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DISCUSSION ON α-CONTRACTIONS AND RELATED FIXED POINT THEOREMS IN HAUSDORFF b-GAUGE SPACES

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ABSTRACT. In this paper, we have used the structure of b-gauge spaces to prove fixed point theorems for multivalued mappings in the lights of α-contraction. Towards the end, the validity of our results have been insured by discussing a possible application and example.

1. Introduction and Preliminaries

Characterization of gauge spaces can be made by the fact that the distance between two distinct points of the space may be zero, which has been the center of interest for many researchers world wide. For details on gauge spaces, we refer[1]. As we know that the Banach contraction principle is considered as the most fundamental entity and its inception has opened a new era in metric fixed point theory. The generalization of this famous result in different dimensions has shown to be promising. Frigon [2] and Chis and Precup [3] generalized the Banach contraction principle on gauge spaces. For more intersecting results on gauge spaces, the readers can look into [4, 5, 6, 7, 8, 9].

Another interesting result is the introduction of the notion of b-metric space by Czerwik [10]. Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is said to be a b-metric on X, if there exists $s \geq 1$ such that for each $x, y, z \in X$, we have (i)

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$d(x,y)=0$ if and only if $x=y$ (ii) $d(x,y)=d(y,x)$ (iii) $d(x,z)\leq s[d(x,y)+d(y,z)]$. The triplet (X,d,s) is said to be a b-metric space. Note that every metric space is a b-metric but converse is not true. Convergence of a sequence in a b-metric space is defined in a similar fashion as in a metric space. A sequence $\{x_n\}\subset X$ is a Cauchy sequence in (X,d,s), if for each $\epsilon>0$ there exists a natural number $N(\epsilon)$ such that $d(x_n,x_m)<\epsilon$ for each $m,n\geq N(\epsilon)$. A b-metric space (X,d,s) is complete if each Cauchy sequence in X converges to some point of X. Czerwik [10] extended Banach contraction principle for self mappings on b metric spaces. Czerwik [11] further extended the notion of a b-metric space (X,d,s) by defining Hausdorff metric for the space of all nonempty closed and bounded subsets of the b-metric space (X,d,s). Let (X,d,s) be a b metric space, for $x\in X$ and $A\subset X$, $d(x,A)=\inf\{d(x,a):a\in A\}$. Denote $CB(X)$ as the class of all nonempty closed and bounded subsets of X and $CL(X)$ as the class of all nonempty closed subsets of X. For $A,B\in CB(X)$, the function $H:CB(X)\times CB(X)\to[0,\infty)$ defined by $H(A,B)=\max\left\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\right\}$ is said to be a Hausdorff b-metric induced by a b-metric space (X,d,s). A Hausdorff b metric space enjoys the same properties as a Hausdorff metric, expect for triangular inequality which in Hausdorff b metric space takes the form $H(A,B)\leq s[H(A,C)+H(C,B)]$. Czerwik [11] extended Nadler's fixed point theorem in the setting of Hausdorff b metric spaces. On other hand, Semat *et al*. [12] also succeeded to generalized Banach contraction condition by introducing α-ψ-contraction. Many authors appreciate this condition, which can be seen in [13, 15, 16, 17, 18, 19, 20, 21, 22, 23].

By using b-metric spaces, in this paper, we will first discuss the notion of b_s-gauge spaces introduced by [14]. Then, we extend this notion to define b_s-gauge structure on the space of nonempty closed subsets of the b metric space and prove some fixed point theorems for multivalued $\alpha-$contractions. To substantiate our main results, we construct an example. Moreover, we also discuss a possible application of our results to solve an integral equation.

