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SOME STUDIES ON ORDERED KRASNER HYPERRINGS WITH RESPECT TO DERIVATIONS

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Abstract. The concept of ordered Krasner hyperrings is a generalization of the concept of ordered rings. In this paper, we study subhyperrings, d-subhyperrings and injective subhyperrings of ordered Krasner hyperrings. Also, the notion of convex ordered Krasner hyperrings is introduced and some related results are studied. Moreover, we prove that for an injective strong derivation d of a convex ordered Krasner hyperring $(R, +, \cdot, \leq)$ associated to a d-strongly regular relation σ, there exists an injective strong derivation on $(R/\sigma, \oplus, \odot, \preceq)$.

1. Introduction and prerequisites

Hyperstructure theory was first introduced in 1934, when Marty [16] defined hypergroups. Several books on hyperstructure theory have been published [3, 4, 5, 8, 21]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. There are different types of hyperrings. Davvaz and Leoreanu-Fotea studied hyperrings in more details in [8]. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. Krasner hyperrings are generalizations of rings in which the addition is multivalued, i.e. the sum of two elements is no longer an element but a subset.

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Krasner (m,n)-hyperrings were introduced and analyzed by Mirvakili and Davvaz [17]. Krasner (m,n)-hyperrings are a suitable generalization of Krasner hyperrings. Davvaz [6] defined fuzzy Krasner (m,n)-hyperrings and obtained some results in this respect.

A preorder on an arbitrary non-empty set X is a binary relation on X which is reflexive and transitive. An antisymmetric preorder is said to be an order. Ganesamoorthy and Karpagavalli [11] introduced and analyzed congruence relations in partially ordered sets. The concept of ordering hypergroups studied by Chvalina [2] as a special type of hypergroups. The concept of an ordered semihypergroup is a generalization of the concept of an ordered semigroup. The notion of ordered semihypergroup was first introduced in 2011 by Heidari and Davvaz in [12]. After, Davvaz et al. [10] studied the concept of pseudoorders in ordered semihypergroups. Recall from [12] that an *ordered semihypergroup* $(S,\circ,\leq)$ is a semihypergroup $(S,\circ)$ together with a partial order $\leq$ that is compatible with the hyperoperation $\circ$, meaning that for any $x,y,z \in S$,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Posner [19] studied derivations in rings. In 2013, Asokkumar [1] introduced the notion of derivation in Krasner hyperrings as follows: Let $(R,+,\cdot)$ be a Krasner hyperring. A function $d : R \to R$ is called a *derivation* of R if (1) $d(a+b) \subseteq d(a)+d(b)$ and (2) $d(a \cdot b) \in d(a) \cdot b + a \cdot d(b)$ for all $a,b \in R$. Also, Kamali Ardekani and Davvaz studied the main properties of derivations of multiplicative hyperrings [13] and Krasner hyperrings [14]. Recently, Wang et al. [22] proved some results in bounded hyperlattices using derivations. In [20], Rafi et al. introduced the concept of d-ideals on almost distributive lattices. The concept of an ordered semihyperring was first given by Davvaz and Omidi [9]. The concept of ordered Krasner hyperring was introduced and studied in [18].

Let H be a non-empty set. A mapping $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of H, is called a *hyperoperation* on H. The couple $(H, \circ)$ is called a *hyperstructure*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we denote

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b,\ A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hyperstructure $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A non-empty subset K of a semihypergroup $(H, \circ)$ is called a *subsemihypergroup* of H if $K \circ K \subseteq K$. Let $(H, \circ)$ be a semihypergroup. Then, H is called a *hypergroup* if it satisfies the reproduction axiom, for all $x \in H$, $H \circ x = x \circ H = H$. A non-empty subset K of H is a *subhypergroup* of H if $K \circ a = a \circ K = K$, for all $a \in K$.

