

AN ASPECT OF LOCAL PROPERTY OF FOURIER SERIES

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ABSTRACT. In this paper, a general theorem concerning $|\bar{N}, p_n, \theta_n|_k$ summability factors of Fourier series is obtained. Using this result we study a localization problem for Fourier series.

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with the partial sums (s_n) . By u_n^α and t_n^α we denote the n th Cesàro means of order α , with $\alpha > -1$, of the sequences (s_n) and (na_n) , respectively, that is (see [6])

$$(1.1) \quad u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^n A_{n-v}^{\alpha-1} s_v \text{ and } t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^n A_{n-v}^{\alpha-1} v a_v, \quad (t_n^1 = t_n)$$

where

$$(1.2) \quad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \text{for } n > 0.$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k, k \geq 1$, if (see [7])

$$(1.3) \quad \sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty.$$

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If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$(1.4) \quad P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$(1.5) \quad v_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

defines the sequence (v_n) of the Riesz mean or simply the (\bar{N}, p_n) mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) (see [8]). Let (θ_n) be any sequence of positive constants. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n, \theta_n|_k$, $k \geq 1$, if (see [14])

$$(1.6) \quad \sum_{n=1}^{\infty} \theta_n^{k-1} |v_n - v_{n-1}|^k < \infty.$$

If we take $\theta_n = \frac{p_n}{p_n}$, then $|\bar{N}, p_n, \theta_n|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability (see [2]). Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of n , then we have $|C, 1|_k$ summability. Furthermore, if we take $\theta_n = n$, then $|\bar{N}, p_n, \theta_n|_k$ summability reduces to $|R, p_n|_k$ summability (see [3]).

2. KNOWN RESULTS FOR $|\bar{N}, p_n|_k$ SUMMABILITY

Let f be a periodic function with period 2π , and integrable (L) over $[-\pi, \pi]$. Let the Fourier series of f be

$$f(t) \sim \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_0^{\infty} C_n(t).$$

We write

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\}.$$

The local property problem of the factored Fourier series plays an important role in many areas of applied mathematics and mechanics. It is familiar that the convergence

of the Fourier series at any point $t = x$ is a local property of the generating function f . That is, it depends only on the behaviour of f in an arbitrarily small neighbourhood of x . Therefore the summability of this series at the point by any regular linear summability method is also a local property of f . Lal [10] proved the following theorem concerning summability factors for $|\bar{N}, p_n|$ summability.

Theorem 2.1. *If $\{s_n\}$ is a bounded sequence and $\{\lambda_n\}$ satisfies*

$$(2.1) \quad \sum_1^{\infty} \frac{p_n}{P_n} |\lambda_n| < \infty,$$

and

$$(2.2) \quad \sum_1^{\infty} |\Delta \lambda_n| < \infty,$$

then $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|$.

He deduced from Theorem 2.1 that under the conditions (2.1) and (2.2) the summability $|\bar{N}, p_n|$ of $\sum \lambda_n C_n(t)$ is a local property of the generating function.

Later on, Bor [4] extended Theorem 2.1 to Theorem 2.2 concerning the summability $|\bar{N}, p_n|_k$, $k \geq 1$ (see also [5]).

Theorem 2.2. *If $\{s_n\}$ is a bounded sequence, and*

$$(2.3) \quad \sum_1^{\infty} \frac{p_n}{P_n} |\lambda_n|^k < \infty,$$

and (2.2) holds, then $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Mazhar [11] generalized Theorem 2.2 dealing with $|\bar{N}, p_n|_k$ summability factors of infinite series to Theorem 2.3.

Theorem 2.3. *If*

$$(2.4) \quad \sum_1^{\infty} \frac{p_n}{P_n} |s_n \lambda_n|^k < \infty$$

and

$$(2.5) \quad \sum_1^{\infty} |s_n| |\Delta \lambda_n| < \infty,$$

then $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

In [11], an interesting result obtained from Theorem 2.3 is the following form.

