spec}(\overline{R})$ which is impossible. Thus $\phi^{\mathbf{R}}(W)$ is a proper subset of q.Spec$(\overline{R})$. □

Theorem 3.1. *Let M be a quasi-primaryful R-module. Then the following statements are equivalent:*

(1) q.Spec(M) *together with quasi-Zariski topology is a connected space;*

(2) q.Spec$(\overline{R})$ *together with quasi-Zariski topology is a connected space;*

(3) Spec$(\overline{R})$ *together with Zariski topology is a connected space;*

(4) $\mathrm{Spec}(M)$ *together with Zariski topology is a connected space;*

(5) *The ring* $\overline{R}$ *contains no idempotent other than* $\overline{0}$ *and* $\overline{1}$.

Consequently, if R *is a quasi-local ring or* $\mathrm{Ann}(M)$ *is a prime ideal of* R*, then both* $\mathrm{q.Spec}(M)$ *and* $\mathrm{q.Spec}(\overline{R})$ *are connected.*

Proof. (1) $\Rightarrow$ (3) follows since $\varphi = \phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is a surjective and continuous map of the connected space $\mathrm{q.Spec}(M)$. To prove (3) $\Rightarrow$ (1), we assume that $\mathrm{Spec}(\overline{R})$ is connected. If $\mathrm{q.Spec}(M)$ is disconnected, then $\mathrm{q.Spec}(M)$ must contain a non-empty proper subset Y that is both open and closed. Accordingly, $\varphi(Y)$ is a non-empty subset of $\mathrm{Spec}(\overline{R})$ that is both open and closed by Proposition 3.1. To complete the proof, it suffices to show that $\varphi(Y)$ is a proper subset of $\mathrm{Spec}(\overline{R})$ so that $\mathrm{Spec}(\overline{R})$ is disconnected, a contradiction.

Since Y is open, $Y = \mathrm{q.Spec}(M) - \nu(N)$ for some submodule N of M whence $\varphi(Y) = \mathrm{Spec}(\overline{R}) - V(\overline{\sqrt{(N:M)}})$ by Proposition 3.1. Therefore, if $\varphi(Y) = \mathrm{Spec}(\overline{R})$, then $V(\overline{\sqrt{(N:M)}}) = \emptyset$, and so $\overline{\sqrt{(N:M)}} = \overline{R}$, i.e., $N = M$. It follows that $Y = \mathrm{q.Spec}(M) - \nu(M) = \mathrm{q.Spec}(M)$ which is impossible. Thus $\varphi(Y)$ is a proper subset of $\mathrm{Spec}(\overline{R})$.

By Lemma 3.3, (2) and (3) are equivalent and (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) may be obtained by using [5, Theorem 3.1.] and [13, Corollary 3.8]. □

A topological space $(X;\tau)$ is said to be a T_0-space if for each pair of distinct points a, b in X, either there exists an open set containing a and not b, or there exists an open set containing b and not a. It has been shown that a topological space is T_0 if and only if the closures of distinct points are distinct. Also, a topological space $(X;\tau)$ is called a T_1-space if every singleton set $\{x\}$ is closed in $(X;\tau)$. Clearly every T_1-space is a T_0-space.

Proposition 3.2. *Let* M *be an* R*-module,* $Y \subseteq \mathrm{q.Spec}(M)$ *and let* $Q \in \mathrm{q.Spec}_p(M)$. *Then*

(1) $\nu(\xi(Y)) = cl(Y)$. *In particular,* $cl(\{Q\}) = \nu(Q)$.

(2) *If* $(0) \in Y$*, then* Y *is dense in* q.Spec(M).

(3) *The set* $\{Q\}$ *is closed in* q.Spec(M) *if and only if*

  (i) p *is a maximal element in* $\{\sqrt{(N:M)} \mid N \in \text{q.Spec}(M)\}$*, and*

  (ii) $\text{q.Spec}_p(M) = \{Q\}$.

(4) *If* $\{Q\}$ *is closed in* q.Spec(M)*, then* Q *is a maximal element of* q.Spec(M).

(5) q.Spec(M) *is a* T_0*-space if and only if any of the equivalent statements* (1)-(4) *in Theorem 2.1 hold.*

(6) q.Spec(M) *is a* T_1*-space if and only if* q.Spec(M) *is a* T_0*-space and for every element* $Q \in \text{q.Spec}(M)$*,* $\sqrt{(Q:M)}$ *is a maximal element in* $\{\sqrt{(N:M)} \mid N \in \text{q.Spec}(M)\}$.