Definition 1.1. [14] *Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called b_s-pseudo metric on X if there exists $s \geq 1$ such that for each $x, y, z \in X$, we have*

: *(i) $d(x, x) = 0$ for each $x \in X$;*

: *(ii) $d(x, y) = d(y, x)$;*

: *(iii) $d(x, z) \leq s[d(x, y) + d(y, x)]$.*

Remark 1.1. [14] *Every b-metric space (X, d, s) is a b_s-pseudo metric space, but the converse is not true.*

Example 1.1. [14] *Let $X = C([0, \infty), \mathbb{R})$. Define a function $d : X \times X \to [0, \infty)$ by $d(x(t), y(t)) = \max_{t\in[0,1]}(x(t) - y(t))^2$. Then*

: *(i) Clearly d is not a metric on X;*

: *(ii) d is not a pseudo metric on X, since $x, y, z \in C([0, \infty), \mathbb{R})$ defined by*

$$x(t) = \begin{cases} 0 \text{ if } 0 \leq t \leq 1 \\ t - 1 \text{ if } t > 1, \end{cases}$$

$y(t) = 3$ for each $t \geq 0$ and $z(t) = -3$ for each $t \geq 0$. Then $d(y, z) = 36 \nleq 18 = d(y, x) + d(x, z)$.

: *(iii) d is not a b-metric on X, since $u, v \in C([0, \infty), \mathbb{R})$ defined by*

$$u(t) = \begin{cases} 0 \text{ if } 0 \leq t \leq 1 \\ t - 1 \text{ if } t > 1, \end{cases}$$

and

$$v(t) = \begin{cases} 0 \text{ if } 0 \leq t \leq 1 \\ 2t - 2 \text{ if } t > 1. \end{cases}$$

Then $u \neq v$, but $d(u,v) = 0$.

: *(iv) d is b_2-pseudo metric on X with $s = 2$.*

In order to define gauge spaces in the setting of b_s-pseudo metrics, we need to define the following definitions.

Definition 1.2. [14] *Let X be a nonempty set endowed with the b_s-pseudo metric d. The d_s-ball of radius $\epsilon > 0$ centered at $x \in X$ is the set*

$$B(x,d,\epsilon) = \{y \in X : d(x,y) < \epsilon\}.$$

Definition 1.3. [14] *A family $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ of b_s-pseudo metrics is said to be separating if for each pair (x,y) with $x \neq y$, there exists $d_\nu \in \mathfrak{F}$ with $d_\nu(x,y) \neq 0$.*

Definition 1.4. [14] *Let X be a nonempty set and $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ be a family of b_s-pseudo metrics on X. The topology $\mathfrak{T}(\mathfrak{F})$ having subbases in the family*

$$\mathfrak{B}(\mathfrak{F}) = \{B(x,d_\nu,\epsilon) : x \in X, d_\nu \in \mathfrak{F} \text{ and } \epsilon > 0\}$$

of balls is called topology induced by the family $\mathfrak{F}$ of b_s-pseudo metrics. The pair $(X,\mathfrak{T}(\mathfrak{F}))$ is called a b_s-gauge space. Note that $(X,\mathfrak{T}(\mathfrak{F}))$ is Hausdorff if $\mathfrak{F}$ is separating.

Definition 1.5. [14] *Let $(X,\mathfrak{T}(\mathfrak{F}))$ be a b_s-gauge space with respect to the family $\mathfrak{F} = \{d_\nu : \nu \in \mathfrak{A}\}$ of b_s-pseudo metrics on X and $\{x_n\}$ is a sequence in X and $x \in X$. Then:*

: *(i): the sequence $\{x_n\}$ converges to x if for each $\nu \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_\nu(x_n,x) < \epsilon$ for each $n \geq N_0$. We denote it as $x_n \to^{\mathfrak{F}} x$;*

: *(ii) the sequence $\{x_n\}$ is a Cauchy sequence if for each $\nu \in \mathfrak{A}$ and $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d_\nu(x_n,x_m) < \epsilon$ for each $n,m \geq N_0$;*

: *(iii) $(X,\mathfrak{T}(\mathfrak{F}))$ is complete if each Cauchy sequence in $(X,\mathfrak{T}(\mathfrak{F}))$ is convergent in X;*

: *(iv) a subset of X is said to be closed if it contains the limit of each convergent sequence of its elements.*

Remark 1.2. [14] *When $s = 1$, then all the above definitions reduce to the corresponding definitions in a gauge space.*