Corollary 2.1. *If*

$$(2.6) \quad \sum_1^{\infty} \frac{p_n}{P_n} |s_n|^k < \infty, \quad k \geq 1,$$

then $\sum a_n$ is summable $|\bar{N}, p_n|_k$.

We will use this corollary in Section 4 to obtain a result on local property of the summability $|\bar{N}, p_n, \theta_n|_k$ of Fourier series.

3. KNOWN RESULTS FOR LOCAL PROPERTY OF FOURIER SERIES

Izumi [9] and Mohanty [12] independently proved that summability $|R, \log n, 1|$ of a Fourier series is not a local property of the generating function. Izumi showed that if

$$(3.1) \quad C_n(x) = O(\log n^{-2}),$$

then the summability $|R, \log n, 1|$ of the Fourier series $\sum C_n(t)$ at $t = x$ is a local property, that is the summability $|R, \log n, 1|$ of the Fourier series depends only on the behaviour of the generating function in the immediate neighbourhood of the point concerned. Later on, Mohanty and Izumi [13] improved this result by establishing the following theorem.

Theorem 3.1. *If*

$$(3.2) \quad \sum_1^{\infty} \frac{|C_n(x)|}{n} \log \log n < \infty,$$

then the summability $|R, \log n, 1|$ of $\sum C_n(x)$ depends only on a local condition.

Theorem 3.1 was further generalized by Bhatt [1] in the following manner.

Theorem 3.2. *If*

$$(3.3) \quad \sum_1^{\infty} \frac{|C_n(x)|}{n \log n} < \infty,$$

then the summability $|R, \log n, 1|$ of $\sum C_n(t)$ depends only on the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the point $t = x$.

Mazhar [11] proved the following theorem dealing with local property of the summability $|\bar{N}, p_n|_k$ of a Fourier series. This result includes all the previous results.

Theorem 3.3. *Let $\{p_n\}$ satisfy the following conditions:*

$$(3.4) \quad \left\{ \frac{p_n}{P_n} \right\} \downarrow$$

$$(3.5) \quad \frac{P_{2n}}{p_{2n}} = O\left(\frac{P_n}{p_n}\right)$$

$$(3.6) \quad \sum_{n=1}^{\infty} \frac{p_n}{nP_n} < \infty,$$

and

$$(3.7) \quad \sum_{n=1}^{\infty} \frac{|C_n(x)|^k p_n}{P_n} < \infty,$$

then the summability $|\bar{N}, p_n|_k$ of $\sum C_n(t)$ depends only on the behaviour of the generating function at the point $t = x$.

4. MAIN RESULTS

In this section, we propose to examine the local property of the summability $|\bar{N}, p_n, \theta_n|_k$ of a Fourier series. We will prove the following theorem.

Theorem 4.1. *Let $\left(\frac{\theta_n p_n}{P_n}\right)$ be a non-increasing sequence. If the conditions of Theorem 3.3 are satisfied with the condition (3.5) replaced by:*

$$(4.1) \quad \left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{P_{2n}}{p_{2n}} = O\left(\frac{P_n}{p_n}\right)$$

then, the summability $|\bar{N}, p_n, \theta_n|_k$ of $\sum C_n(t)$ depends only on the behaviour of the generating function at the point $t = x$.

If we put $\theta_n = \frac{P_n}{p_n}$ in Theorem 4.1, then we have Theorem 3.3. In this case condition (4.1) reduces to condition (3.5) and the condition $\left(\frac{\theta_n p_n}{P_n}\right)$ which is a non-increasing sequence is automatically satisfied.

