BEST APPROXIMATION, FIXED POINTS AND INVARIANT APPROXIMATION IN LINEAR METRIC SPACES

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ABSTRACT. If G is a nonempty subset of a metric space (X, d) and $x \in X$, a point $g_0 \in G$ is called a best approximation to x in G if $d(x, g_0) = dist(x, G) \equiv \inf\{d(x, g) : g \in G\}$. The set of all best approximations to x in G is denoted by $P_G(x)$. The set-valued map P_G is called a best approximation map. This paper shows how the best approximation map helps in finding fixed points of certain mappings in linear metric spaces. The results proved in the paper generalize and extend several known results on the subject.

1. BASIC CONCEPTS

Suppose G is a nonempty subset of a metric space (X, d) and $x \in X$ be arbitrary. An element $g_0 \in G$ satisfying

$$d(x, g_0) = inf\{d(x, g) : g \in G\} \equiv d(x, G)$$

is called a **nearest point** or a **best approximation** to x in G. The set of all such $g_0 \in G$, denoted by $P_G(x)$, is called the set of best approximations to x in G and the set-valued map $P_G : X \to G$ is called a **nearest point map** or a **best approximation map** or **a metric projection**. The set G is said to be **proximinal**

if $P_G(x)$ is nonempty for each $x \in X$. The set G is called **Chebyshev** if $P_G(x)$ is exactly a singleton for each $x \in X$. For Chebyshev sets G, P_G is a single-valued map.

The set G is said to be **approximatively compact** [7] if for every $x \in X$ and every sequence $< g_n >$ in G satisfying $\lim_{n \to \infty} d(x, g_n) = d(x, G)$, there exists a subsequence $< g_{n_i} >$ converging to an element of G.

A mapping $T : X \to X$ is said to be **nonexpansive** if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$.

A point $x \in X$ is called a **fixed point** of a mapping $T : X \to X$ if $Tx = x$.

A metric space (X, d) is called a **linear metric space** if (i) X is a linear space (ii) addition and scalar multiplication are continuous, and (iii) d is translation invariant i.e. $d(x + z, y + z) = d(x, y)$ for all $x, y \in X$.

A linear metric space (X, d) is said to satisfy **property(*)** (see [12]) if

$$d(\lambda x + (1 - \lambda)y, z) \leq \lambda d(x, z) + (1 - \lambda)d(y, z)$$

for all $x, y, z \in X$ and $0 \leq \lambda \leq 1$.

Clearly, every normed linear space satisfies property (*).

A linear metric space (X, d) is said to be **Uniformly convex** [1] if there corresponds to each pair of positive numbers (ϵ, r) a positive number δ such that if x and y lie in X with $d(x, y) \geq \epsilon, d(x, r) < r + \delta, d(y, r) < r + \delta$, then $d(\frac{x+y}{2}, 0) < r$.

A subset K of a linear space X is said to be a **convex set** if $\lambda x + (1 - \lambda)y \in K$ for all $x, y \in K$ and $0 \leq \lambda \leq 1$.

A topological vector space (X, d) is said to be **locally convex** if it has a base of convex neighbourhoods.

A mapping $T : X \to X$, where X is a topological vector space, is said to be **convex continuous** [5] at $x_0 \in X$ if for any open neighbourhood $N(Tx_0)$ of Tx_0, there exists an open neighbourhood $V(x_0)$ of x_0 such that $conv(TV(x_0)) \subset N(Tx_0)$, where $conv(TV(x_0))$ represents the convex hull of $TV(x_0)$.

It is known (see[5]) that if X is locally convex topological vector space then $T : X \to X$ is convex continuous if and only if T is continuous. Moreover, if C is a compact convex subset of a linear metric space (X, d) and $T : C \to C$ is convex continuous then T has a fixed point (see[5]).

Let T be a point-valued and f a set-valued mappings on a topological space X. We say, as in the sense of Itoh and Takahashi [9], that T and f **commute** if $T[f(x)] \subseteq f[T(x)]$ for each $x \in X$.

2. Main Results

Using fixed point theory, Brosowski [2] proved the following theorem on invariant approximation (points of best approximation which are also fixed points of a mapping).

Theorem 2.1. *Let T be a linear and non-expansive mapping on a normed linear space E. Let G be a T invariant (i.e. $T(G) \subset G$) subset of E and x a T-invariant point (i.e $x = Tx$). If the set $P_G(x)$ is nonempty compact and convex, then it contains a T-invariant point.*

Subsequently, various generalizations of Brosowski's result appeared in the literature (see [4],[13],[15] and [17]).

The strength of the following theorem and subsequent corollaries lie in the fact that the nonexpansiveness and the linearity of T may be dropped from Brosowski's theorem if we assume that T commutes with P_G.

Theorem 2.2. *Let T be a convex continuous mapping on a linear metric space (X, d), G a subset of X invariant under T and p a fixed point of T. If $P_G(p)$ is nonempty compact and convex and if T commutes with P_G, then there is a fixed point in G for T which is an element of best approximation of p in G.*

Proof. Since $T[P_G(p)] \subset P_G[T(p)]$, T maps $P_G(p)$ into itself. Since $P_G(p)$ is a compact convex subset of a linear metric space and T is convex continuous, T has a fixed point in $P_G(p)$ [5]. □

Since for a locally convex space X, a mapping $T : X \to X$ is convex continuous if and only if T is continuous, we have

Corollary 2.1. [12] *Let T be a continuous mapping on a locally convex linear metric space (X, d), G a subset of X invariant under T and p a fixed point of T. If $P_G(p)$ is nonempty compact and convex, and T commutes with P_G, then there is a fixed point in G for T which is an element of best approximation to p in G.*

