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SOME CHARACTERIZATIONS OF CAMPANATO SPACES VIA THE COMMUTATOR OF FRACTIONAL INTEGRAL OPERATOR

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ABSTRACT. In this paper, we give some characterizations of Campanato spaces via the commutators or higher order commutators of fractional integral operator.

1. INTRODUCTION

In 1976, Coifman, Rochberg and Weiss [1] obtained the characterization of BMO spaces; they proved that the commutator T_b is bounded in $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$, where T is well-known Calderón-Zygmund singular integral operator .

In 1978, Janson [2] gave a characterization of Lipschitz space; he proved that T_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$, where $1 < p < q < \infty, 0 < \beta < 1, \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ and $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is the Lipschitz space with the equivalent norm

$$\begin{aligned} ||b||_{\mathrm{Lip}_\beta(\mathbb{R}^n)} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |b - b_Q| \\ &\approx \sup_Q \left(\frac{1}{|Q|^{1+\frac{q\beta}{n}}} \int_Q |b - b_Q|^q \right)^{\frac{1}{q}} \\ &\approx \sup_{x,h\in\mathbb{R}^n, h\neq 0} \frac{|b(x+h) - b(x)|}{|h|^\beta}. \end{aligned}$$

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where $b_Q = \frac{1}{|Q|}\int_Q b(x)dx$, Q denotes any cube contained in $\mathbb{R}^n$ and $|Q|$ is the Lebesgue measure of Q.

In 1997, Ding [3] proved that T_b is bounded from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{p,\beta}(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$, where $1 < p < \infty, -\frac{n}{p} \leq \beta < 0$ and $M^{p,\beta}(\mathbb{R}^n)$ is the classical Morrey space with norm

$$\|b\|_{M^{p,\beta}(\mathbb{R}^n)} = \sup_Q \|b\|_{M^{p,\beta}(Q)} = \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\left(\frac{1}{|Q|}\int_Q |b|^p\right)^{\frac{1}{p}}.$$

In 2013, Shi [4] obtained that T_b is bounded from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{q,\tilde{\beta}}(\mathbb{R}^n)$ if and only if $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ for some suitable conditions. He also proved that T_b is bounded from $M^{p_2,\beta_2}(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$ if and only if $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$ for some suitable conditions.

Let $-\frac{n}{p} \leq \beta < 1$ and $1 \leq p < \infty$. A locally intergrable function f is said to belong to Campanato spaces $C^{p,\beta}(\mathbb{R}^n)$ if

$$\|f\|_{C^{p,\beta}(\mathbb{R}^n)} = \sup_Q \|f\|_{C^{p,\beta}(Q)} := \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}\left(\frac{1}{|Q|}\int_Q |f - f_Q|^p\right)^{\frac{1}{p}} < \infty.$$

Shi [4] also proved that

$$C^{p,\beta}(\mathbb{R}^n)\begin{cases} = \mathrm{BMO}(\mathbb{R}^n), & for\ \beta = 0,\\ = \mathrm{Lip}_\beta(\mathbb{R}^n), & for\ 0 < \beta < 1,\\ \supset M^{p,\beta}(\mathbb{R}^n), & for\ -\frac{n}{p} \leq \beta < 0.\end{cases}$$

Shi [5] obtained some results about commutator of fractional integral.

In 1982, Chanillo [6] obtained that $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$, where $1 < p < q < \infty,\ 0 < \alpha < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

Let $0 < \alpha < n$ and $b \in L_{loc}(\mathbb{R}^n)$, m-order commutator , $[b, I_\alpha]^m$, generated by I_α and b is defined by

$$[b, I_\alpha]^m f(x) = \int_{\mathbb{R}^n} \frac{[b(x) - b(y)]^m}{|x - y|^{n-\alpha}} f(y)dy.$$

For $0 < \alpha < n$ and $f \in L_{loc}(\mathbb{R}^n)$, define M_α by

$$M_\alpha(f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|\le r} |f(x-y)|dy.$$

An equivalent definition of M_α is

$$M_\alpha(f)(x) = \sup_{Q_x} \frac{1}{|Q_x|^{1-\frac{\alpha}{n}}} \int_{Q_x} |f(y)|dy,$$

where the supremum is taken over all cubes Q_x in $\mathbb{R}^n$ with the center at x and with the sides parallel to the axes.

Lemma 1.1. [7] *Let* $1 < p < \frac{n}{\alpha}$, $0 < \alpha < n$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 < 1 + \frac{p\beta}{n} < \frac{p}{q}$, $-\frac{n}{p} \le \beta < 0$ *and* $\widetilde{\beta} = \alpha + \beta$. *Then the fractional integral operator*

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

is bounded from $M^{p,\beta}(\mathbb{R}^n)$ *to* $M^{q,\tilde{\beta}}(\mathbb{R}^n)$.