(7) q.Spec(M) *is a* T_1*-space if and only if* q.Spec(M) *is a* T_0*-space and every quasi-primary submodule of* M *satisfying the primeful property is a maximal element of* q.Spec(M).

(8) *Let* $(0) \in \text{q.Spec}(M)$*. Then* q.Spec(M) *is a* T_1*-space if and only if* (0) *is the only quasi-primary submodule of* M *satisfying the primeful property.*

Proof. (1). Suppose $L \in Y$. Then $\xi(Y) \subseteq L$. Therefore $\sqrt{(L:M)} \supseteq \sqrt{(\xi(Y):M)}$. Thus $L \in \nu(\xi(Y))$ and so $Y \subseteq \nu(\xi(Y))$. Next, let $\nu(N)$ be any closed subset of q.Spec(M) containing Y. Then $\sqrt{(L:M)} \supseteq \sqrt{(N:M)}$ for every $L \in Y$ so that $\sqrt{(\xi(Y):M)} \supseteq \sqrt{(N:M)}$. Hence, for every $L' \in \nu(\xi(Y))$; $\sqrt{(L':M)} \supseteq \sqrt{(\xi(Y):M)} \supseteq \sqrt{(N:M)}$. Then $\nu(\xi(Y)) \subseteq \nu(N)$. Thus $\nu(\xi(Y))$ is the smallest closed subset of q.Spec(M) containing Y , hence $\nu(\xi(Y)) = cl(Y)$.

(2) is trivial by (1).

(3). Suppose that $\{Q\}$ is closed. Then $\{Q\} = \nu(Q)$ by (1). Let $N \in \text{q.Spec}(M)$ such that $\sqrt{(N:M)} \supseteq p = \sqrt{(Q:M)}$. Hence, $N \in \nu(Q) = \{Q\}$, and so $\text{q.Spec}_p(M) = \{Q\}$. Conversely, assume that (i) and (ii) hold. Let $N \in cl(\{Q\})$. Hence by (1),

$\sqrt{(N:M)} \supseteq \sqrt{(Q:M)}$. Thus by (i), $\sqrt{(N:M)} = \sqrt{(Q:M)} = p$ and therefore $Q = N$ by (ii). This yields $cl(\{Q\}) = \{Q\}$.

(4). Suppose $Q' \in$ q.Spec(M) such that $Q' \supseteq Q$. Then $\sqrt{(Q':M)} \supseteq \sqrt{(Q:M)}$. i.e., $Q' \in \nu(Q) = cl(\{Q\}) = \{Q\}$. Hence, $Q' = Q$, and so Q is a maximal element of q.Spec(M).

(5). The result follows from the part (1).

(6). The result is easy to check from the parts (3), (5).

(7). The sufficiency is trivial by part (4). Conversely, suppose $Q, N \in$ q.Spec(M) such that $Q \in cl(\{N\}) = \nu(N)$. Thus $\sqrt{(Q:M)} \supseteq \sqrt{(N:M)}$. Since Q satisfies the primeful property, $\sqrt{(Q:M)}$ is a proper ideal of R and hence by maximality of N we have $\sqrt{(Q:M)} = \sqrt{(N:M)}$; i.e. $\nu(Q) = \nu(N)$. Now, by Theorem 2.1, we conclude that $Q = N$. Thus $cl(\{N\}) = \{N\}$; i.e. every singleton subset of q.Spec(M) is closed. So, q.Spec(M) is a T_1-space.

(8). Use part (7). □

Example 3.1. *Consider the* $\mathbb{Z}$*-module* $M = \prod_p \mathbb{Z}/p\mathbb{Z}$ *where* p *runs through the set* Ω *of all prime integers of* $\mathbb{Z}$*. We claim that* q.Spec$(M) = \{pM \mid p \in \Omega\}$*. Let* $p \in \Omega$*. By* [11, Example 1(3) p. 136]*,* pM *is a* p*-prime submodule of* M *and hence by* [11, Proposition 4.5] pM *satisfies the primeful property. Thus* $\{pM \mid p \in \Omega\} \subseteq$ q.Spec(M)*. For the reverse inclusion, let* $Q \in$ q.Spec(M)*. By the argument in the Example* [5, Example 3.1]*,* $\sqrt{(Q:M)}$ *is a nonzero prime ideal of* $\mathbb{Z}$*. Take* $\sqrt{(Q:M)} = p\mathbb{Z}$*. So* $p\mathbb{Z} = \sqrt{(Q:M)} = (\text{rad}Q : M)$ *implies that* rad$Q$ *is a prime submodule of* M*. Thus* rad$Q = pM$*. Since the ring of integers is Noetherian, there is* $n \in \mathbb{N}$ *such that* $p^n = (\sqrt{(Q:M)})^n \subseteq (Q:M)$*. Hence* $p^nM \subseteq Q \subseteq pM$*. It is easy to see that* $p^nM = pM$ *and so* $Q = pM$*. Now by Proposition 3.2(3),* q.Spec(M) *is a* T_1*-space.*