Let $(H, \circ)$ be a semihypergroup and σ be an equivalence relation on H. If A and B are nonempty subsets of H, then

$$\begin{aligned} A\overline{\sigma}B &\text{ means that } \forall a \in A, \exists b \in B \text{ such that } a\sigma b \text{ and} \\ &\qquad\qquad \forall b' \in B, \exists a' \in A \text{ such that } a'\sigma b'; \\ A\overline{\overline{\sigma}}B &\text{ means that } \forall a \in A, \forall b \in B, \text{ we have } a\sigma b. \end{aligned}$$

The equivalence relation σ is called

(1) *regular on the right* (*on the left*) if for all x of H, from $a\sigma b$, it follows that $(a \circ x)\overline{\sigma}(b \circ x)$ $((x \circ a)\overline{\sigma}(x \circ b)$ respectively);

(2) *strongly regular on the right* (*on the left*) if for all x of H, from $a\sigma b$, it follows that $(a \circ x)\overline{\overline{\sigma}}(b \circ x)$ $((x \circ a)\overline{\overline{\sigma}}(x \circ b)$ respectively);

(3) σ is called *regular* (*strongly regular*) if it is regular (strongly regular) on the right and on the left.

In [7], Davvaz gave the fundamental homomorphism theorem of Krasner hyperrings. Let us introduce some definitions on Krasner hyperrings that will be used in the next section. A *Krasner hyperring* [15] is an algebraic hypersructure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup, i.e., (i) for any $x, y, z \in R, x + (y + z) = (x + y) + z$, (ii) for any $x, y \in R, x + y = y + x$, (iii) there exists $0 \in R$ such that $0 + x = x + 0 = x$, for any $x \in R$, (iv) for every $x \in R$, there exists a unique element $x' \in R$, such that $0 \in x + x'$ (we shall write $-x$ for x' and we call it the opposite of x), (v) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in R$, that is $(R, +)$ is reversible;

(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$, for all $x \in R$;

(3) The multiplication is distributive with respect to the hyperoperation $+$.

A non-empty subset A of a Krasner hyperring $(R, +, \cdot)$ is called a *subhyperring* of R if $(A, +, \cdot)$ itself is a Krasner hyperring. Equivalently, a non-empty subset A of a Krasner hyperring $(R, +, \cdot)$ is a subhyperring of R if and only if, for all $x, y \in A$, $x + y \subseteq A$, $-x \in A$ and $x \cdot y \in A$.

Theorem 1.1. [18] *Let $(R, +, \cdot, \leq)$ be a preordered Krasner hyperring and σ a strongly regular relation on R. Then, $(R/\sigma, \oplus, \odot, \preceq)$ is a preordered Krasner hyperring with respect to the following hyperoperations on the quotient set R/σ:*

$$[a]_\sigma \oplus [b]_\sigma = \{[c]_\sigma \mid c \in a + b\},$$

$$[a]_\sigma \odot [b]_\sigma = [a \cdot b]_\sigma,$$

where for all $[a]_\sigma, [b]_\sigma \in R/\sigma$ a preorder relation $\preceq$ is defined by:

$$[a]_\sigma \preceq [b]_\sigma \Leftrightarrow \forall a_1 \in [a]_\sigma \exists b_1 \in [b]_\sigma \text{ such that } a_1 \leq b_1.$$

2. Definitions and examples

Definition 2.1. Let $(R, +, \cdot)$ be a Krasner hyperring. We say that $(R, +, \cdot, \leq)$ is an *ordered Krasner hyperring* if the following axioms are fulfilled:

(1) $(R, \leq)$ is a partially ordered set.

(2) For every $a, b, c \in R$, $a \leq b$ implies $a + c \leq b + c$, that is for any $u \in a + c$, there exists $v \in b + c$ such that $u \leq v$.

(3) For any $a, b, c \in R$, $a \leq b$ and $0 \leq c$ imply $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

Remark 1. An ordered Krasner hyperring R is *positive* if $0 \leq x$ for each $x \in R$.

If we remove the restriction $0 \leq c$ from (3), then Definition 2.1 is equivalent to positive ordered Krasner hyperring.

Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. For $A \subseteq R$, we denote

$$(A] = \{x \in R \mid x \leq a, \text{ for some } a \in A\}.$$

It is clear that $(R] = R$.

Definition 2.2. Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. A non-empty subset A of R is said to be a *subhyperring* of R if the following conditions hold:

(1) $(A, +)$ is a canonical subhypergroup of $(R, +)$ and $A \cdot A \subseteq A$;

(2) $(A] = A$.