Lemma 1.2. [4] *Suppose* $Q_* \subset Q$ *and* $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$ *with* $1 < p_1 < \infty$ *and* $-\frac{n}{p_1} \le \beta_1 < 0$. *Then*

$$|b_{Q_*} - b_Q| \le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} |Q_*|^{\frac{\beta_1}{n}}.$$

2. MAIN RESULTS

Now we formulate our results as follows:

Theorem 2.1. *Let* $1 < p < \infty$, $0 < \alpha < \min\{1, \frac{n}{2}\}$, $-\frac{n}{p} \le \beta < 0$, $0 < 1 + \frac{p\beta}{n} < \frac{p}{q}$, $\frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{n}$ *and* $\widetilde{\beta} = 2\alpha + \beta$. *The following statements are equivalent:*

(1) $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

(2) $[b, I_\alpha]$ *is a bounded operator from* $M^{p,\beta}(\mathbb{R}^n)$ *to* $M^{q,\widetilde{\beta}}(\mathbb{R}^n)$.

Theorem 2.2. *Let* $1 < p < \infty$, $0 < \alpha < \min\{1, \frac{n}{m+1}\}$, $-\frac{n}{p} \le \beta < 0$, $\frac{1}{q} = \frac{1}{p} - \frac{(m+1)\alpha}{n}$, $\widetilde{\beta} = (m+1)\alpha + \beta$ *and* $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, *then* $[b, I_\alpha]^m$ *is a bounded operator from* $M^{p,\beta}(\mathbb{R}^n)$ *to* $M^{q,\widetilde{\beta}}(\mathbb{R}^n)$.

Conversely, if m *is an even integer,* $[b, I_\alpha]^m$ *is a bounded operator from* $M^{p,\beta}(\mathbb{R}^n)$ *to* $M^{q,\widetilde{\beta}}(\mathbb{R}^n)$ *, then* $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Theorem 2.3. *Let* $\max\{1, \frac{n}{1-\beta_3}\} < q < \frac{n}{\alpha}$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\beta = \beta_1 + \beta_2$, $\frac{1}{p_1} + \frac{1}{p_3} = \frac{1}{q}$, $\beta_3 = \beta_2 - \alpha$, $\frac{1}{p_2} = \frac{1}{p_3} - \frac{\alpha}{n}$, $1 < p_3 < \frac{n}{\alpha}$, $0 < 1 + \frac{p_3\beta_3}{n} < \frac{p_3}{p_2}$, $1 < p < \infty$, $-\frac{n}{p_i} \le \beta_i < 0 (i = 1, 3)$, $-\frac{n}{p} \le \beta < 0$, $1 < p_1 < \infty$ *and* $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$, *then* $[b, I_\alpha]$ *is a bounded operator from* $M^{p_3,\beta_3}(\mathbb{R}^n)$ *to* $C^{p,\beta}(\mathbb{R}^n)$.

Conversely, if p_1 *is an even integer,* $[b, I_\alpha]$ *is a bounded operator from* $M^{p_3,\beta_3}(\mathbb{R}^n)$ *to* $C^{p,\beta}(\mathbb{R}^n)$ *and there exist a constant* $C > 0$ *such that for that any* $Q \subset \mathbb{R}^n$,

$$\sup_Q |b - b_Q| \le \frac{C}{|Q|} \int_Q |b - b_Q|, \tag{2.1}$$

then $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$.

Theorem 2.4. *Let* $\max\{1, \frac{n}{1-\beta_3}\} < q < \frac{n}{\alpha}$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\beta = m\beta_1 + \beta_2$, $\frac{m}{p_1} + \frac{1}{p_3} = \frac{1}{q}$, $\beta_3 = \beta_2 - \alpha$, $\frac{1}{p_2} = \frac{1}{p_3} - \frac{\alpha}{n}$, $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$, $1 < p_3 < \frac{n}{\alpha}$, $0 < 1 + \frac{p_3\beta_3}{n} < \frac{p_3}{p_2}$, $1 < p < \infty$, $-\frac{n}{p_i} \le \beta_i < 0 (i = 1, 3)$, $-\frac{n}{p} \le \beta < 0$ *and* $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$, *then* $[b, I_\alpha]^m$ *is a bounded operator from* $M^{p_3,\beta_3}(\mathbb{R}^n)$ *to* $C^{p,\beta}(\mathbb{R}^n)$.

Conversely, if m *is an even integer,* $m > p_1$, $[b, I_\alpha]^m$ *is a bounded operator from* $M^{p_3,\beta_3}(\mathbb{R}^n)$ *to* $C^{p,\beta}(\mathbb{R}^n)$, *then* $b \in C^{p_1,\beta_1}(\mathbb{R}^n)$.