2. Main results

Through out this paper, $\mathfrak{A}$ is directed set and X is a nonempty set endowed with a separating complete b_s-gauge structure $\{d_\nu : \nu \in \mathfrak{A}\}$. Further, $\alpha : X \times X \to [0, \infty)$ is a mapping. For each $d_\nu \in \mathfrak{F}$, $CL_\nu(X)$ denote the set of all nonempty closed subsets of X with respect to d_v. For each $\nu \in \mathfrak{A}$ and $A, B \in CL_\nu(X)$, the function $H_\nu : CL_\nu(X) \times CL_\nu(X) \to [0, \infty)$ defined by

$$H_\nu(A,B) = \begin{cases} \max\Big\{ \sup_{x\in A} d_\nu(x,B), \sup_{y\in B} d_\nu(y,A)\Big\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

is a generalized Hausdorff b_s-pseudo metric on $CL_\nu(X)$. We denote $CL(X)$ as the set of all nonempty closed subsets in the b_s-gauge space $(X, \mathfrak{T}(\mathfrak{F}))$.

Theorem 2.1. *Let $T : X \to CL(X)$ be a mapping such that for each $\nu \in \mathfrak{A}$, we have*

$$\begin{aligned} H_\nu(Tx,Ty) \;\; &\leq \;\; a_\nu d_\nu(x,y) + b_\nu d_\nu(x,Tx) + c_\nu d_\nu(y,Ty) + e_\nu d_\nu(x,Ty) \\ &\quad + L_\nu d_\nu(y,Tx) \;\forall\; \alpha(x,y) \geq 1 \end{aligned} \tag{2.1}$$

where, $a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \geq 0$, and $s^2 a_\nu + s^2 b_\nu + s^2 c_\nu + 2s^3 e_\nu < 1$. Further, assume that the following conditions hold:

: *(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*

: *(ii) if $\alpha(x,y) \geq 1$ then for $u \in Tx$ and $v \in Ty$, we have $\alpha(u,v) \geq 1$;*

: *(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$;*

: *(iv) for each* $\{q_\nu : q_\nu > 1\}_{\nu\in\mathfrak{A}}$ *and* $x \in X$ *there exists* $y \in Tx$ *such that*

$$d_\nu(x, y) \leq q_\nu d_\nu(x, Tx) \;\forall\; \nu \in \mathfrak{A}.$$

Then T *has a fixed point.*

Proof. By hypothesis (i), there exist $x_0, x_1 \in X$ such that $x_1 \in Tx_0$ and $\alpha(x_0, x_1) \geq 1$. Now, it follows from (2.1) that

$$\begin{aligned} H_\nu(Tx_0, Tx_1) \;\leq\;& a_\nu d_\nu(x_0, x_1) + b_\nu d_\nu(x_0, Tx_0) + c_\nu d_\nu(x_1, Tx_1) + e_\nu d_\nu(x_0, Tx_1) \\ &+ L_\nu d_\nu(x_1, Tx_0) \;\forall\; \nu \in \mathfrak{A}. \end{aligned} \tag{2.2}$$

Since $d_\nu(x_1, Tx_1) \leq H_\nu(Tx_0, Tx_1)$ and $d_\nu(x_0, Tx_1) \leq s[d_\nu(x_0, x_1)+d_\nu(x_1, Tx_1)]$, therefore from (2.2), we get

$$d_\nu(x_1, Tx_1) \leq \frac{1}{\xi_\nu} d_\nu(x_0, x_1) \;\forall\; \nu \in \mathfrak{A} \tag{2.3}$$

where, $\xi_\nu = \frac{1-c_\nu-se_\nu}{a_\nu+b_\nu+se_\nu} > 1$. Using hypothesis (iv), there exists $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq \sqrt{\xi_\nu} d_\nu(x_1, Tx_1) \;\forall\; \nu \in \mathfrak{A}. \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$d_\nu(x_1, x_2) \leq \frac{1}{\sqrt{\xi_\nu}} d_\nu(x_0, x_1) \;\forall\; \nu \in \mathfrak{A}. \tag{2.5}$$

Hypothesis (ii), implies that $\alpha(x_1, x_2) \geq 1$. Continuing in the same way, we get a sequence $\{x_m\}$ in X such that $\alpha(x_m, x_{m+1}) \geq 1$ and

$$d_\nu(x_m, x_{m+1}) \leq \Big(\frac{1}{\sqrt{\xi_\nu}}\Big)^m d_\nu(x_0, x_1) \;\forall\; \nu \in \mathfrak{A}.$$