5. PROOF OF THEOREM 4.1

Proof. Let $s_n(x)$ denote the n -th partial sum of $\sum C_n(x)$. We assume that the constant term is zero, and $0 < \eta < \pi$ then

$$\begin{aligned} s_n(x) &= \sum_{v=1}^n C_v(x) \\ &= \frac{1}{2\pi} \int_0^\pi \varphi(u) \frac{\sin(n + \frac{1}{2})u}{\sin \frac{u}{2}} du, \end{aligned}$$

where $\varphi(u)$ is the appropriate function that makes the equation equal to $\sum_{v=1}^n C_v(x)$,

so

$$\begin{aligned} &= \frac{1}{2\pi} \left[\int_0^\eta \frac{\varphi(u) \sin \frac{u}{2}}{\sin^2 \frac{\eta}{2}} \sin(n + \frac{1}{2})u du + \int_\eta^\pi \varphi(u) \frac{\sin(n + \frac{1}{2})u}{\sin \frac{u}{2}} du \right] \\ &+ \frac{1}{2\pi} \int_0^\eta \varphi(u) \left[1 - \left(\frac{\sin \frac{u}{2}}{\sin \frac{\eta}{2}} \right)^2 \right] \frac{\sin(n + \frac{1}{2})u}{\sin \frac{u}{2}} du \\ &= \frac{1}{2\pi} [Q_n + R_n]. \end{aligned}$$

Thus to prove the summability $|\bar{N}, p_n, \theta_n|_k$ of $\sum C_n(x)$, it is enough, in view of Corollary 2.1 to prove that

$$(5.1) \quad \sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |Q_n|^k < \infty,$$

$$(5.2) \quad \sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |R_n|^k < \infty, \quad k \geq 1.$$

The convergence of (5.2) depends on the behaviour of the generating function in the neighbourhood of the point x . Therefore to prove Theorem 4.1 it is sufficient to show (5.1).

Proceeding as in [1] we have

$$\begin{aligned} Q_n &= O(1) \left[\sum_{v=-\infty}^0 + \sum_{v=1}^{n-1} + \sum_{v=n+1}^{m+1} + \sum_{v=n+m+1}^{\infty} \right] \times \frac{C_v}{(n-v)^2} + O(|C_n(x)|) \\ &= O(1) [M_1 + M_2 + M_3 + M_4] + O(|C_n(x)|), \end{aligned}$$

where we write $C_{-v}(x) = C_v(x) = C_v$. To prove (5.1), we have to show that

$$(5.3) \quad \sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |M_r|^k < \infty, \quad r = 1, 2, 3, 4$$

in view of (3.7). Firstly by applying Hölder's inequality we have

$$\begin{aligned} \sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |M_1|^k &= \sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\sum_{v=0}^{\infty} \frac{|C_v|}{(n-v)^2} \right)^k \\ &\leq K \sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{n^k} \leq K \sum_{n=1}^{\infty} \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} \left(\frac{p_n}{n P_n} \right) \leq K \left(\frac{\theta_1 p_1}{P_1} \right)^{k-1} \sum_{n=1}^{\infty} \left(\frac{p_n}{n P_n} \right) < \infty, \end{aligned}$$

since $|C_v| = O(1)$ and by virtue of the hypotheses of Theorem 4.1. Also,

$$\begin{aligned}
\sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |M_2|^k &= O(1) \sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\sum_{v=1}^{n-1} \frac{|C_v|}{(n-v)^2} \right)^k \\
&= O(1) \sum_{n=1}^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\sum_1^{n-1} \frac{|C_{n-m}|}{m^2} \right)^k = O(1) \sum_1^{\infty} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \sum_{m=1}^{n-1} \frac{|C_{n-m}|^k}{m^2} \\
&= O(1) \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=m+1}^{\infty} \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{|C_{n-m}|^k p_n}{P_n} = O(1) \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=m+1}^{\infty} \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{|C_{n-m}|^k p_{n-m}}{P_{n-m}} \\
&= O(1) \sum_{m=1}^{\infty} \frac{1}{m^2} \left(\frac{\theta_1 p_1}{P_1} \right)^{k-1} \sum_{n=m+1}^{\infty} \frac{|C_{n-m}|^k p_{n-m}}{P_{n-m}} = O(1),
\end{aligned}$$