Remark 2.1. *Inequalities (2.1) can be thought of as a form of mean value inequality. Besides polynominal functions, mean value inequalities also characterize harmonic functions(see [8]). Solutions to a large class of elliptic second order PDEs satify the mean value inequality.*

Proof of Theorem 2.1.

(1) $\Rightarrow$ (2). We can obtain

$$\begin{aligned}|[b,I_\alpha]f(x)| &= \left|\int_{\mathbb{R}^n}\frac{[b(x)-b(y)]}{|x-y|^{n-\alpha}}f(y)dy\right|\\ &\le \int_{\mathbb{R}^n}\frac{|b(x)-b(y)|}{|x-y|^{\alpha}}\frac{|f(y)|}{|x-y|^{n-2\alpha}}dy\\ &\le CI_{2\alpha}(|f|)(x).\end{aligned}$$

By Lemma 1.1, $[b,I_\alpha]$ is a bounded operator from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{q,\widetilde{\beta}}(\mathbb{R}^n)$.

(2) $\Rightarrow$ (1). From [2] and [9], we can choose $z_0\in\mathbb{R}^n$, $\delta>0$ such that in the neighborhood $\{z:|z-z_0|<\delta\sqrt{n}\}$, function $|z|^{n-\alpha}$ can be represented as a Fourier series which absolutely converges. That is

$$|z|^{n-\alpha}=\sum_{m=0}^{\infty}a_m e^{i\langle v_m,z\rangle}.$$

Let $z_1=\delta^{-1}z_0$. Choose any cube $Q=Q\left(x_0,r\right)$. Set $y_0=x_0-rz_1$ and $Q'=Q'\left(y_0,r\right)$. Thus, if $x\in Q$ and $y\in Q'$, we have

$$\left|\frac{x-y}{r}-z_1\right|\le\left|\frac{x-x_0}{r}\right|+\left|\frac{y-y_0}{r}\right|\le\sqrt{n}.$$

Now set $s(x)=sgn\,[b(x)-b_{Q'}]$, then

$$\begin{aligned}\int_Q|b(x)-b_{Q'}|dx &= \int_Q(b(x)-b_{Q'})s(x)dx\\ &= \frac{1}{|Q'|}\int_Q\int_{Q'}(b(x)-b(y))s(x)dydx\\ &= \delta^{\alpha-n}r^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}\left|\frac{\delta(x-y)}{r}\right|^{n-\alpha}s(x)\chi_Q(x)\chi_{Q'}(y)dydx\\ &= Cr^{-\alpha}\sum_m a_m\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}e^{i\left\langle v_m,\frac{\delta}{r}(x-y)\right\rangle}s(x)\chi_Q(x)\chi_{Q'}(y)dydx.\end{aligned}$$

Set

$$f_m(y)=e^{-i\frac{\delta}{r}\langle v_m,y\rangle}\chi_{Q'}(y)$$

and

$$g_m(x) = e^{i\frac{\delta}{r}\langle v_m, x\rangle} s(x)\chi_Q(x),$$

then

$$\begin{aligned}\int_Q |b(x) - b_{Q'}|dx &\le Cr^{-\alpha}\sum_m a_m \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} f_m(y)dy g_m(x)dx \\ &\le Cr^{-\alpha}\sum_m |a_m| \int_{\mathbb{R}^n} |[b, I_\alpha](f_m)(x)|\,|g_m(x)|\,dx \\ &\le Cr^{-\alpha}\sum_m |a_m| \int_Q |[b, I_\alpha](f_m)(x)|dx.\end{aligned}$$

Applying the Hölder's inequality,

$$\begin{aligned}\int_Q |[b, I_\alpha](f_m)(x)|dx &\le \left(\frac{1}{|Q|}\int_Q |[b, I_\alpha](f_m)(x)|^q dx\right)^{\frac{1}{q}} |Q| \\ &= |Q|^{1+\frac{\tilde{\beta}}{n}} \|[b, I_\alpha](f_m)(x)\|_{M^{q,\tilde{\beta}}(\mathbb{R}^n)} \\ &\le |Q|^{1+\frac{\tilde{\beta}}{n}} \|f_m(x)\|_{M^{p,\beta}(\mathbb{R}^n)} \\ &= |Q|^{1+\frac{\tilde{\beta}}{n}} |Q|^{-\frac{\beta}{n}} \\ &= |Q|^{1+\frac{2\alpha}{n}}.\end{aligned}$$

Therefore we can obtain that

$$\begin{aligned}\int_Q |b(x) - b_{Q'}|dx &\le Cr^{-\alpha}|Q|^{1+\frac{2\alpha}{n}} \\ &= C|Q|^{1+\frac{\alpha}{n}}.\end{aligned}$$

This implies that

$$\frac{1}{|Q|^{1+\alpha/n}}\int_Q |b(x) - b_{Q'}|dx \le C.$$

Thus we have $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Proof of Theorem 2.2.