For convenience we assume that $\eta_\nu = \frac{1}{\sqrt{\xi_\nu}}$ for each $\nu \in \mathfrak{A}$. Now we show that $\{x_m\}$ is a Cauchy sequence. For each $m, p \in \mathbb{N}$ and $\nu \in \mathfrak{A}$, we have

$$\begin{aligned} d_\nu(x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\nu(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{m+p-1} s^i (\eta_\nu)^i d_\nu(x_0, x_1) \\ &\leq \sum_{i=m}^{\infty} (s\eta_\nu)^i d_\nu(x_0, x_1) < \infty \ (since\ s\eta_\nu < 1). \end{aligned}$$

This implies that $\{x_m\}$ is a Cauchy sequence in X. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. By using hypothesis (iii), triangular inequality and (2.1), we have

$$\begin{aligned} d_\nu(x^*, Tx^*) &\leq sd_\nu(x^*, x_{m-1}) + sd_\nu(x_{m-1}, Tx^*) \\ &\leq sd_\nu(x^*, x_{m-1}) + sH_\nu(Tx_m, Tx^*) \\ &\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, Tx_m) + \\ &\quad sc_\nu d_\nu(x^*, Tx^*) + se_\nu d_\nu(x_m, Tx^*) + sL_\nu d_n(x^*, Tx_m) \\ &\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, x_{m+1}) + \\ &\quad sc_\nu d_\nu(x^*, Tx^*) + se_\nu d_\nu(x_m, Tx^*) + sL_\nu d_\nu(x^*, x_{m+1}) \\ &\leq sd_\nu(x^*, x_{m-1}) + sa_\nu d_\nu(x_m, x^*) + sb_\nu d_\nu(x_m, x_{m+1}) + \\ &\quad sc_\nu d_\nu(x^*, Tx^*) + se_\nu[sd_\nu(x_m, x^*) + sd_\nu(x^*, Tx^*)] \\ &\quad + sL_\nu d_\nu(x^*, x_{m+1}) \ \forall\ \nu \in \mathfrak{A}. \end{aligned}$$

Letting $m \to \infty$, we get

$$d_\nu(x^*, Tx^*) \leq (sc_\nu + s^2 e_\nu) d_\nu(x^*, Tx^*) \ \forall\ \nu \in \mathfrak{A}.$$

Which is only possible if $d_\nu(x^*, Tx^*) = 0$. Since the structure $\{d_\nu : \nu \in \mathfrak{A}\}$ on X is separating, we have $x^* \in Tx^*$. □

In case of single valued mapping $T : X \to X$, we have the following result:

Corollary 2.1. *Let $T : X \to X$ be a mapping such that for each $\nu \in \mathfrak{A}$ we have*

$$\begin{aligned} d_\nu(Tx, Ty) \;\; &\leq \;\; a_\nu d_\nu(x, y) + b_\nu d_\nu(x, Tx) + c_\nu d_\nu(y, Ty) + e_\nu d_\nu(x, Ty) \\ &\quad + L_\nu d_\nu(y, Tx) \; \forall \; \alpha(x, y) \geq 1 \end{aligned} \tag{2.6}$$

where, $a_\nu, b_\nu, c_\nu, e_\nu, L_\nu \geq 0$, and $sa_\nu + sb_\nu + sc_\nu + 2s^2 e_\nu < 1$. Further, assume that the following conditions hold:

: *(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

: *(ii) if $\alpha(x, y) \geq 1$, then $\alpha(Tx, Ty) \geq 1$;*

: *(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$;*

Then T has a fixed point.

We denoted Ψ_{s^2} as the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that:

: (ψ_1) $\psi(0) = 0$;

: (ψ_2) $\psi(\rho t) = \rho\psi(t) < \rho t$ for each $\rho, t > 0$;

: (ψ_3) $\sum_{i=1}^{\infty} s^{2i}\psi^i(t) < \infty$, where $s \geq 1$.

Note that in the following theorems we have used the structure with $s > 1$.