Theorem 3.2. *Let* M *be a finitely generated* R*-module. The following statements are equivalent:*

(1) q.Spec(M) *is a* T_1*-space;*

(2) q.Spec(M) *is a* T_0*-space and* q.Spec$(M) = $ Max(M)*;*

(3) M *is a multiplication module and* q.Spec$(M) = $ Max(M)*.*

Proof. (1) $\Rightarrow$ (2). Since M is finitely generated, every submodule of M satisfies the primeful property by [11, Theorem 2.2]. Thus Max$(M) \subseteq$ q.Spec(M). The reverse inclusion is obtained by using Proposition 3.2(7) and the fact that every proper submodule, in particular every quasi-primary submodule, of a finitely generated module is contained in a maximal submodule.

(2) $\Rightarrow$ (1) is clear by Proposition 3.2(7).

(2) $\Rightarrow$ (3). By [11, Theorem 2.2], we may assume that Spec(M) is a subspace of q.Spec(M) and hence $\mid \text{Spec}_p(M) \mid \leq 1$ for every prime ideal p of R, by Proposition 3.2(5). Now, it follows from [15, Theorem 3.5] that M is multiplication.

(3) $\Rightarrow$ (2). Suppose M is a multiplication module and q.Spec$(M) = $ Max(M). Thus every quasi-primary submodule of M is of the form pM for some maximal ideal p of R, by [4, Theorem 2.5(ii)]. Now, let $\nu(pM) = \nu(p'M)$ for some $pM, p'M \in$ q.Spec(M). Hence $\sqrt{(pM : M)} = \sqrt{(p'M : M)}$. It implies that $(\text{rad}(pM) : M) = (\text{rad}(p'M) : M)$ and so $\text{rad}(pM) = \text{rad}(p'M)$. Since pM and $p'M$ are prime, we have $pM = p'M$. Thus q.Spec(M) is a T_0-space by Proposition 3.2(5). □

Corollary 3.1. *Let M be an R-module.*

(1) *Let R be a domain. If* q.Spec(R) *is a* T_1*-space, then R is a field.*

(2) *If M is Noetherian and* q.Spec(M) *is a* T_1*-space, then M is Artinian cyclic.*

Proof. (1). Since R is a domain, $(0) \in$ q.Spec(R). But by Theorem 3.2, we have q.Spec$(R) = $ Max(R). Thus, R is a field.

(2). By Theorem 3.2, M is multiplication and every quasi-primary submodule and hence every prime submodule of M is maximal. By [2, Theorem 4.9], M is Artinian and the result follows from [4, Corollary 2.9]. □

A topological space X is called irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in X intersect. A subset A of a topological space X is irreducible if for every pair of closed subsets A_i $(i = 1, 2)$ of X with $A \subseteq A_1 \cup A_2$, we have $A \subseteq A_1$ or $A \subseteq A_2$. An irreducible component of a topological space A is a maximal irreducible subset of X. A singleton subset and its closure in q.Spec(M) are both irreducible. Now, we can apply Proposition 3.2(1) to achieve the following result:

Lemma 3.4. *$\nu(Q)$ is an irreducible closed subset of* q.Spec(M) *for every quasi-primary submodule Q of M satisfying the primeful property.*

As we mentioned before, it is easily seen that if Q is a quasi-primary submodule of M satisfying the primeful property, then $(Q : M)$ is a quasi-primary ideal of R. The converse is also true when M is a multiplication module. Indeed if $(Q : M)$ is a quasi-primary ideal of R, then $p = \sqrt{(Q : M)} = (\text{rad}Q : M)$ is a prime ideal of R. Thus by [4, Corollary 2.11], rad$Q$ is a prime submodule and so Q is a quasi-primary submodule of M. Using this fact, some assertions will be proved in the following.