We first give the proof of sufficiency. We can obtain

$$\begin{aligned}|[b,I_\alpha]^m f(x)| &= \left|\int_{\mathbb{R}^n} \frac{[b(x)-b(y)]^m}{|x-y|^{n-\alpha}} f(y)dy\right| \\ &\le \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|^m}{|x-y|^{n-\alpha}} |f(y)|dy \\ &= \int_{\mathbb{R}^n} \left[\frac{|b(x)-b(y)|}{|x-y|^{\alpha}}\right]^m \frac{|f(y)|}{|x-y|^{n-(m+1)\alpha}}dy \\ &\le CI_{(m+1)\alpha}(|f|)(x).\end{aligned}$$

By Lemma 1.1, $[b,I_\alpha]^m$ is a bounded operator from $M^{p,\beta}(\mathbb{R}^n)$ to $M^{q,\widetilde{\beta}}(\mathbb{R}^n)$.

Next we will give the proof of necessity.

Let z_0, δ, x and y be the same as in Theorem 2.1. We have

$$\begin{aligned}\int_Q |b(x)-b_{Q'}|^m dx &= \int_Q \left|\frac{1}{|Q'|}\int_{Q'}(b(x)-b(y))dy\right|^m dx \\ &\le \int_Q \left(\frac{1}{|Q'|}\int_{Q'}|b(x)-b(y)|\,dy\right)^m dx \\ &\le \int_Q \frac{1}{|Q'|}\int_{Q'}|b(x)-b(y)|^m dydx\end{aligned}$$

$$\begin{aligned}&= \frac{1}{|Q'|}\int_Q\int_{Q'}(b(x)-b(y))^m dydx \\ &= \delta^{\alpha-n}r^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(b(x)-b(y))^m}{|x-y|^{n-\alpha}}\left|\frac{\delta(x-y)}{r}\right|^{n-\alpha}\chi_Q(x)\chi_{Q'}(y)dydx \\ &= Cr^{-\alpha}\sum_k a_k\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(b(x)-b(y))^m}{|x-y|^{n-\alpha}}e^{i\left\langle v_k,\frac{\delta}{r}(x-y)\right\rangle}\chi_Q(x)\chi_{Q'}(y)dydx.\end{aligned}$$

Set

$$f_k(y) = e^{-i\frac{\delta}{r}\langle v_k,y\rangle}\chi_{Q'}(y)$$

and

$$g_k(x) = e^{i\frac{\delta}{r}\langle v_k,x\rangle}\chi_Q(x),$$

We can obtain

$$\begin{aligned}\int\limits_Q |b(x)-b_{Q'}|^m dx &\leq Cr^{-\alpha}\sum_k a_k \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(b(x)-b(y))^m}{|x-y|^{n-\alpha}}f_k(y)g_k(x)dx\\ &\leq Cr^{-\alpha}\sum_k |a_k|\int_Q |[b,I_\alpha]^m f_k(x)|\,|g_k(x)|dx.\\ &\leq Cr^{-\alpha}\sum_k |a_k|\int_Q |[b,I_\alpha]^m f_k(x)|dx.\end{aligned}$$

Applying the Hölder's inequality,

$$\begin{aligned}\int_Q |[b,I_\alpha]^m f_k(x)|dx &\leq \left(\frac{1}{|Q|}\int_Q |[b,I_\alpha]^m (f_k)(x)|^q dx\right)^{\frac{1}{q}}|Q|\\ &= |Q|^{1+\frac{\tilde{\beta}}{n}}\|[b,I_\alpha]^m(f_k)(x)\|_{M^{q,\tilde{\beta}}(\mathbb{R}^n)}\\ &\leq |Q|^{1+\frac{\tilde{\beta}}{n}}\|f_k(x)\|_{M^{p,\beta}(\mathbb{R}^n)}\\ &= |Q|^{1+\frac{\tilde{\beta}}{n}}|Q|^{-\frac{\beta}{n}}\\ &= |Q|^{1+\frac{(m+1)\alpha}{n}}.\end{aligned}$$

Therefore we can obtain that

$$\begin{aligned}\int_Q |b(x)-b_{Q'}|^m dx &\leq Cr^{-\alpha}|Q|^{1+\frac{(m+1)\alpha}{n}}\\ &= C|Q|^{1+\frac{(m+1)\alpha}{n}-\frac{\alpha}{n}}\\ &= C|Q|^{1+\frac{m\alpha}{n}}.\end{aligned}$$

Thus we have obtained

$$\frac{1}{|Q|^{1+\frac{m\alpha}{n}}}\int_Q |b(x)-b_{Q'}|^m dx \leq C.$$

Thus we have $b\in \mathrm{Lip}_\alpha(\mathbb{R}^n)$.

Proof of Theorem 2.3.