In the following, we define the notion of derivations and provide some examples.

Definition 2.3. Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. A function $d : R \to R$ is called a *derivation* of R if it satisfies the following conditions:

(1) $d(a + b) \subseteq d(a) + d(b)$ for all $a, b \in R$;

(2) $d(a \cdot b) \in d(a) \cdot b + a \cdot d(b)$ for all $a, b \in R$;

(3) d is isotone, that is, for any $a, b \in R$, $a \leq b$ implies $d(a) \leq d(b)$.

A function $d : R \to R$ is called *strong derivation* if for all $a, b \in R$, it satisfies (2), (3) and (1) $d(a+b) = d(a) + d(b)$.

Example 2.1. *Let $R = \{0, a, b, c\}$ be a set with the hyperoperation $+$ and the multiplication $\cdot$ defined as follows:*

$+$	0	a	b	c
0	0	a	b	c
a	a	$\{0,a\}$	c	$\{b,c\}$
b	b	c	0	a
c	c	$\{b,c\}$	a	$\{0,a\}$

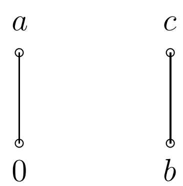
$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	b
c	0	0	b	b

Then, $(R, +, \cdot)$ is a Krasner hyperring [17]*. We have $(R, +, \cdot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:*

$$\leq := \{(0,0), (a,a), (b,b), (c,c), (0,a), (b,c)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(0,a), (b,c)\}.$$



Define a map $d : R \to R$ by $d(0) = 0, d(a) = a, d(b) = 0, d(c) = a$. Now, $d(a+a) = d(\{0,a\}) = \{0,a\} = a+a = d(a)+d(a)$ and $d(a \cdot a) = d(0) = 0 = 0+0 = a \cdot a + a \cdot a = d(a) \cdot a + a \cdot d(a)$. Also, $d(a+b) = d(c) = a = a+0 = d(a)+d(b)$, $d(a \cdot b) = d(0) = 0 = 0+0 = a \cdot b + a \cdot 0 = d(a) \cdot b + a \cdot d(b)$, $d(a+c) = d(\{b,c\}) = \{0,a\} = a+a = d(a)+d(c)$, $d(a \cdot c) = d(0) = 0 = 0+0 = a \cdot c + a \cdot a = d(a) \cdot c + a \cdot d(c)$, $d(b+b) = d(0) = 0 = 0+0 = d(b)+d(b)$, $d(b \cdot b) = d(b) = 0 = 0+0 = 0 \cdot b + b \cdot 0 = d(b) \cdot b + b \cdot d(b)$, $d(b+c) = d(a) = a = 0+a = d(b)+d(c)$, $d(b \cdot c) = d(b) = 0 =$

$0+0 = 0 \cdot c + b \cdot a = d(b) \cdot c + b \cdot d(c)$, $d(c+c) = d(\{0,a\}) = \{0,a\} = a+a = d(c)+d(c)$ *and* $d(c \cdot c) = d(b) = 0 = 0+0 = a \cdot c + c \cdot a = d(c) \cdot c + c \cdot d(c)$. *We can verify that* $x \leq y$ *implies* $d(x) \leq d(y)$, *for all* $x, y \in R$. *Hence,* d *is a strong derivation of* R.

Example 2.2. *Consider the hyperring* $R = \{0, a, b\}$ *with the hyperaddition* $+$ *and the multiplication* $\cdot$ *defined as follows:*

$+$	0	a	b
0	0	a	b
a	a	$\{a,b\}$	R
b	b	R	$\{a,b\}$

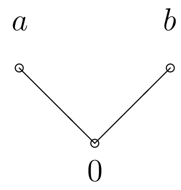
$\cdot$	0	a	b
0	0	0	0
a	0	b	a
b	0	a	b

Then, $(R, +, \cdot)$ *is a Krasner hyperring* [1]. *We have* $(R, +, \cdot, \leq)$ *is an ordered Krasner hyperring, where the order relation* $\leq$ *is defined by:*

$$\leq := \{(0,0), (a,a), (b,b), (0,a), (0,b)\}.$$

The covering relation and the figure of R *are given by:*

$$\prec = \{(0,a), (0,b)\}.$$



Define a map $d : R \to R$ *by* $d(0) = 0, d(a) = b, d(b) = a$. *Now, it is easy to see that* d *is a strong derivation of* R.