Theorem 3.3. *Let M be an R-module and $Y \subseteq$* q.Spec(M)*. If $\xi(Y)$ is a quasi-primary submodule of M, then Y is an irreducible space. The converse is true, if M is a multiplication module and $\xi(Y)$ satisfies the primeful property.*

Proof. Suppose $\xi(Y)$ is a quasi-primary submodule of M. Let $Y \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are two closed subsets of q.Spec(M). Then there exist two submodules N and K of M such that $Y_1 = \nu(N)$ and $Y_2 = \nu(K)$. Thus, $Y \subseteq \nu(N) \cup \nu(K) = \nu(N \cap K)$ and so by Lemma 3.2(6), $\sqrt{((N \cap K) : M)} \subseteq \sqrt{(\xi(Y) : M)}$. Since $\sqrt{(\xi(Y) : M)}$ is a prime ideal, either $\sqrt{(N : M)} \subseteq \sqrt{(\xi(Y) : M)}$ or $\sqrt{(K : M)} \subseteq \sqrt{(\xi(Y) : M)}$. Again by using Lemma 3.2(6), either $Y \subseteq \nu(N) = Y_1$ or $Y \subseteq \nu(K) = Y_2$. Thus we conclude that Y is irreducible. Conversely, assume that M is a multiplication module and Y is an irreducible space. By the above argument, it suffices to show that $(\xi(Y) : M)$ is a quasi-primary ideal of R. Let $ab \in (\xi(Y):M)$ for some $a, b \in R$. Suppose, on the contrary, that $Ra \nsubseteq \sqrt{(\xi(Y):M)}$ and $Rb \nsubseteq \sqrt{(\xi(Y):M)}$. Then $\sqrt{(RaM:M)} \nsubseteq \sqrt{(\xi(Y):M)}$ and $\sqrt{(RbM:M)} \nsubseteq \sqrt{(\xi(Y):M)}$. By Lemma 3.2(6), $Y \nsubseteq \nu(RaM)$ and $Y \nsubseteq \nu(RbM)$. Let $Q \in Y$. Then $\sqrt{(Q:M)} \supseteq \sqrt{(\xi(Y):M)} \supseteq Rab$. This means that either $RaM \subseteq \sqrt{(Q:M)}M$ or $RbM \subseteq \sqrt{(Q:M)}M$. So, by Lemma 3.2(1),(3), either $\nu(Q) \subseteq \nu(RaM)$ or $\nu(Q) \subseteq \nu(RbM)$. Therefore, $Y \subseteq \nu(RaM) \cup \nu(RbM)$ and hence $Y \subseteq \nu(RaM)$ or $Y \subseteq \nu(RbM)$ as Y is irreducible. It is a contradiction. $\square$

Corollary 3.2. *Let M be a multiplication R-module.*

(1) *If M is finitely generated and N is a submodule of M. Then $V(N)$ is irreducible if and only if $N \in$ q.Spec(M).*

(2) *Let R be a domain, M be a faithful module and $\xi($q.Spec$(M))$ satisfies the primeful property. Then* q.Spec(M) *is irreducible.*

Proof. (1). It is clear that $\text{rad}(N) = \xi(V(N)) \neq M$. Since M is finitely generated, [11, Theorem 2.2] follows that every proper submodule of M satisfies the primeful property and hence we have $V(N) \subseteq$ q.Spec(M). Now by Theorem 3.3, $V(N)$ is an irreducible space if and only if rad$N \in$ q.Spec(M). On the other hand, by the argument before Theorem 3.3, rad$N \in$ q.Spec(M) if and only if $N \in$ q.Spec(M).
(2). Since (0) is a prime ideal of R, we have $\text{rad}(\mathbf{0}) = \text{rad}(\mathbf{0}M) = \sqrt{(0)}M = \mathbf{0}$ by [4, Theorem 2.12]. Now, $(\xi(\text{q.Spec}(M)) : M) \subseteq (\xi(\text{Spec}(M)) : M) = (\bigcap_{P \in \text{Spec}(M)} P : M) = (\mathbf{0} : M) = (0)$. Thus $\xi(\text{q.Spec}(M))$ is a quasi-primary submodule of M and hence the result follows from Theorem 3.3. $\square$

Let Y be a closed subset of a topological space. An element $y \in Y$ is said to be a generic point of Y if $Y = cl(\{y\})$. Proposition 3.2(1) follows that every element Q of q.Spec(M) is a generic point of the irreducible closed subset $\nu(Q)$ of q.Spec(M). Note that a generic point of a closed subset Y of a topological space is unique if the topological space is a T_0-space.

