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ON THE CONNECTIONS BETWEEN PADOVAN NUMBERS AND FIBONACCI p-NUMBERS

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Abstract. In this paper, we define the Fibonacci-Padovan p-sequence and then we discuss the connection of the Fibonacci-Padovan p-sequence with the Padovan sequence and Fibonacci p-sequence. In addition, we obtain miscellaneous properties of the Fibonacci-Padovan p-numbers such as the Binet formulas, the exponential, combinatorial, permanental and determinantal representations, and the sums of certain matrices.

1. Introduction

The Padovan sequence is the sequence of the integer $\{P(n)\}$ defined by the initial values $P(0) = P(1) = P(2) = 1$ and the recurrence relation:

$$P(n) = P(n-2) + P(n-3)$$

for all $n \geq 3$.

There are many important generalizations of the Fibonacci sequence. The Fibonacci p-sequence $\{F_p(n)\}$ (see detailed information in [19, 20]) is the one of them:

$$F_p(n) = F_p(n-1) + F_p(n-p-1)$$

for $n > p$ and $p = 1, 2, 3, \ldots$, in which $F_p(0) = 0$, $F_p(1) = \cdots F_p(p) = 1$. When $p = 1$, the Fibonacci p-sequence $\{F_p(n)\}$ is reduced to the usual Fibonacci sequence $\{F_n\}$.

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It is easy to see that the characteristic polynomials of the Padovan sequence and Fibonacci p-sequence are $q_1(x) = x^3 - x - 1$ and $q_2(x) = x^{p+1} - x^p - 1$, respectively. We use these in the next section.

Suppose that the $(n+k)$th term of a sequence be defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. In [12], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method as follows:

Let the matrix A be defined by

$$A = [a_{i,j}]_{k\times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

Number theoretic properties such as these obtained from homogeneous linear recurrence relations relevant to this paper have been studied recently by many authors: see, for example, [2, 5, 10, 11, 17, 18, 21]. In [1, 6, 7, 8, 9, 14, 22], the authors defined some linear recurrence sequences and gave their various properties by matrix methods. In this paper, we discuss connections between the Padovan and Fibonacci p-numbers. Firstly, we define the Fibonacci-Padovan p-sequence and then we give recurrence relation among this sequence, the Padovan sequence and Fibonacci p-sequence. Also, using the roots of the characteristic polynomial of the Fibonacci-Padovan p-sequence, we produce the Binet formula for the Fibonacci-Padovan p-sequence. Finally, we give the exponential, combinatorial, permanental and determinantal representations, the generating function, and the sums of the Fibonacci-Padovan p-numbers.

2. On The Connections Between Padovan Numbers and Fibonacci p-Numbers

Now we define the Fibonacci-Padovan p-sequence $\left\{F_n^{Pa,p}\right\}$ by the following homogeneous linear recurrence relation for any given $p\,(4,5,6,\ldots)$ and $n\geq 0$

$$F_{n+p+4}^{Pa,p}=F_{n+p+3}^{Pa,p}+F_{n+p+2}^{Pa,p}-F_{n+p}^{Pa,p}+F_{n+3}^{Pa,p}-F_{n+1}^{Pa,p}-F_{n}^{Pa,p} \tag{2.1}$$

in which $F_0^{Pa,p}=\cdots=F_{p+2}^{Pa,p}=0$ and $F_{p+3}^{Pa,p}=1$.

First, we present relationships between the above the Fibonacci-Padovan p-sequence, Padovan sequence, and Fibonacci p-sequence.

Theorem 2.1. *Let $P\,(n)$, $F_p\,(n)$ and $F_n^{Pa,p}$ be the nth Padovan number, Fibonacci p-number, and Fibonacci-Padovan p-numbers, respectively. Then,*

$$P\,(n+2)=F_p\,(n+p-1)+F_p\,(n)+\sum_{i=n+2}^{n+p-3}F_i^{Pa,p}$$

for $n\geq 0$ and $p\geq 4$.