Example 2.3. *In Example 2.2, the identity function,* $d(x) = x$ *for every* $x \in R$, *is a strong derivation of* R.

Example 2.4. *Let $R = \{0, a, b, c\}$ be a set with the hyperoperation $+$ and the multiplication $\cdot$ defined as follows:*

$+$	0	a	b	c
0	0	a	b	c
a	a	$\{0, b\}$	$\{a, c\}$	b
b	b	$\{a, c\}$	$\{0, b\}$	a
c	c	b	a	0

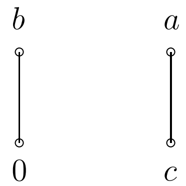
$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Then, $(R, +, \cdot)$ is a Krasner hyperring [1]. *We have $(R, +, \cdot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:*

$$\leq := \{(0, 0), (a, a), (b, b), (c, c), (0, b), (c, a)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(0, b), (c, a)\}.$$



Define a map $d : R \to R$ by $d(0) = 0, d(a) = b, d(b) = b, d(c) = 0$. Now, it is easy to see that d is a non strong derivation of R.

Definition 2.4. Let d be a derivation of an ordered Krasner hyperring $(R, +, \cdot, \leq)$. Furthermore, we assume that A is a subhyperring of R. The subhyperring A is a *d-subhyperring* of R if $d(a) \in A$, for all $a \in A$.

Note that $\{0\}$ and R are the d-subhyperrings of R.

Example 2.5. (1) *In Example 2.1, $\{0, a\}$ and $\{0, b\}$ are d-subhyperrings of R.*

(2) *In Example 2.4, $\{0, b\}$ and $\{0, c\}$ are d-subhyperrings of R.*

Remark 2. Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. If $d : R \to R$ is a function defined by $d(x) = 0$ for each $x \in R$, then every subhyperring is a d-subhyperring.

Definition 2.5. Let d be a derivation of an ordered Krasner hyperring $(R, +, \cdot, \leq)$. and let A be a subhyperring of R. Then A is an *injective subhyperring with respect to* d if for all $a, b \in R$, the following holds:

$$d(a) = d(b) \text{ and } a \in A \Rightarrow b \in A.$$

Example 2.6. *In Example 2.4,* $\{0, c\}$ *is an injective subhyperring of* R.

Example 2.7. *In Example 2.4,* $\{0, b\}$ *is a d-subhyperring but not injective with respect to d. Indeed:*

$$d(a) = d(b) = b \textit{ and } b \in \{0, b\} \textit{ but } a \notin \{0, b\}.$$

Remark 3. Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. If d is an injective derivation of R, then every subhyperring is injective with respect to d.

Note that $\{0\}$ is not necessarily an injective subhyperring. This is shown by the following example.

Example 2.8. *Consider the ordered Krasner hyperring* $(R, +, \cdot, \leq)$ *defined in Example 2.4. Then,* $\{0\}$ *is not an injective subhyperring of* R*. Indeed:*

$$d(0) = d(c) = 0 \textit{ and } 0 \in \{0\} \textit{ but } c \notin \{0\}.$$

In the following, we introduce the notion of convex ordered Krasner hyperrings associated to strongly regular relations.

Definition 2.6. Let σ be a strongly regular relation on an ordered Krasner hyperring $(R, +, \cdot, \leq)$. We say that R is a *convex ordered Krasner hyperring associated to* σ if the following correlation takes place:

$$(x, z) \in \sigma \text{ and } x \leq y \leq z \Rightarrow (x, y) \in \sigma.$$

Remark 4. If we consider the identity strongly regular relation on an ordered Krasner hyperring R, then R is a convex ordered Krasner hyperring.