Corollary 2.2. *Let (X, d) be a locally convex linear metric space, G an approximatively compact subset of X. Let T be a continuous mapping which has a fixed point p in X. If $P_G(p)$ is convex and T commutes with P_G, then T has a fixed point in G which is also a best approximation of p in G.*

Proof. Since G is approximatively compact, $P_G(p)$ is nonempty[7] and compact [13]. The result follows. □

Corollary 2.3. *Let (X, d) be a locally convex linear metric space satisfying property (*), G an approximatively compact convex subset of X. Let T be a continuous mapping which has a fixed point p in X. If T commutes with P_G, then T has a fixed point in G which is also a best approximation of p in G.*

Proof. Since G is approximatively compact, $P_G(p)$ is non-empty [7] and compact [13]. Since G is a convex set and the space has property(*), $P_G(p)$ is a convex set [15]. The result follows. □

Corollary 2.4. *Let (X, d) be a locally convex linear metric space which is uniformly convex, G a complete convex subset of X. Let T be a continuous mapping which has*

a fixed point p in X. If T commutes with P_G, then T has a unique fixed point in G which is also a unique element of best approximation of p in G.

Proof. Since G is a complete convex subset of a uniformly convex linear metric space (X, d), G is approximatively compact and Chebyshev [1]. Since G is Chebyshev, $P_G(p)$ is a singleton and so convex. The result now follows from Corollary 2.2. $\square$

We have the following version of Brosowski's theorem.

Theorem 2.3. *Let (X, d) be a locally convex linear metric space satisfying property (*) and G an approximatively compact T-invariant convex subset of X. If T is a non-expansive mapping which has a fixed point in X, then T has a fixed point in G which is also an element of best approximation of p in G.*

Proof. Since T is nonexpansive, for $x \in P_G(p)$,

$$d(Tx, p) = d(Tx, Tp) \le d(x, p) = d(x, G),$$

This shows that T maps $P_G(p)$ into $P_G(p)$.

Since G is approximatively compact, $P_G(p)$ is nonempty and compact. Since G is a convex subset of the linear metric space satisfying property (*), $P_G(p)$ is convex. Since T is non-expansive, T is continuous on the compact convex subset of the locally convex linear metric space and hence T has a fixed point in $P_G(p)$ (Schauder fixed point theorem, see [6]). $\square$

Remarks: 1 Since in a linear metric space satisfying property (*), $P_G(p) \subset \partial G \cap G$ (see [15]), the condition that G is invariant under T in Theorem 2.3 can be weakened to $T : \partial G \to G$ as the only use made of $T : G \to G$ is to prove $T : P_G(p) \to P_G(p)$ (Here ∂G is the boundary of G).

2. In the proof of Theorem 2.3, the non-expansiveness of T is required only on $P_G(p) \cup \{p\}$.

3. Theorem 2.2, Corollaries 2.1, 2.2, 2.4 and Theorem 2.3 have been proved in [10] when the underlying spaces are normed linear spaces.

Let C be a nonempty subset of a metric space (X, d) and $f : C \to X$. Consider the problem of seeking a point $x \in C$ which is a best approximation for $f(x)$ i.e. find an $x \in C$ such that

$$d(f(x), x) = dist(f(x), C) \equiv inf\{d(f(x), y) : y \in C\} \tag{1}$$

It is easy to show that x is a solution of (1) if and only if x is a fixed point of the mapping Pof, where P is the metric projection onto C.

Suppose x is a fixed point of Pof i.e.

$$(Pof)(x) = x \ \ i.e. \ \ P[f(x)] = x$$

This gives $d(f(x), x) = d(f(x), C)$.

Conversely, suppose that x is a solution of (1) i.e. $d(f(x), C) = d(f(x), x)$. This gives

$$P[f(x)] = x \ \ i.e. \ \ (Pof)(x) = x$$

i.e. x is a fixed point of Pof.

It may be noted that if $f : C \to C$ then (1) implies $d(f(x), x) = 0$ i.e. $f(x) = x$ i.e. x is a fixed point of f.

The following well known Ky Fan's best approximation theorem (see [8], [9])has been of great importance in various fields, like nonlinear analysis, approximation theory, game theory, fixed point theory (see [3]).

Theorem 2.4. *Let C ba a nonempty compact convex subset of a normed linear space X and $f : C \to X$ a continuous function. Then there exists an $x \in C$ such that*

$$\| x - f(x) \| = d(f(x), C)$$

In case $f : C \to C$ then f has a fixed point.

Various generalizations of Ky Fan's theorem have appeared in the litearture (see [3], [11], [14] and [17]). The following theorem gives its generalization in locally convex linear metric spaces:

Theorem 2.5. *Let C be an approximatively compact convex subset of a locally convex linear metric space (X, d) satisfying property (*) and let $f : C \to X$ be continuous. If $f(C)$ is relatively compact, then there is a point $y \in C$ such that*

$$d(f(y), C) = d(f(y), y)$$

In particular, if $f(C) \subset C$ then y is a fixed point of f.

Proof. Consider $F : C \to C$ defined by

$$F(x) = P_C[f(x)]$$

Then F is u.s.c [16]. Also $F(x)$ is a nonempty compact [13] and convex [12] subset of C for each $x \in C$. If $f(C)$ is relatively compact, then $F(C)$ is also relatively compact because the image of a compact set under an u.s.c mapping with compact point images is compact. Therefore F has a fixed point $y \in C$ (see [14]) i.e. $F(y) = P_C[f(y)] = y$. Hence $d(f(y), y) = d(f(y), C)$. $\square$

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