Proof. The assertion may be proved by induction on n. It is clear that $P\,(2)=F_p\,(p-1)+F_p\,(0)+\sum_{i=2}^{p-3}F_i^{Pa,p}=1$. Suppose that the equation holds for $n\geq 1$. Then we must show that the equation holds for $n+1$. Since the characteristic polynomial of Fibonacci-Padovan p-sequence $\left\{F_n^{J,p}\right\}$, is

$$q\,(x)=x^{p+4}-x^{p+3}-x^{p+2}+x^{p}-x^{3}+x+1$$

and

$$q\,(x)=q_1\,(x)\,q_2\,(x)\,,$$

where $q_1\,(x)$ and $q_2\,(x)$ are the characteristic polynomials of the Padovan sequence and Fibonacci p-sequence, respectively, we obtain the following relations:

$$P\,(n+p+4)=P\,(n+p+3)+P\,(n+p+2)-P\,(n+p)+P\,(n+3)-P\,(n+1)-P\,(n)$$

and

$$F_p(n+p+4)=F_p(n+p+3)+F_p(n+p+2)-F_p(n+p)+F_p(n+3)-F_p(n+1)-F_p(n)$$

for $n \geq 1$. Thus, by a simple calculation, we have the conclusion. □

By the recurrence relation (2.1), we have

$$\begin{bmatrix} F_{n+p+4}^{Pa,p} \\ F_{n+p+3}^{Pa,p} \\ F_{n+p+2}^{Pa,p} \\ \vdots \\ F_{n+1}^{Pa,p} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+p+3}^{Pa,p} \\ F_{n+p+2}^{Pa,p} \\ F_{n+p}^{Pa,p} \\ \vdots \\ F_{n}^{Pa,p} \end{bmatrix}$$

for the Fibonacci-Padovan p-sequence $\left\{F_n^{Pa,p}\right\}$. Letting

$$B_p = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(p+4)\times(p+4)}.$$

The companion matrix $B_p = [b_{i,j}]_{(p+4)\times(p+4)}$ is said to be the Fibonacci-Padovan p-

matrix. For detailed information about the companion matrices, see [15, 16]. It can be readily established by mathematical induction that for $p \geq 4$ and $n \geq 2p+1$

$$(B_p)^n = \begin{bmatrix} F_{n+p+3}^{Pa,p} & F_{n+p+4}^{Pa,p} - F_{n+p+3}^{Pa,p} & F_p(n-p+2) - F_{n+p+1}^{Pa,p} & F_p(n-p+3) - F_{n+p+2}^{Pa,p} & F_p(n-p+4) & \cdots & \\ F_{n+p+2}^{Pa,p} & F_{n+p+3}^{Pa,p} - F_{n+p+2}^{Pa,p} & F_p(n-p+1) - F_{n+p}^{Pa,p} & F_p(n-p+2) - F_{n+p+1}^{Pa,p} & F_p(n-p+3) & \cdots & \\ F_{n+p+1}^{Pa,p} & F_{n+p+2}^{Pa,p} - F_{n+p+1}^{Pa,p} & F_p(n-p) - F_{n+p-1}^{Pa,p} & F_p(n-p+1) - F_{n+p}^{Pa,p} & F_p(n-p+2) & \cdots & B_p^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ F_{n+1}^{Pa,p} & F_{n+2}^{Pa,p} - F_{n+1}^{Pa,p} & F_p(n-2p) - F_{n-1}^{Pa,p} & F_p(n-2p+1) - F_{n}^{Pa,p} & F_p(n-2p+2) & \cdots & \\ F_{n}^{Pa,p} & F_{n+1}^{Pa,p} - F_{n}^{Pa,p} & F_p(n-2p-1) - F_{n-2}^{Pa,p} & F_p(n-2p) - F_{n-1}^{Pa,p} & F_p(n-2p+1) & \cdots & \end{bmatrix},$$

where

$$B_p^* = \begin{bmatrix} F_p(n) & F_p(n+1) - F_{n+p+3}^{Pa,p} & F_p(n+2) - F_{n+p+4}^{Pa,p} & -F_{n+p+2}^{Pa,p} \\ F_p(n-1) & F_p(n) - F_{n+p+2}^{Pa,p} & F_p(n+1) - F_{n+p+3}^{Pa,p} & -F_{n+p+1}^{Pa,p} \\ F_p(n-2) & F_p(n-1) - F_{n+p+1}^{Pa,p} & F_p(n) - F_{n+p+2}^{Pa,p} & -F_{n+p}^{Pa,p} \\ \vdots & \vdots & \vdots & \vdots \\ F_p(n-p-2) & F_p(n-p-1) - F_{n+1}^{Pa,p} & F_p(n-p) - F_{n+2}^{Pa,p} & -F_{n}^{Pa,p} \\ F_p(n-p-3) & F_p(n-p-2) - F_{n}^{Pa,p} & F_p(n-p-1) - F_{n+1}^{Pa,p} & -F_{n-1}^{Pa,p} \end{bmatrix}.$$