We first give the proof of sufficiency. For a cube $Q = Q(x_Q, r) \subset \mathbb{R}^n$ and $y \in Q$, take $f \in M^{p_3,\beta_3}(\mathbb{R}^n)$ and set $f_1 = f\chi_{2Q}$ and $f_2 = f - f_1$. Noticing that

$$[b, I_\alpha]f = [b - b_Q, I_\alpha]f,$$

we have

$$\begin{aligned}
&\frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |[b, I_\alpha]f(y) - ([b, I_\alpha]f)_Q|^p dy\right)^{\frac{1}{p}} \\
&= \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |[b - b_Q, I_\alpha]f(y) - ([b, I_\alpha]f)_Q|^p dy\right)^{\frac{1}{p}} \\
&= \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |(b(y) - b_Q)I_\alpha f(y) - I_\alpha(b - b_Q)f(y) - ([b, I_\alpha]f)_Q|^p dy\right)^{\frac{1}{p}} \\
&\le \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |(b(y) - b_Q)I_\alpha f(y)|^p dy\right)^{\frac{1}{p}} + \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |I_\alpha(b - b_Q)f_1(y)|^p dy\right)^{\frac{1}{p}} \\
&+ \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |I_\alpha(b - b_Q)f_2(y) - I_\alpha(b - b_Q)f_2(x_Q)|^p dy\right)^{\frac{1}{p}} \\
&= I + II + III.
\end{aligned}$$

The Hölder's inequality and Lemma 1.1 imply

$$\begin{aligned}
I &= \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |(b(y) - b_Q)I_\alpha f(y)|^p (y)\right)^{\frac{1}{p}} \\
&\le \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |b(y) - b_Q|^{p_1} dy\right)^{\frac{1}{p_1}}\left(\int_Q |I_\alpha f(y)|^{p_2} dy\right)^{\frac{1}{p_2}} \\
&\le \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\,\|I_\alpha f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)} \\
&\le C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}.
\end{aligned}$$

The Hölder's inequality implies

$$
\begin{aligned}
II &= \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left(\int_Q |I_\alpha (b-b_Q) f_1(y)|^p dy \right)^{\frac{1}{p}} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \|(b-b_Q) f_1\|_{L^q(\mathbb{R}^n)} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left(\int_Q |b(y)-b_Q|^{p_1} dy \right)^{\frac{1}{p_1}} \left(\int_Q |f(y)|^{p_3} dy \right)^{\frac{1}{P_3}} \\
&\leq C \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}.
\end{aligned}
$$

We now turn to the estimate for the term III, it may be concluded that

$$
\begin{aligned}
& |I_\alpha (b-b_Q) f_2(y) - I_\alpha (b-b_Q) f_2(x_Q)| \\
&= \left| \int_{\mathbb{R}^n} \left(\frac{1}{|y-z|^{n-\alpha}} - \frac{1}{|x_Q - z|^{n-\alpha}} \right) (b(z)-b_Q) f_2(z) dz \right| \\
&\leq \int_{(2Q)^c} \frac{1}{|x_Q - z|^{n-\alpha+1}} |y-x_Q| |b(z)-b_Q| |f_2(z)| dz \\
&\leq \sum_{k=2}^{\infty} \int_{2^k Q \setminus 2^{k-1} Q} \frac{1}{2^k |2^k Q|^{1-\frac{\alpha}{n}}} \left(|b(z)-b_{2^k Q}| + |b_Q - b_{2^k Q}| \right) |f(z)| dz \\
&\leq \sum_{k=2}^{\infty} \frac{1}{2^k |2^k Q|^{1-\frac{\alpha}{n}}} \int_{2^k Q} \left(|b(z)-b_{2^k Q}| + |b_Q - b_{2^k Q}| \right) |f(z)| dz,
\end{aligned}
$$

which yields

$$
\begin{aligned}
III &\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left(\int_Q \left| \sum_{k=2}^{\infty} \frac{1}{2^k |2^k Q|^{1-\frac{\alpha}{n}}} \int_{2^k Q} |b(z)-b_{2^k Q}| |f(z)| dz \right|^p \right)^{\frac{1}{p}} \\
&+ \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left(\int_Q \left| \sum_{k=2}^{\infty} \frac{1}{2^k |2^k Q|^{1-\frac{\alpha}{n}}} \int_{2^k Q} |b_Q - b_{2^k Q}| |f(z)| dz \right|^p \right)^{\frac{1}{p}} \\
&=: III_1 + III_2.
\end{aligned}
$$