Example 2.9. *Let $R = \{0, a, b, c, d, e\}$ be a set with the hyperaddition $+$ and the multiplication $\cdot$ defined as follows:*

$+$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	a	$\{0, a, b\}$	d	d	$\{c, d, e\}$
b	b	$\{0, a, b\}$	b	e	$\{c, d, e\}$	e
c	c	d	e	0	a	b
d	d	d	$\{c, d, e\}$	a	a	$\{0, a, b\}$
e	e	$\{c, d, e\}$	e	b	$\{0, a, b\}$	b

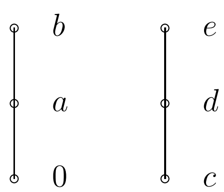
$\cdot$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	a	b	0	a	b
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	a	b	c	d	e
e	0	a	b	c	d	e

Clearly, $(R, +, \cdot)$ is a Krasner hyperring [14]. *We have $(R, +, \cdot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:*

$$\leq \ := \{(0,0), (a,a), (b,b), (c,c), (d,d), (e,e), (0,a), (0,b), (a,b), (c,d), (c,e), (d,e)\}.$$

The covering relation and the figure of R are given by:

$$\prec \ = \{(0,a), (a,b), (c,d), (d,e)\}.$$



Let σ be strongly regular relation on R define as follows:

$$\begin{aligned}\sigma &= \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(0,a),\\ &(a,0),(0,b),(b,0),(a,b),(b,a),(c,d),(d,c),\\ &(c,e),(e,c),(d,e),(e,d)\}.\end{aligned}$$

We can easily verify that R is a convex ordered Krasner hyperring associated to σ.

Definition 2.7. Let d be a derivation of an ordered Krasner hyperring $(R,+,\cdot,\leq)$. A strongly regular relation σ is said to be a *d-strongly regular relation* if it satisfies the following property:

$$(x,y)\in\sigma \Rightarrow (d(x),d(y))\in\sigma.$$

Example 2.10. *In Example 2.1,*

$$\sigma = \{(0,0),(a,a),(b,b),(c,c),(0,a),(a,0),(b,c),(c,b)\}$$

is a d-strongly regular relation of R.

Remark 5. Let $(R,+,\cdot,\leq)$ be an ordered Krasner hyperring such that $x\in x+x$ for every $x\in R$. If d is the identity function, $d(x)=x$ for all $x\in R$, then every strongly regular relation is a d-strongly regular relation of R.

3. Main results

Lemma 3.1. *Let $(R,+,\cdot,\leq)$ be an ordered Krasner hyperring and $A,B\subseteq R$. Then*

(1) $A\subseteq (A]$.
(2) $((A]]=(A]$.
(3) $(A]\subseteq(B]$ *for any* $A\subseteq B\subseteq R$.
(4) $(A]\cdot(B]\subseteq(A\cdot B]$.

Proof. (1): Let $a\in A$. Since $a\leq a$ and $a\in R$, we get $a\in(A]$. Hence $A\subseteq(A]$.

(2): From (1), we have $(A]\subseteq((A]]$. Let $a\in((A]]$. Then $a\leq u$ for some $u\in(A]$.

Since $u \in (A]$, it follows that $u \leq a'$ for some $a' \in A$. So, $a \leq a'$ for some $a' \in A$. This implies that $a \in (A]$. Thus, $((A]] = (A]$.

(3): It is obvious.

(4): Let $x \in (A] \cdot (B]$. Then, $x = a \cdot b$, where $a \in (A]$ and $b \in (B]$. Since $a \in (A]$, it follows that $a \leq u$ for some $u \in A$. Similarly, $b \leq v$ for some $v \in B$. By hypothesis, we have $a \cdot b \leq u \cdot b \leq u \cdot v$. So, $x \leq u \cdot v$ for some $u \cdot v \in A \cdot B$. Hence $x \in (A \cdot B]$. Therefore, $(A] \cdot (B] \subseteq (A \cdot B]$. □

Theorem 3.1. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. If $\{A_i \mid i \in I\}$ is a family of subhyperrings of R, then $\bigcap_{i \in I} A_i$ is a subhyperring of R.*