Theorem 3.4. *Let M be a quasi-primaryful R-module and $Y \subseteq \text{q.Spec}(M)$.*

(1) *Y is an irreducible closed subset of* $\text{q.Spec}(M)$ *if and only if $Y = \nu(Q)$ for some $Q \in \text{q.Spec}(M)$. In particular every irreducible closed subset of* $\text{q.Spec}(M)$ *has a generic point.*

(2) *The set of all irreducible components of* $\text{q.Spec}(M)$ *is of the form*

$$T = \{\nu(\sqrt{q}M) \mid q \in V^{\mathbf{q}}(\text{Ann}(M)) \textit{ and } \sqrt{q} \textit{ is a minimal element of } V(\text{Ann}(M)) \textit{ with respect to inclusion}\}.$$

(3) *Let R be a Laskerian ring and M be a nonzero R-module. Then* $\text{q.Spec}(M)$ *has finitely many irreducible components.*

Proof. By Lemma 3.4, $Y = \nu(Q)$ is an irreducible closed subset of $\text{q.Spec}(M)$ for some $Q \in \text{q.Spec}(M)$. Conversely, let Y be an irreducible space. Hence $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}(Y) = Y'$ is an irreducible subset of $\text{Spec}(\overline{R})$ because $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is continuous by Proposition 3.1(4). It follows from [3, P. 129, Proposition 14] that $\xi(Y') = \overline{\sqrt{(\xi(Y) : M)}}$ is a prime ideal of $\overline{R}$. Therefore $\sqrt{(\xi(Y) : M)}$ is a prime ideal of R. Since the map $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ is surjective, there exists $Q \in \text{q.Spec}(M)$ such that $\sqrt{(Q : M)} = \sqrt{(\xi(Y) : M)}$. Since Y is closed, there exists a submodule N of M such that $Y = \nu(N)$. It means that $\sqrt{(\xi(\nu(N)) : M)} = \sqrt{(Q : M)}$ and hence $\nu(\xi(Y)) = \nu(\xi(\nu(N))) = \nu(Q)$ by Lemma 3.2(6). Thus $Y = \nu(Q)$ by Proposition 3.2(1).

(2). Suppose Y is an irreducible component of $\text{q.Spec}(M)$. By part (1), $Y = \nu(Q)$ for some $Q \in \text{q.Spec}(M)$. Hence, $Y = \nu(Q) = \nu(\sqrt{(Q : M)}M)$ by Lemma 3.2(3). Let $q = (Q : M)$. Now, it suffices to show that $\sqrt{q}$ is a minimal element of $V(\text{Ann}(M))$ with respect to inclusion. To see this let $q' \in V(\text{Ann}(M))$ and $q' \subseteq \sqrt{q}$. Then there exists an element $Q' \in \text{q.Spec}(M)$ such that $\sqrt{(Q' : M)} = q'$ because M is quasi-primaryful. So, $Y = \nu(Q) \subseteq \nu(Q')$. Hence, $Y = \nu(Q) = \nu(Q')$ due to the maximality of $\nu(Q)$. It implies that $\sqrt{q} = q'$. Conversely, let $Y \in T$. Then there exists $q \in V^{\mathbf{q}}(\text{Ann}(M))$ such that $\sqrt{q}$ is a minimal element in $V(\text{Ann}(M))$ and

$Y = \nu(\sqrt{q}M)$. Since M is quasi-primaryful, there exists an element $Q \in \text{q.Spec}(M)$ such that $\sqrt{(Q:M)} = \sqrt{q}$. So, $Y = \nu(\sqrt{q}M) = \nu(\sqrt{(Q:M)}M) = \nu(Q)$, and so Y is irreducible by part (1). Suppose that $Y = \nu(Q) \subseteq \nu(Q')$, where $Q' \in \text{q.Spec}(M)$. Since $Q \in \nu(Q')$ and $\sqrt{q}$ is minimal, it follows that $\sqrt{(Q:M)} = \sqrt{(Q':M)}$. Now, by Lemma 3.2(3), we have