We can obtain

$$
\begin{aligned}
III_1 &= \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left\| \sum_{k=2}^{\infty} \frac{1}{2^k|2^kQ|^{1-\frac{\alpha}{n}}} \int_{2^kQ} |b(z)-b_{2^kQ}||f(z)\chi_{2^kQ}(z)|dz \right\|_{L^p} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left\| \frac{1}{|2^kQ|^{1-\frac{\alpha}{n}}} \int_{2^kQ} |(b(z)-b_{2^kQ}||f(z)\chi_{2^kQ}(z)|dz \right\|_{L^p} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left\| M_\alpha \left((b(z)-b_{2^kQ})f(z)\chi_{2^kQ}(z) \right) \right\|_{L^p} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left\| (b(z)-b_{2^kQ})f(z)\chi_{2^kQ}(z) \right\|_{L^q} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left(\int_{2^kQ} |b(z)-b_{2^kQ}|^{p_1} \right)^{\frac{1}{p_1}} \left(\int_{2^kQ} |f(z)|^{p_3} \right)^{\frac{1}{p_3}} \\
&\leq C||b||_{C^{p_1,\beta_1}(\mathbb{R}^n)}||f||_{M^{p_3,\beta_3}(\mathbb{R}^n)}.
\end{aligned}
$$

$$
\begin{aligned}
III_2 &= \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left\| \sum_{k=2}^{\infty} \frac{1}{2^k|2^kQ|^{1-\frac{\alpha}{n}}} \int_{2^kQ} |b_Q-b_{2^kQ}||f(z)\chi_{2^kQ}(z)|dz \right\|_{L^p} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left\| \frac{1}{|2^kQ|^{1-\frac{\alpha}{n}}} \int_{2^kQ} |b_Q-b_{2^kQ}||f(z)\chi_{2^kQ}(z)|dz \right\|_{L^p} \\
&\leq \sum_{k=2}^{\infty} \frac{1}{2^k} \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta_2}{n}}} \left\| M_\alpha(f(z)\chi_{2^kQ}(z)) \right\|_{L^p} \\
&\leq \sum_{k=2}^{\infty} \frac{1}{2^k} \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta_2}{n}}} \|f(z)\chi_{2^kQ}(z)\|_{L^q} \\
&\leq \sum_{k=2}^{\infty} \frac{1}{2^k} \|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta_2}{n}}} \left(\int_{2^kQ} |f|^{p_3} \right)^{\frac{1}{p_3}} |2^kQ|^{\frac{1}{p_1}} \\
&\leq C\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}.
\end{aligned}
$$

Next we will give the proof of necessity.

We first claim that for fixed $Q \subset \mathbb{R}^n$, $b \in C^{p_1,\beta_1}(Q)$ and $f \in M^{p_3,\beta_3}(Q)$ with $\|f\|_{M^{p_3,\beta_3}(Q)} = |Q|^{-\frac{\beta_3}{n}}$, we have

$$\|[b, I_\alpha]^m f\|_{C^{p,\beta}(Q)} \leq C|Q|^{\frac{\beta_1(m-1)}{n}}\|f\|_{M^{p_3,\beta_3}(Q)} \leq |Q|^{\frac{\beta_1 m-\beta+\alpha}{n}}. \tag{2.2}$$

We shall prove (2.2) by induction.

When $m = 1$,

$$\|[b, I_\alpha]f\|_{C^{p,\beta}(Q)} \le C\|f\|_{M^{p_3,\beta_3}(Q)} = C|Q|^{-\frac{\beta_3}{n}}.$$

We now assume that for any $b \in C^{p_1,\beta_1}(Q)$, we have

$$\|[b, I_\alpha]^{m-1}f\|_{C^{p,\beta}(Q)} \le C|Q|^{\frac{\beta_1(m-2)}{n}}\|f\|_{M^{p_3,\beta_3}(Q)} \le C|Q|^{\frac{\beta_1(m-2)-\beta+\alpha}{n}}.$$

Next we show the case m.

$$\begin{aligned}
|[b, I_\alpha]^m f| &= \left|\int_{\mathbb{R}^n}(b(x)-b(y))^{m-1}\frac{1}{|x-y|^{n-\alpha}}f(y)(b(x)-b(y))dy\right| \\
&\le \left|\int_{\mathbb{R}^n}(b(x)-b(y))^{m-1}\frac{1}{|x-y|^{n-\alpha}}f(y)(b(x)-b_Q)dy\right| \\
&+ \left|\int_{\mathbb{R}^n}(b(x)-b(y))^{m-1}\frac{1}{|x-y|^{n-\alpha}}f(y)(b(y)-b_Q)dy\right| \\
&\le |b-b_Q||[b, I_\alpha]^{m-1}f(x)| + \left|[b, I_\alpha]^{m-1}\left((b-b_Q)f\right)(x)\right| \\
&=: J_1 + J_2.
\end{aligned}$$