Proof. Clearly, $(\bigcap_{i \in I} A_i, +)$ is a subhypergroup of $(R, +)$ and $(\bigcap_{i \in I} A_i) \cdot (\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} A_i$. By (1) of Lemma 3.1, $\bigcap_{i \in I} A_i \subseteq (\bigcap_{i \in I} A_i]$. Now, let $x \in (\bigcap_{i \in I} A_i]$. Then, $x \leq u$ for some $u \in \bigcap_{i \in I} A_i$. Thus, $x \leq u$, where $u \in A_i$, for all $i \in I$. So, $x \in (A_i] = A_i$, for all $i \in I$. Hence, $x \in \bigcap_{i \in I} A_i$. Therefore, $(\bigcap_{i \in I} A_i] \subseteq \bigcap_{i \in I} A_i$. Hence the proof is completed. □

In general, the union of subhyperrings of an ordered Krasner hyperring is not a subhyperring.

Example 3.1. *Let $R = \{0, a, b, c\}$ be a set with the hyperoperation $+$ and the multiplication $\cdot$ defined as follows:*

$+$	0	a	b	c
0	0	a	b	c
a	a	$\{0, a\}$	c	b
b	b	c	$\{0, b\}$	a
c	c	b	a	$\{0, c\}$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Then, $(R, +, \cdot)$ is a Krasner hyperring [23]*. We have $(R, +, \cdot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:*

$$\leq := \{(0, 0), (a, a), (b, b), (c, c)\}.$$

Here, $A = \{0, a\}$ and $B = \{0, b\}$ are subhyperrings of R. Since $a + b = c \notin A \cup B$, it follows that $A \cup B$ is not a subhyperring of R.

Now, we show that under an additional hypothesis the union of subhyperrings is a subhyperring.

Theorem 3.2. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. If $\{A_i \mid i \in I\}$ is a family of subhyperrings of R such that $A_i \subseteq A_j$ or $A_j \subseteq A_i$, for all $i, j \in I$, then $\bigcup_{i \in I} A_i$ is a subhyperring of R.*

Proof. Since $0 \in \bigcup_{i\in I} A_i$, it follows that $\bigcup_{i\in I} A_i \neq \emptyset$. Let $a, b \in \bigcup_{i\in I} A_i$. Then $a \in A_s$ and $b \in A_t$ for some $s, t \in I$. If, say, $A_s \subseteq A_t$, then both a, b are inside A_t, so we have $a + b \subseteq A_t$. This implies that $a + b \subseteq \bigcup_{i\in I} A_i$. Similarly, we have $-a, a \cdot b, b \cdot a \in \bigcup_{i\in I} A_i$.

Now, let $a \in \bigcup_{i\in I} A_i$, $x \in R$ and $x \leq a$. Then $a \in A_i$ for some $i \in I$. Since A_i is a subhyperring of R, it follows that $x \in A_i \subseteq \bigcup_{i\in I} A_i$. This completes the proof. □

The kernel of a derivation is defined as $Ker(d) = \{x \in R \mid d(x) = 0\}$.

Theorem 3.3. *Let $(R, +, \cdot, \leq)$ be a positive ordered Krasner hyperring. Then $Ker(d)$ is a d-subhyperring of R.*

Proof. Since $0 \in Ker(d)$, it follows that $Ker(d) \neq \emptyset$. Let $a, b \in Ker(d)$. Then $d(a) = 0 = d(b)$. Since d is a derivation, it follows that $d(a + b) \subseteq d(a) + d(b) = 0 + 0 = \{0\}$. So, $a + b \subseteq Ker(d)$. Also, $d(-a) = -d(a) = -0 = 0$. Hence, $-a \in Ker(d)$. On the other hand, $d(a \cdot b) \in d(a) \cdot b + a \cdot d(b) = 0 \cdot b + a \cdot 0 = 0 + 0 = \{0\}$. So, we have $a \cdot b \in Ker(d)$. Now, let $a \in Ker(d)$, $x \in R$ and $x \leq a$. Since d is a derivation, it follows that $d(x) \leq d(a) = 0$. Thus $d(x) = 0$. So, we have $x \in Ker(d)$. Therefore, $Ker(d)$ is a subhyperring of R. Now, it remains to show that $d(y) \in Ker(d)$ for all $y \in Ker(d)$. If $y \in Ker(d)$, then $d(y) = 0 \in Ker(d)$. This completes the proof. □

Theorem 3.4. *Let $(R, +, \cdot, \leq)$ be a positive ordered Krasner hyperring. Then $Ker(d)$ is an injective subhyperring of R.*

Proof. The proof is straightforward. □

Lemma 3.2. *Let d be a derivation of an ordered Krasner hyperring $(R, +, \cdot, \leq)$. Then the following statements are equivalent:*

(1) $\{0\}$ *is an injective subhyperring of R with respect to d.*

(2) $Ker(d) = \{0\}$.