Theorem 2.2. *Let $T : X \to CL(X)$ be a mapping such that for each $\nu \in \mathfrak{A}$ we have*

$$\begin{aligned} H_\nu(Tx, Ty) \;\; &\leq \;\; \psi_\nu(\max\{d_\nu(x, y), d_\nu(x, Tx), d_\nu(y, Ty), \frac{1}{2s}[d_\nu(x, Ty) + d_\nu(y, Tx)]\}) \\ &\quad + L_\nu d_\nu(y, Tx) \; \forall \; \alpha(x, y) \geq 1 \end{aligned} \tag{2.7}$$

where, $\psi_\nu \in \Psi_{s^2}$ and $L_\nu \geq 0$. Further, assume that the following conditions hold:

: *(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*

: *(ii) if $\alpha(x, y) \geq 1$, for $u \in Tx$ and $v \in Ty$, we have $\alpha(u, v) \geq 1$;*

: *(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$;*

: *(iv) for each $x \in X$, we have $y \in Tx$ such that*

$$d_\nu(x, y) \leq s d_\nu(x, Tx) \ \forall \ \nu \in \mathfrak{A}.$$

Then T has a fixed point.

Proof. By hypothesis, we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. From (2.7), we get

$$\begin{aligned} d_\nu(x_1, Tx_1) &\leq H_\nu(Tx_0, Tx_1) \\ &\leq \psi_\nu(\max\{d_\nu(x_0, x_1), d_\nu(x_0, Tx_0), d_\nu(x_1, Tx_1), \\ &\quad \frac{1}{2s}[d_\nu(x_0, Tx_1) + d_\nu(x_1, Tx_0)]\}) + L_\nu d_\nu(x_1, Tx_0) \\ &\leq \psi_\nu(\max\{d_\nu(x_0, x_1), d_\nu(x_0, x_1), d_\nu(x_1, Tx_1), \\ &\quad \frac{1}{2s}[s(d_\nu(x_0, x_1) + d_\nu(x_1, Tx_1))]\}) + L_\nu.0 \\ &= \psi_\nu(d_\nu(x_0, x_1)) \ \forall \ \nu \in \mathfrak{A}. \end{aligned} \tag{2.8}$$

By hypothesis (iv), for $x_1 \in X$, we have $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq s d_\nu(x_1, Tx_1) \leq s\psi_\nu(d_\nu(x_0, x_1)) \ \forall \ \nu \in \mathfrak{A}. \tag{2.9}$$

Applying ψ_ν, we have

$$\psi_\nu(d_\nu(x_1, x_2)) \leq \psi_\nu(s\psi_\nu(d_\nu(x_0, x_1))) = s\psi_\nu^2(d_\nu(x_0, x_1)) \ \forall \ \nu \in \mathfrak{A}.$$

By hypothesis (ii), it is clear that $\alpha(x_1, x_2) \geq 1$. Again from (2.7), we reach the following inequality after some simplification.

$$d_\nu(x_2, Tx_2) \leq H_\nu(Tx_1, Tx_2) \leq \psi_\nu(d_\nu(x_1, x_2)) \ \forall \ \nu \in \mathfrak{A}. \tag{2.10}$$

By hypothesis (iv), for $x_2 \in X$, we have $x_3 \in Tx_2$ such that

$$d_\nu(x_2, x_3) \leq sd_\nu(x_2, Tx_2) \leq s\psi_\nu(d_\nu(x_1, x_2)) \leq s^2\psi_\nu^2(d_\nu(x_0, x_1)) \ \forall\ \nu \in \mathfrak{A}. \tag{2.11}$$

Clearly, $\alpha(x_2, x_3) \geq 1$. Continuing in the same way, we get a sequence $\{x_m\}$ in X such that $\alpha(x_m, x_{m+1}) \geq 1$ and

$$d_\nu(x_m, x_{m+1}) \leq s^m\psi_\nu^m(d_\nu(x_0, x_1)) \ \forall\ \nu \in \mathfrak{A}.$$

Now, we show that $\{x_m\}$ is a Cauchy sequence. For $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_\nu(x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\nu(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{m+p-1} s^{2i}\psi_\nu^i(d_\nu(x_0, x_1)) < \infty \ \forall\ \nu \in \mathfrak{A}. \end{aligned}$$