$$Y = \nu(Q) = \nu(\sqrt{(Q:M)}M) = \nu(\sqrt{(Q':M)}M) = \nu(Q').$$

(3). Suppose $q \in V^{\mathbf{q}}(\text{Ann}(M))$ and $\sqrt{q}$ is a minimal element of $V(\text{Ann}(M))$. Let $\text{Ann}(M) = \bigcap\limits_{i=1}^{t} q_i$ be a minimal primary decomposition of $\text{Ann}(M)$. Then $\sqrt{q_i} \subseteq \sqrt{q}$ for some $1 \leq i \leq t$, since $\sqrt{q}$ is prime. By minimality of $\sqrt{q}$, we get $\sqrt{q} = \sqrt{q_i}$. Therefore, irreducible components of $\text{q.Spec}(M)$ are of the form $\nu(\sqrt{q_i}M)$, by part (2). $\square$

For any submodule N of M, we define $\Lambda_M(N) = \text{q.Spec}(M) - \nu(N)$ as an open set of $\text{q.Spec}(M)$. Also, $\Lambda_M(a) = \Lambda_M(aM)$ for any $a \in R$. Clearly, $\Lambda_M(0) = \emptyset$ and $\Lambda_M(1) = \text{q.Spec}(M)$. The following result shows that the set $B = \{\Lambda_M(a) \mid a \in R\}$ is a base for the quasi-Zariski topology on $\text{q.Spec}(M)$.

Theorem 3.5. *Let M be an R-module. The set $B = \{\Lambda_M(a) \mid a \in R\}$ forms a base for the quasi-Zariski topology on* $\text{q.Spec}(M)$.

Proof. We may assume that $\text{q.Spec}(M) \neq \emptyset$. We will show that every open subset of $\text{q.Spec}(M)$ is a union of members of B. Let O be an open subset in $\text{q.Spec}(M)$.

Thus $O = \text{q.Spec}(M) - \nu(N)$ for some submodule N of M. Therefore

$$\begin{aligned} O &= \text{q.Spec}(M) - \nu(N) = \text{q.Spec}(M) - \nu(\sqrt{(N:M)}M) \\ &= \text{q.Spec}(M) - \nu(\sum_{a\in\sqrt{(N:M)}} aM) \\ &= \text{q.Spec}(M) - \nu(\sum_{a\in\sqrt{(N:M)}} (aM:M)M) \\ &= \text{q.Spec}(M) - \bigcap_{a\in\sqrt{(N:M)}} \nu(aM) \\ &= \bigcup_{a\in\sqrt{(N:M)}} \Lambda_M(a) \end{aligned}$$

□

Theorem 3.6. *Let R be a ring and $a, b \in R$.*

1. *$\Lambda_R(a) = \emptyset$ if and only if a is a nilpotent element of R.*
2. *$\Lambda_R(a) = \text{q.Spec}(R)$ if and only if a is a unit element of R.*
3. *For each pair of ideals I and J of R, $\Lambda_R(I) = \Lambda_R(J)$ if and only if $\sqrt{I} = \sqrt{J}$.*
4. *$\Lambda_R(ab) = \Lambda_R(a) \cap \Lambda_R(b)$.*
5. *$\text{q.Spec}(R)$ is quasi-compact.*
6. *$\text{q.Spec}(R)$ is a T_0-space.*

Proof. (1). Let $a \in R$. Then

$$\begin{aligned} \emptyset &= \Lambda_R(a) = \text{q.Spec}(R) - V^{\mathfrak{q}}(Ra) \\ &\Leftrightarrow V^{\mathfrak{q}}(Ra) = \text{q.Spec}(R) \\ &\Leftrightarrow \sqrt{q} \supseteq Ra \ for\ every\ q \in \text{q.Spec}(R) \\ &\Leftrightarrow a\ is\ in\ every\ prime\ ideal\ of\ R \\ &\Leftrightarrow a\ is\ a\ nilpotent\ element\ of\ R. \end{aligned}$$

(2). Let $a \in R$. Then

$$\begin{aligned} \Lambda_R(a) = \text{q.Spec}(R) &\Leftrightarrow a \notin \sqrt{q} \ for\ all\ q \in \text{q.Spec}(R) \\ &\Rightarrow a \notin q \ for\ all\ q \in \text{Max}(R) \\ &\Rightarrow a \ is\ unit. \end{aligned}$$

Conversely, if a is a unit, then clearly a is not in any quasi-primary ideal. That is, $\Lambda_R(a) = \text{q.Spec}(R)$.