(3) $d(x) = 0$ *implies that $x = 0$, for all $x \in R$.*

Proof. (1) ⇒ (2): Assume that (1) holds. Let $a \in Ker(d)$. Then $d(a) = 0 = d(0)$. Since $\{0\}$ is an injective subhyperring, we obtain $a \in \{0\}$. So, we have $Ker(d) = \{0\}$.

(2) ⇒ (3): Let $x \in R$ and suppose that $d(x) = 0$, for all $x \in R$. Then $x \in Ker(d) = \{0\}$. So, we have $x = 0$.

(3) ⇒ (1): Assume that (3) holds. Let $d(a) = d(b)$ and $a \in \{0\}$. Then $d(b) = d(a) = d(0) = 0$. By hypothesis, we have $b = 0$. This means that $b \in \{0\}$. Therefore, $\{0\}$ is an injective subhyperring of R . □

Theorem 3.5. *Let $(R, +, \cdot, \leq)$ be a positive ordered Krasner hyperring. Then $Ker(d)$ is the smallest injective subhyperring of R.*

Proof. By Theorem 3.4, $Ker(d)$ is an injective subhyperring of R. Let A be an injective subhyperring of R with respect to d. We claim that $Ker(d) \subseteq A$. Consider $x \in Ker(d)$. Then $d(x) = 0 = d(0)$. Since A is an injective subhyperring, it follows that $x \in A$. Hence, $Ker(d) \subseteq A$. □

Now, we discuss the link between convex ordered Krasner hyperrings associated to strongly regular relations and ordered rings.

Theorem 3.6. *Let us follow the notations and definitions used in Theorem 1.1. If $(R, +, \cdot, \leq)$ is a convex ordered Krasner hyperring associated to σ, then $(R/\sigma, \oplus, \odot, \preceq)$ is an ordered ring.*

Proof. By Theorem 1.1, $(R/\sigma, \oplus, \odot, \preceq)$ is a preordered ring. It remains to show only that $\preceq$ is also an antisymmetric relation. Let $[a]_\sigma \preceq [b]_\sigma$ and $[b]_\sigma \preceq [a]_\sigma$ in R/σ. Take $a \in [a]_\sigma$; then there exists $b_1 \in [b]_\sigma$ such that $a \leq b_1$. For this $b_1 \in [b]_\sigma$ there exists $a_1 \in [a]_\sigma$ such that $b_1 \leq a_1$. This implies that $a \leq b_1 \leq a_1$. Since R is a convex ordered Krasner hyperring, we get $[a]_\sigma = [b]_\sigma$. Hence the proof is completed. □

At the end of the paper, we obtain the following theorem.

Theorem 3.7. *Let us follow the notations and definitions used in Theorem 1.1. Let $(R, +, \cdot, \leq)$ be a convex ordered Krasner hyperring associated to a d-strongly regular relation σ. If d is a strong derivation of R satisfies the following condition*

$$d(a)\sigma d(b) \;\Rightarrow\; a\sigma b,$$

then there exists an injective strong derivation on $(R/\sigma, \oplus, \odot, \preceq)$.