by virtue of the hypotheses of Theorem 4.1. Again, we have that

$$\begin{aligned}
\sum_1^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |M_3|^k &= O(1) \sum_1^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\sum_{v=n+m+1}^{\infty} \frac{|C_v|}{(n-v)^2} \right)^k \\
&= O(1) \sum_1^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{(m+1)^k} = O(1) \frac{1}{m+1} \sum_1^m \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{p_n}{P_n} \\
&= O(1) \frac{1}{m+1} \left(\frac{\theta_1 p_1}{P_1} \right)^{k-1} \sum_1^m \frac{p_n}{P_n} = O(1),
\end{aligned}$$

by virtue of the hypotheses of Theorem 4.1. Finally, we have that

$$\begin{aligned}
\sum_1^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |M_4|^k &= O(1) \sum_1^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \left(\sum_{v=1}^m \frac{|C_{n+v}|}{v^2} \right)^k \\
&= O(1) \sum_1^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \sum_{v=1}^m \frac{|C_{n+v}|^k}{v^2} = O(1) \sum_{v=1}^m \frac{1}{v^2} \sum_{n=1}^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |C_{n+v}|^k \\
&= O(1) \sum_{v=1}^m \frac{1}{v^2} \left(\sum_{n=1}^v + \sum_{n=v+1}^m \right) \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |C_{n+v}|^k = L_1 + L_2.
\end{aligned}$$

We have that

$$\begin{aligned}
 L_1 &= O(1) \sum_{v=1}^m \frac{1}{v^2} \sum_{n=1}^v \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |C_{n+v}|^k = O(1) \sum_{v=1}^m \frac{1}{v^2} \sum_{n=1}^v \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \\
 &= O(1) \sum_{n=1}^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \sum_{v=n}^m \frac{1}{v^2} = O(1) \sum_{n=1}^{\infty} \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} \left(\frac{p_n}{n P_n} \right) \\
 &= O(1) \left(\frac{\theta_1 p_1}{P_1} \right)^{k-1} \sum_{n=1}^{\infty} \left(\frac{p_n}{n P_n} \right) = O(1),
 \end{aligned}$$

by virtue of the hypotheses of Theorem 4.1. Also, we have that

$$\begin{aligned}
 L_2 &= O(1) \sum_{v=1}^m v^{-2} \left(\sum_{n=v+1}^m \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |C_{n+v}|^k \right) \\
 &= O(1) \sum_{v=1}^m v^{-2} \left(\sum_{n=v+1}^{\infty} \frac{p_{n+v}}{P_{n+v}} |C_{n+v}|^k \cdot \frac{P_{n+v}}{p_{n+v}} \theta_n^{k-1} \left(\frac{p_n}{P_n} \right)^k \right) \\
 &= O(1) \sum_{v=1}^m v^{-2} \left(\sum_{n=v+1}^{\infty} \frac{p_{n+v}}{P_{n+v}} |C_{n+v}|^k \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} \frac{P_{2n}}{p_{2n}} \left(\frac{p_n}{P_n} \right) \right) \\
 &= O(1) \sum_{v=1}^m v^{-2} \sum_{n=v+1}^{\infty} \frac{p_{n+v}}{P_{n+v}} |C_{n+v}|^k = O(1),
 \end{aligned}$$

in view of (4.1) and by virtue of the hypotheses of Theorem 4.1. Therefore we obtain that (5.1). This completes the proof of the Theorem 4.1. \square

6. CONCLUSIONS

1. If we take $\theta_n = n$ and $p_n = 1$ for all values of n , then we have a new result about $|C, 1|_k$ summability factors of Fourier series.
2. If we take $\theta_n = n$, $p_n = 1$ for all values of n and if $\sum_1^{\infty} \frac{|C_n(x)|^k}{n} < \infty$, then the summability $|C, 1|_k$ of $\sum C_n(x)$ is a local property.
3. If we take $p_n = 1$ for all values of n , then we have a new result for $|C, 1, \theta_n|_k$ summability factors.

4. If we take $\theta_n = n$, then we have another new result for $|R, p_n|_k$ summability factors.

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