(3) Suppose that $\Lambda_R(I) = \Lambda_R(J)$. Let p be a prime ideal of R containing I. Since p is a quasi-primary ideal of R and $p \supseteq \sqrt{I}$, we have $p \in \nu(I)$. Thus, by assumption, $p \supseteq \sqrt{J} \supseteq J$ and so every prime ideal of R containing I is also a prime ideal of R containing J, and vice versa. Therefore $\sqrt{I} = \sqrt{J}$. The converse is trivialy true.

(4). To prove (4), it suffices to show that $\nu(Rab) = \nu(Ra) \cup \nu(Rb)$. Let $q \in \nu(Rab)$. Then

$$\begin{aligned} \sqrt{q} \supseteq \sqrt{Rab} = \sqrt{Ra} \cap \sqrt{Rb} &\Leftrightarrow (\sqrt{q} \supseteq \sqrt{Ra} \ or\ \sqrt{q} \supseteq \sqrt{Rb}) \\ &\Leftrightarrow (q \in \nu(Ra) \ or\ q \in \nu(Rb)) \\ &\Leftrightarrow q \in \nu(Ra) \cup \nu(Rb). \end{aligned}$$

(5). Let $\text{q.Spec}(R) = \underset{i \in I}{\cup} \Lambda_R(J_i)$, where $\{J_i\}_{i \in I}$ is a family of ideals of R. We clearly have $\Lambda_R(R) = \text{q.Spec}(R) = \Lambda_R(\sum_{i \in I} J_i)$. Thus, by part (3), we have $R = \sqrt{\sum_{i \in I} J_i}$ and hence, $1 \in \sum_{i \in I} J_i$. So there are $i_1, i_2, \cdots, i_n \in I$ such that $1 \in \sum_{k=1}^{n} J_{i_k}$, that is $R = \sum_{k=1}^{n} J_{i_k}$. Consequently $\text{q.Spec}(R) = \Lambda_R(R) = \Lambda_R(\sum_{k=1}^{n} J_{i_k}) = \overset{n}{\underset{k=1}{\cup}} \Lambda_R(J_{i_k})$.

(6). Let q_1, q_2 be two distinct points of $\text{q.Spec}(R)$. If $q_1 \nsubseteq q_2$, then obviously $q_2 \in \Lambda_R(q_1)$ and $q_1 \notin \Lambda_R(q_1)$. □

Proposition 3.3. *Let M be an R-module and $a, b \in R$.*

(1) $(\psi^{\mathbf{q}})^{-1}(\Lambda_{\overline{R}}(\overline{a})) = \Lambda_M(a)$.

(2) $\psi^{\mathbf{q}}(\Lambda_M(a)) \subseteq \Lambda_{\overline{R}}(\overline{a})$ *and the equality holds if* $\psi^{\mathbf{q}}$ *is surjective.*

(3) $\Lambda_M(ab) = \Lambda_M(a) \cap \Lambda_M(b)$.

(4) *If* $\psi^{\mathbf{q}}$ *is surjective, then the open set* $\Lambda_M(Ra)$ *in* q.Spec(M) *is quasi-compact. In particular, the space* q.Spec(M) *is quasi-compact.*

Proof. (1). Since $\psi^{\mathbf{q}}$ is continuous, by Proposition 3.1(3), we have

$$\begin{aligned}(\psi^{\mathbf{q}})^{-1}(\Lambda_{\overline{R}}(\overline{a})) &= (\psi^{\mathbf{q}})^{-1}(\text{q.Spec}(\overline{R}) - \nu(\overline{a}\overline{R})) \\ &= \text{q.Spec}(M) - (\psi^{\mathbf{q}})^{-1}(\nu(\overline{a}\overline{R})) \\ &= \text{q.Spec}(M) - \nu(aM) \\ &= \Lambda_M(a).\end{aligned}$$

(2) follows immediately from part (1).