$$\begin{aligned}
\|J_1\|_{C^{p,\beta}(Q)} &\le \left\||b-b_Q||[b, I_\alpha]^{m-1}f|\right\|_{C^{p,\beta}(Q)} \\
&\le \|b-b_Q\|_{L^\infty}\left\|[b, I_\alpha]^{m-1}f\right\|_{C^{p,\beta}(Q)} \\
&\le C\frac{1}{|Q|}\int_Q |b-b_Q|dx|Q|^{\frac{\beta_1(m-2)}{n}}\|f\|_{M^{p_3,\beta_3}(Q)} \\
&\le C\|b\|_{C^{p_1,\beta_1}(Q)}\|f\|_{M^{p_3,\beta_3}(Q)}|Q|^{\frac{\beta_1(m-1)}{n}} \\
&\le C|Q|^{\frac{\beta_1 m+\alpha-\beta}{n}}.
\end{aligned}$$

$$\begin{aligned}
\|J_2\|_{C^{p,\beta}(Q)} &\le C|Q|^{\frac{\beta_1(m-2)}{n}}\|(b-b_Q)f\|_{M^{p_3,\beta_3}(Q)} \\
&\le C|Q|^{\frac{\beta_1(m-1)}{n}}\|b\|_{C^{p_1,\beta_1}(Q)}\|f\|_{M^{p_3,\beta_3}(Q)} \\
&\le C|Q|^{\frac{\beta_1 m+\alpha-\beta}{n}}.
\end{aligned}$$

Similar to Theorem 2.1, we can obtain that

$$\begin{aligned}\int_Q |b(x)-b_{Q'}|^{p_1}dx &\le Cr^{-\alpha}\sum_m |a_m| \int_Q |[b,I_\alpha]^{p_1}(f_m)(x)|\,dx\\ &\le \|[b,I_\alpha]^{p_1}(f_m)\|_{C^{p,\beta}(Q)}\,|Q|^{1+\frac{\beta-\alpha}{n}}\\ &\le |Q|^{1+\frac{\beta_1 p_1}{n}}.\end{aligned}$$

So

$$\frac{1}{|Q|^{1+\frac{\beta_1 p_1}{n}}}\int_Q |b(x)-b_{Q'}|^{p_1} \le C.$$

Which completes the proof of Theorem 2.3.

Proof of Theorem 2.4.

We first give the proof of sufficiency. The proof is similar to that of Theorem 2.2.

$$\begin{aligned}\int_Q |b(x)-b_{Q'}|^{p_1}dx &\le \left(\int_Q |b(x)-b_{Q'}|^m\right)^{\frac{p_1}{m}} |Q|^{1-\frac{p_1}{m}}\\ &\le C\left(r^{-\alpha}\int_Q |[b,I_\alpha]^m(f_k)(x)|dx\right)^{\frac{p_1}{m}} |Q|^{1-\frac{p_1}{m}}\\ &\le C\left(r^{-\alpha}\|[b,I_\alpha]^m f_k\|_{C^{p,\beta}(\mathbb{R}^n)}|Q|^{1+\frac{\beta}{n}}\right)^{\frac{p_1}{m}} |Q|^{1-\frac{p_1}{m}}\\ &\le C\left(|Q|^{-\frac{\alpha}{n}}\|f_k\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}|Q|^{1+\frac{\beta}{n}}\right)^{\frac{p_1}{m}} |Q|^{1-\frac{p_1}{m}}\\ &\le C|Q|^{1+\frac{p_1\beta_1}{n}},\end{aligned}$$

So we can obtain

$$\frac{1}{|Q|^{1+\frac{p_1\beta_1}{n}}}\int_Q |b(x)-b_{Q'}|^{p_1}dx \le C.$$

Next we will give the proof of necessity.

For a cube $Q=Q(x_Q,r)\subset\mathbb{R}^n$ and $y\in Q$, take $f\in M^{p_2,\beta_2}(\mathbb{R}^n)$ and set $f_1=f\chi_{2Q}$ and $f_2=f-f_1$. Noticing that

$$[b,I_\alpha]^m f=[b-b_Q,I_\alpha]^m f,$$

We have

$$
\begin{aligned}
&\frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |[b,I_\alpha]^m f(y)-([b,I_\alpha]^m f)_Q|^p dy\right)^{\frac{1}{p}}\\
&\le \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |(b(y)-b_Q)^m I_\alpha f(y)|^p dy\right)^{\frac{1}{p}}\\
&\quad+\frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |I_\alpha(b-b_Q)^m f_1(y)|^p dy\right)^{\frac{1}{p}}\\
&\quad+\frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |I_\alpha(b-b_Q)^m f_2(y)-I_\alpha(b-b_Q)^m f_2(x_Q)|^p dy\right)^{\frac{1}{p}}\\
&=VII+VIII+IX.
\end{aligned}
$$