This implies that $\{x_m\}$ is a Cauchy sequence in X. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. Using hypothesis (iv), triangular inequality and (2.7), we have

$$\begin{aligned} d_\nu(x^*, Tx^*) &\leq sd_\nu(x^*, x_{m-1}) + sd_\nu(x_{m-1}, Tx^*) \\ &\leq sd_\nu(x^*, x_{m-1}) + sH_\nu(Tx_m, Tx^*) \\ &\leq sd_\nu(x^*, x_{m-1}) + s\psi_\nu(\max\{d_\nu(x_m, x^*), d_\nu(x_m, Tx_m), d_\nu(x^*, Tx^*), \\ &\quad \frac{1}{2s}[d_\nu(x_m, Tx^*) + d_\nu(x^*, Tx_m)]\}) + sL_\nu d_\nu(x^*, Tx_m) \\ &< sd_\nu(x^*, x_{m-1}) + s\max\{d_\nu(x_m, x^*), d_\nu(x_m, x_{m+1}), d_\nu(x^*, Tx^*), \\ &\quad \frac{1}{2s}[d_\nu(x_m, Tx^*) + d_\nu(x^*, x_{m+1})]\} + sL_\nu d_\nu(x^*, x_{m+1}) \\ &\leq sd_\nu(x^*, x_{m-1}) + s\max\{d_\nu(x_m, x^*), d_\nu(x_m, x_{m+1}), d_\nu(x^*, Tx^*), \\ &\quad \frac{1}{2s}[sd_\nu(x_m, x^*) + sd_\nu(x^*, Tx^*) + d_\nu(x^*, x_{m+1})]\} \\ &\quad +sL_\nu d_\nu(x^*, x_{m+1}) \ \forall\ \nu \in \mathfrak{A}. \end{aligned}$$

Letting $m \to \infty$ in the above inequality, we get

$$d_\nu(x^*, Tx^*) \leq s d_\nu(x^*, Tx^*).$$

This is not possible, if $d_\nu(x^*, Tx^*) > 0$. Thus, $d_\nu(x^*, Tx^*) = 0$ for each $\nu \in \mathfrak{A}$. Since the structure $\{d_\nu : \nu \in \mathfrak{A}\}$ on X is separating, we have $x^* \in Tx^*$. □

By considering $T : X \to X$ in the above theorem, we get the following one.

Corollary 2.2. *Let $T : X \to X$ be a mapping such that for each $\nu \in \mathfrak{A}$ we have*

$$\begin{aligned} d_\nu(Tx, Ty) \;\leq\; & \psi_\nu(\max\{d_\nu(x, y), d_\nu(x, Tx), d_\nu(y, Ty), \frac{1}{2s}[d_\nu(x, Ty) + d_\nu(y, Tx)]\}) \\ & + L_\nu d_\nu(y, Tx) \;\forall\; \alpha(x, y) \geq 1 \end{aligned} \tag{2.12}$$

where $\psi_\nu \in \Psi_{s^2}$. Further, assume that the following conditions hold:

: *(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

: *(ii) if $\alpha(x, y) \geq 1$, then $\alpha(Tx, Ty) \geq 1$;*

: *(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Now we introduce a fixed point theorem containing Feng-liu type contraction:

Theorem 2.3. *Let $T : X \to CL(X)$ be a mapping such that for each $\nu \in \mathfrak{A}$ we have*

$$d_\nu(y, Ty) \leq \psi_\nu(d_\nu(x, y)) \;\forall x \in X \text{ and } y \in Tx \text{ with } \alpha(x, y) \geq 1 \tag{2.13}$$

where, $\psi_\nu \in \Psi_{s^2}$. Further, assume that the following conditions hold:

: *(i) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*

: *(ii) for $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, v) \geq 1$ for each $v \in Ty$;*

: *(iii) for each $x \in X$, we have $y \in Tx$ such that*

$$d_\nu(x, y) \leq s d_\nu(x, Tx) \ \forall \ \nu \in \mathfrak{A}.$$

Then T has a fixed point, provided $d_\nu(x, Tx)$ is lower semi continuous, for each $\nu \in \mathfrak{A}$.