(3). Let $a, b \in R$. Then

$$\begin{aligned}\Lambda_M(ab) &= (\psi^{\mathbf{q}})^{-1}(\Lambda_{\overline{R}}(\overline{ab})) \text{ by part } (1) \\ &= (\psi^{\mathbf{q}})^{-1}(\Lambda_{\overline{R}}(\overline{a}) \cap \Lambda_{\overline{R}}(\overline{a})) \text{ by Theorem } 3.6(4) \\ &= (\psi^{\mathbf{q}})^{-1}(\Lambda_{\overline{R}}(\overline{a})) \cap (\psi^{\mathbf{q}})^{-1}(\Lambda_{\overline{R}}(\overline{a})) \\ &= \Lambda_M(a) \cap \Lambda_M(a).\end{aligned}$$

(4). Since $B = \{\Lambda_M(a) \mid a \in R\}$ forms a base for the quasi-Zariski topology on q.Spec(M) by Theorem 3.5, for any open cover of $\Lambda_M(a)$, there is a family $\{a_i \in R \mid i \in I\}$ of elements of R such that $\Lambda_M(a) \subseteq \underset{i\in I}{\cup}\Lambda_M(a_i)$. By part (2), $\Lambda_{\overline{R}}(\overline{a}) = \psi^{\mathbf{q}}(\Lambda_M(a)) \subseteq \underset{i\in I}{\cup}\psi^{\mathbf{q}}(\Lambda_M(a_i)) = \underset{i\in I}{\cup}\Lambda_{\overline{R}}(\overline{a_i})$. It follows that there exists a finite subset I' of I such that $\Lambda_{\overline{R}}(\overline{a}) \subseteq \underset{i\in I'}{\cup}\Lambda_{\overline{R}}(\overline{a_i})$ as $\Lambda_{\overline{R}}(\overline{a})$ is quasi-compact, since $\phi^{\mathbf{R}}$ is surjective, whence $\Lambda_M(a) = (\psi^{\mathbf{q}})^{-1}(\Lambda_{\overline{R}}(\overline{a})) \subseteq \underset{i\in I'}{\cup}\Lambda_M(a_i)$ by part (1). □

Theorem 3.7. *Let M be an R-module. If the map $\psi^{\mathfrak{q}}$ is surjective, then the quasi-compact open sets of* q.Spec(M) *are closed under finite intersection and form an open base.*

Proof. It suffices to show that the intersection $C = C_1 \cap C_2$ of two quasi-compact open sets C_1 and C_2 of q.Spec(M) is a quasi-compact set. Each C_j , $j = 1$ or 2, is a finite union of members of the open base $B = \{\Lambda_M(a) \mid a \in R\}$, hence so is C due to Proposition 3.3. Put $C = \bigcup_{i=1}^{n} \Lambda_M(a_i)$ and let Ω be any open cover of C. Then Ω also covers each $\Lambda_M(a_i)$ which is quasi-compact by Proposition 3.3 (4). Hence, each $\Lambda_M(a_i)$ has a finite subcover of Ω and so does C. The other part of the theorem is trivially true due to the existence of the open base B. □

Following [10], we say that a topological space W is a spectral space in case W is homeomorphic to Spec(S), with the Zariski topology, for some ring S. Spectral spaces have been characterized by Hochster [10, p.52, Proposition 4] as the topological spaces W which satisfy the following conditions:

(1) W is a T_0-space;

(2) W is quasi-compact;

(3) The quasi-compact open subsets of W are closed under finite intersection and form an open base;

(4) Each irreducible closed subset of W has a generic point.

In the end of this paper, we observe q.Spec(M) from the point of view of spectral topological spaces; we will follow the above mentioned Hochster's characterization closely.

The next theorem is obtained by combining Proposition 3.3 (4), Theorem 3.7, and Theorem 3.4 (1).

Theorem 3.8. *Let M be an R-module and the map $\psi^{\mathbf{q}}$ be surjective. Then* q.Spec(M) *fulfills the above conditions* (2), (3), *and* (4), *namely,* q.Spec(M) *satisfies all the conditions to be a spectral space but possibly condition* (1).

Theorem 3.9. *Let M be an R-module and the map $\psi^{\mathbf{q}}$ be surjective. Then the following statements are equivalent:*

(1) q.Spec(M) *is a spectral space;*
(2) q.Spec(M) *is a T_0-space;*
(3) $\phi^{\mathbf{R}} o \psi^{\mathbf{q}}$ *is injective;*
(4) *If $\nu(N) = \nu(K)$, then $N = K$, for any $N, K \in$* q.Spec(M)*;*
(5) $\mid$ q.Spec$_p(M) \mid \leq 1$ *for every $q \in V^{\mathbf{q}}(\mathrm{Ann}(M))$ with $\sqrt{q} = p$;*
(6) $\phi^{\mathbf{M}}$ *is injective.*

Proof. (1) $\Rightarrow$ (2) is trivial and (2) $\Rightarrow$ (1) holds by Theorem 3.8. The equivalence of (2) $-$ (6) is due to Proposition 3.2 (5). □

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