The Hölder's inequality and Lemma 1.1 imply

$$
\begin{aligned}
VII &\le \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}}\left(\int_Q |b(y)-b_Q|^{p_1} dy\right)^{\frac{m}{p_1}}\left(\int_Q |I_\alpha f(y)|^{p_2} dy\right)^{\frac{1}{p_2}}\\
&\le \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\right)^m \|I_\alpha f\|_{M^{p_2,\beta_2}(\mathbb{R}^n)}\\
&\le C\left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}.
\end{aligned}
$$

From Lemma 1.2, it follows that

$$
\begin{aligned}
VIII &\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \|(b-b_Q)^m f\chi_{2Q}\|_{L^q} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left(\|(b_{2Q}-b_Q)^m f\chi_{2Q}\|_{L^q} + \|(b-b_{2Q})^m f\chi_{2Q}\|_{L^q}\right) \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \Bigg(\left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} |Q|^{\frac{\beta_1}{n}}\right)^m \|f\chi_{2Q}\|_{L^q} + \left(\int_{2Q} |b(y)-b_{2Q}|^{p_1} dy\right)^{\frac{m}{p_1}} \\
&\quad \times \left(\int_{2Q} |f(y)|^{p_3} dy\right)^{\frac{1}{p_3}} \Bigg) \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \Bigg(\left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} |Q|^{\frac{\beta_1}{n}}\right)^m \left(\int_{2Q} |f(y)|^{p_3} dy\right)^{\frac{1}{p_3}} |2Q|^{\frac{m}{p_1}} \\
&\quad + \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)} \Bigg) \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \Bigg(\left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)} |2Q|^{\frac{\beta_3}{n}+\frac{1}{q}+\frac{m\beta_1}{n}} \\
&\quad + \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)} \Bigg) \\
&\leq C \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)}\right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}.
\end{aligned}
$$

We now turn to the estimate for the term IX. So

$$
\begin{aligned}
IX &\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left\| \sum_{k=2}^{\infty} \frac{1}{2^k |2^k Q|^{1-\frac{\alpha}{n}}} \int_{2^k Q} |b(z)-b_{2^k Q}|^m |f(z)\chi_{2^k Q}(z)| dz \right\|_{L^p} \\
&\quad + \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \left\| \sum_{k=2}^{\infty} \frac{1}{2^k |2^k Q|^{1-\frac{\alpha}{n}}} \int_{2^k Q} |b_Q-b_{2^k Q}|^m |f(z)\chi_{2^k Q}(z)| dz \right\|_{L^p} \\
&=: IX_1 + IX_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
IX_1 &\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left\| \frac{1}{|2^k Q|^{1-\frac{\alpha}{n}}} \int_{2^k Q} |b(z)-b_{2^k Q}|^m |f(z)\chi_{2^k Q}(z)| dz \right\|_{L^p} \\
&\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left\| (b(z)-b_{2^k Q})^m f(z)\chi_{2^k Q}(z) \right\|_{L^q}
\end{aligned}
$$

$$\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left(\int_{2^kQ} |b(z) - b_{2^kQ}|^{p_1} \right)^{\frac{1}{p_1}} \left(\int_{2^kQ} |f|^{p_3} \right)^{\frac{1}{p_3}}$$

$$= \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)} |2^kQ|^{\frac{m\beta_1}{n}+\frac{\beta_3}{n}+\frac{1}{q}}$$

$$= \sum_{k=2}^{\infty} 2^{k(\beta_1 m+\beta_3+\frac{n}{q}-1)} \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}$$

$$\leq C(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)})^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}.$$

Apply Lemma 1.2 to IX_2, we have

$$IX_2 \leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left\| \frac{1}{|2^kQ|^{1-\frac{\alpha}{n}}} \int_{2^kQ} |b_Q - b_{2^kQ}|^m |f(z)\chi_{2^kQ}(z)| dz \right\|_{L^p}$$

$$\leq \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} |Q|^{\frac{\beta_1}{n}} \right)^m \|M_\alpha(f\chi_{2^kQ})\|_{L^p}$$

$$\leq C \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} |Q|^{\frac{\beta_1}{n}} \right)^m \|f\chi_{2^kQ}\|_{L^q}$$

$$\leq C \frac{1}{|Q|^{\frac{1}{p}+\frac{\beta}{n}}} \sum_{k=2}^{\infty} \frac{1}{2^k} \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} |Q|^{\frac{\beta_1}{n}} \right)^m \left(\int_{2^kQ} |f|^{p_3} \right)^{\frac{1}{p_3}} |2^kQ|^{\frac{m}{p_1}}$$

$$= C \sum_{k=2}^{\infty} 2^{k(\beta_3+\frac{n}{q}-1)} \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}$$

$$\leq C \left(\|b\|_{C^{p_1,\beta_1}(\mathbb{R}^n)} \right)^m \|f\|_{M^{p_3,\beta_3}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 2.4.

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