Proof. By hypothesis, we have $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$. From (2.13), we get

$$d_\nu(x_1, Tx_1) \leq \psi_\nu(d_\nu(x_0, x_1)) \ \forall \ \nu \in \mathfrak{A}. \tag{2.14}$$

By hypothesis (iii), for $x_1 \in X$, we have $x_2 \in Tx_1$ such that

$$d_\nu(x_1, x_2) \leq s d_\nu(x_1, Tx_1) \leq s\psi_\nu(d_\nu(x_0, x_1)) \ \forall \ \nu \in \mathfrak{A}. \tag{2.15}$$

Applying ψ_ν, we have

$$\psi_\nu(d_\nu(x_1, x_2)) \leq \psi_\nu(s\psi_\nu(d_\nu(x_0, x_1))) = s\psi_\nu^2(d_\nu(x_0, x_1)) \ \forall \ \nu \in \mathfrak{A}.$$

By hypothesis (ii), it is clear that $\alpha(x_1, x_2) \geq 1$. Again from (2.13), we have

$$d_\nu(x_2, Tx_2) \leq \psi_\nu(d_\nu(x_1, x_2)) \ \forall \ \nu \in \mathfrak{A}. \tag{2.16}$$

By hypothesis (iii), for $x_2 \in X$, we have $x_3 \in Tx_2$ such that

$$d_\nu(x_2, x_3) \leq s d_\nu(x_2, Tx_2) \leq s\psi_\nu(d_\nu(x_1, x_2)) \leq s^2\psi_\nu^2(d_\nu(x_0, x_1)) \ \forall \ \nu \in \mathfrak{A}. \tag{2.17}$$

Clearly, $\alpha(x_2, x_3) \geq 1$. Continuing in the same way, we get a sequence $\{x_m\}$ in X such that $\alpha(x_m, x_{m+1}) \geq 1$ and

$$d_\nu(x_m, x_{m+1}) \leq s^m \psi_\nu^m(d_\nu(x_0, x_1)) \ \forall \ \nu \in \mathfrak{A}.$$

Now, we show that $\{x_m\}$ is a Cauchy sequence. For $m, p \in \mathbb{N}$, we have

$$\begin{aligned} d_\nu(x_m, x_{m+p}) &\leq \sum_{i=m}^{m+p-1} s^i d_\nu(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{m+p-1} s^{2i} \psi_\nu^i(d_\nu(x_0, x_1)) < \infty \ \forall \ \nu \in \mathfrak{A}. \end{aligned}$$

This implies that $\{x_m\}$ is a Cauchy sequence in X. By completeness of X, we have $x^* \in X$ such that $x_m \to x^*$ as $m \to \infty$. Thus, we have $\lim_{m\to\infty} d_\nu(x_m, Tx_m) = 0$. By lower semi continuity of $d_\nu(x, Tx)$ and last fact, we conclude that $d_\nu(x^*, Tx^*) = 0$ for each $\nu \in \mathfrak{A}$. Since the structure $\{d_\nu : \nu \in \mathfrak{A}\}$ on X is separating, thus $x^* \in Tx^*$. □

For singlevalued mapping, the above theorem reduces as given below:

Corollary 2.3. *Let $T : X \to X$ be a mapping such that for each $\nu \in \mathfrak{A}$ we have*

$$d_\nu(Tx, T^2x) \leq \psi_\nu(d_\nu(x, Tx)) \ \forall x \in X \ \textit{with} \ \ \alpha(x, Tx) \geq 1$$

where, $\psi_\nu \in \Psi_{s^2}$. Further, assume that the following conditions hold:

: *(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

: *(ii) for each $x \in X$ with $\alpha(x, Tx) \geq 1$, we have $\alpha(Tx, T^2x) \geq 1$;*

Then T has a fixed point, provided $d_\nu(x, Tx)$ is lower semi continuous, for each $\nu \in \mathfrak{A}$.

3. Application and Example

Consider the Volterra integral equation of the form:

$$x(t) = \int_a^t K(t, s, x(s))ds, \ t \in I = [0, \infty) \tag{3.1}$$

where $K : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing function.