Proof. By Theorem 3.6, $(R/\sigma, \oplus, \odot, \preceq)$ is an ordered ring. Define $\varphi : R/\sigma \to R/\sigma$ by $\varphi([x]_\sigma) = [d(x)]_\sigma$ for all $[x]_\sigma \in R/\sigma$. Suppose that $[a]_\sigma = [b]_\sigma$, where $[a]_\sigma, [b]_\sigma \in R/\sigma$. Since σ is a d-strongly regular relation, we hava $[d(a)]_\sigma = [d(b)]_\sigma$. Then, $\varphi([a]_\sigma) = \varphi([b]_\sigma)$. Therefore, φ is a well defined map. If $[a]_\sigma$ and $[b]_\sigma$ are two arbitrary elements of R/σ such that $\varphi([a]_\sigma) = \varphi([b]_\sigma)$, then $[d(a)]_\sigma = [d(b)]_\sigma$. So, $d(a)\sigma d(b)$. Hence, we obtain $a\sigma b$. This means that $[a]_\sigma = [b]_\sigma$. Therefore, φ is injective. Now, we show that φ is a strong derivation on $(R/\sigma, \oplus, \odot, \preceq)$. Suppose that $[a]_\sigma$ and $[b]_\sigma$ are two arbitrary elements of R/σ. Then,

$$\begin{aligned} \varphi([a]_\sigma \oplus [b]_\sigma) &= \varphi(\{[u]_\sigma \mid u \in a+b\} \\ &= \{[d(u)]_\sigma \mid u \in a+b\}. \end{aligned}$$

Also, we have

$$\begin{aligned}\varphi([a]_\sigma) \oplus \varphi([b]_\sigma) &= [d(a)]_\sigma \oplus [d(b)]_\sigma \\ &= \{[v]_\sigma \mid v \in d(a) + d(b)\} \\ &= \{[v]_\sigma \mid v \in d(a+b)\} \\ &= \{[d(w)]_\sigma \mid w \in a+b\}.\end{aligned}$$

Hence, $\varphi([a]_\sigma \oplus [b]_\sigma) = \varphi([a]_\sigma) \oplus \varphi([b]_\sigma)$, and so the first condition of the definition of strong derivation is verified. Now, let $[a]_\sigma$ and $[b]_\sigma$ be two arbitrary elements of R/σ. Then,

$$\varphi([a]_\sigma \odot [b]_\sigma) = \varphi([a \cdot b]_\sigma) = [d(a \cdot b)]_\sigma.$$

Also, we have

$$\begin{aligned}(\varphi([a]_\sigma) \odot [b]_\sigma) \oplus ([a]_\sigma \odot \varphi([b]_\sigma)) &= ([d(a)]_\sigma \odot [b]_\sigma) \oplus ([a]_\sigma \odot [d(b)]_\sigma) \\ &= [d(a) \cdot b]_\sigma \oplus [a \cdot d(b)]_\sigma \\ &= \{[t]_\sigma \mid t \in d(a) \cdot b + a \cdot d(b)\}.\end{aligned}$$

Since $d(a \cdot b) \in d(a) \cdot b + a \cdot d(b)$, we get $[d(a \cdot b)]_\sigma \in \{[t]_\sigma \mid t \in d(a) \cdot b + a \cdot d(b)\}$. This implies that $\varphi([a]_\sigma \odot [b]_\sigma) \in (\varphi([a]_\sigma) \odot [b]_\sigma) \oplus ([a]_\sigma \odot \varphi([b]_\sigma))$, and hence the second condition of the definition of strong derivation is verified. Now, let $[a]_\sigma, [b]_\sigma \in R/\sigma$ and $[a]_\sigma \preceq [b]_\sigma$. Take any $a_1 \in [a]_\sigma$; then there exists $b_1 \in [b]_\sigma$ such that $a_1 \leq b_1$. By hypothesis, we have $d(a_1) \leq d(b_1)$. Since σ is a d-strongly regular relation, we get $[d(a_1)]_\sigma = [d(a)]_\sigma$ and $[d(b_1)]_\sigma = [d(b)]_\sigma$. Hence for every $d(a_1) \in [d(a)]_\sigma$ there exists some $d(b_1) \in [d(b)]_\sigma$ such that $d(a_1) \leq d(b_1)$. This means that $[d(a)]_\sigma \preceq [d(b)]_\sigma$. Thus, $\varphi([a]_\sigma) \preceq \varphi([b]_\sigma)$, and so the third condition of the definition of strong derivation is verified. Therefore, φ is a strong derivation and the proof is completed. □

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