Let $X = (C[0, \infty), \mathbb{R})$. Define a family of b_2-pseudo metrics as

$$d_n(x, y) = \max_{t\in[0,n]} (x(t) - y(t))^2 \text{ for each } n \in \mathbb{N}.$$

Clearly, $\mathfrak{F} = \{d_n : n \in \mathbb{N}\}$ defines b_2-gauge structure on X, which is complete and separating. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x(t) \leq y(t) \ \forall t \in I \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 3.1. *Let $X = (C[0,\infty), \mathbb{R})$ and let the operator $T : X \to X$ is define by*

$$Tx(t) = \int_0^t K(t, s, x(s))ds, t \in I = [0, \infty)$$

where $K : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function, that is, for each $t, s \in I$, $x(s) \le y(s)$ implies $K(t, s, x(s)) \le K(t, s, y(s))$. Assume that the following conditions hold:

: *(i) for each $t, s \in [0, n]$ and $x, y \in X$ with $x(s) \le y(s)$, there exists a continuous mapping $p : I \times I \to I$ such that*

$$|K(t, s, x(s)) - K(t, s, y(s))| \le \sqrt{p(t, s)d_n(x, y)} \text{ for each } n \in \mathbb{N};$$

: *(ii)* $\sup_{t\ge 0} \int_0^t \sqrt{p(t, s)}ds = a < \frac{1}{\sqrt{2}}$*;*

: *(iii) there exists $x_0 \in X$ such that*

$$x_0(t) \le \int_0^t K(t, s, x_0(s))ds.$$

Then the integral equation (3.1) *has atleast one solution.*

Proof. First we show that for each $\alpha(x, y) \ge 1$, the inequality in (2.1) hold. For any $\alpha(x, y) \ge 1$ and $t \in [0, n]$ for each $n \ge 1$, we have

$$\begin{aligned} (Tx(t) - Ty(t))^2 &\le \Big(\int_0^t |K(t, s, x(s)) - K(t, s, y(s))|ds\Big)^2 \\ &\le \Big(\int_0^t \sqrt{p(t, s)d_n(x, y)}ds\Big)^2 \\ &= \Big(\int_0^t \sqrt{p(t, s)}ds\Big)^2 d_n(x, y) \\ &= a^2 d_n(x, y). \end{aligned}$$

Thus, we get $d_n(Tx, Ty) \le a^2 d_n(x, y)$ for each $\alpha(x, y) \ge 1$ and $n \in \mathbb{N}$ with $a^2 < 1/2$. This implies that (2.1) holds with $a_n = a^2$, and $b_n = c_n = e_n = L_n = 0$ for each $n \in \mathbb{N}$. As K is nondecreasing, for each $x \le y$, we have $Tx \le Ty$. Hence for $\alpha(x, y) \ge 1$, implies $\alpha(Tx,Ty) \geq 1$. Therefore, by Theorem 2.1, there exists a fixed point of the operator T, that is, integral equation (3.1) has atleast one solution. □

Now, we give an example to support of our result:

Example 3.1. *Let $X = C([0,10],\mathbb{R})$ is the space of twice differentiable functions, endowed with the $d_n(x(t),y(t)) = \max_{t\in[0,n]}(x(t)-y(t))^2$ for each $n \in \{1,2,3,\cdots,10\}$. Consider the operator $T : X \to X$ is defined by $Tx(t) = \frac{d^2x(t)}{dt^2}$ and $\alpha : X\times X \to [0,\infty)$ is defined by*

$$\alpha(x,y) = \begin{cases} 1 \text{ if x,y are linear or constant functions} \\ 0 \text{ otherwise} \end{cases}$$

It is easy to see that 2.12 holds with $a_n = 1/2$ and $b_n = c_n = e_n = L_n = 0$ for each $n \in \{1,2,3,\cdots,10\}$. For $x_0 = t$ and $x_1 = Tx_0 = 0$, we have $\alpha(x_0,Tx_0) = 1$. Further, for each $\alpha(x,y) = 1$, we have $\alpha(Tx,Ty) = 1$. Moreover, for each sequence $\{x_m\}$ in X such that $\alpha(x_m,x_{m+1}) = 1$ for each $m \in \mathbb{N}$ and $x_m \to x$ as $m \to \infty$, then $\alpha(x_m,x) = 1$ for each $m \in \mathbb{N}$. Therefore, all conditions of Theorem 2.1 are satisfied and T has a fixed point.

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