We easily derive that $\det B_p = (-1)^p$. In [19], Stakhov defined the generalized Fibonacci p-matrix Q_p and derived the nth power of the matrix Q_p. In [13], Kılıc gave a Binet formula for the Fibonacci p-numbers by matrix method. Now we concentrate on finding another Binet formula for the Fibonacci-Padovan p-numbers by the aid of the matrix $(B_p)^n$.

Lemma 2.1. *The characteristic equation of all the Fibonacci-Padovan p-numbers $x^{p+4} - x^{p+3} - x^{p+2} + x^p - x^3 + x + 1 = 0$ does not have multiple roots for $p \geq 4$.*

Proof. It is clear that $x^{p+4} - x^{p+3} - x^{p+2} + x^p - x^3 + x + 1 = (x^{p+1} - x^p - 1)(x^3 - x - 1)$. In [13], it was shown that the equation $x^{p+1} - x^p - 1 = 0$ does not have multiple roots for $p > 1$. It is easy to see that the roots of the equation $x^3 - x - 2 = 0$ are

$$\alpha = \frac{1}{3}\sqrt[3]{\frac{27}{2} - \frac{3\sqrt{69}}{2}} + \frac{\sqrt[3]{\frac{1}{2}\left(9+\sqrt{69}\right)}}{3^{\frac{2}{3}}},$$

$$\beta = -\frac{1}{6}\left(1 - i\sqrt{3}\right)\sqrt[3]{\frac{27}{2} - \frac{3\sqrt{69}}{2}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(9+\sqrt{69}\right)}}{2 \times 3^{\frac{2}{3}}}$$

and

$$\gamma=-\frac{1}{6}\left(1+i\sqrt{3}\right)\sqrt[3]{\frac{27}{2}-\frac{3\sqrt{69}}{2}}-\frac{(1-i\sqrt{3})\sqrt[3]{\frac{1}{2}\left(9+\sqrt{69}\right)}}{2\times 3^{\frac{2}{3}}}.$$

Since $(\alpha)^{p+1}-(\alpha)^{p}-1\neq 0$, $(\beta)^{p+1}-(\beta)^{p}-1\neq 0$ and $(\gamma)^{p+1}-(\gamma)^{p}-1\neq 0$ for $p>1$, the equation $x^{p+4}-x^{p+3}-x^{p+2}+x^{p}-x^{3}+x+1=0$ does not have multiple roots for $p\geq 4$. □

Let $q(\lambda)$ be the characteristic polynomial of the Fibonacci-Padovan p-matrix B_p. Then $q(\lambda)=\lambda^{p+4}-\lambda^{p+3}-\lambda^{p+2}+\lambda^{p}-\lambda^{3}+\lambda+1$, which is a well-known fact from the companion matrices. Let $\lambda_1,\lambda_2,\ldots,\lambda_{p+4}$ be the eigenvalues of B_p. Then, by Lemma 2.1, $\lambda_1,\lambda_2,\ldots,\lambda_{p+4}$ are distinct. Define the $(p+4)\times(p+4)$ Vandermonde matrix V_p as follows:

$$V_p=\begin{bmatrix}(\lambda_1)^{p+3} & (\lambda_2)^{p+3} & \ldots & (\lambda_{p+4})^{p+3}\\ (\lambda_1)^{p+2} & (\lambda_2)^{p+2} & \ldots & (\lambda_{p+4})^{p+2}\\ \vdots & \vdots & & \vdots\\ \lambda_1 & \lambda_2 & \ldots & \lambda_{p+4}\\ 1 & 1 & \ldots & 1\end{bmatrix}.$$

Assume that $V_p(i,j)$ is a $(p+4)\times(p+4)$ matrix derived from the Vandermonde matrix V_p by replacing the j^{th} column of V_p by $W_p(i)$, where, $W_p(i)$ is a $(p+4)\times 1$ matrix as follows:

$$W_p(i)=\begin{bmatrix}(\lambda_1)^{n+p+4-i}\\ (\lambda_2)^{n+p+4-i}\\ \vdots\\ (\lambda_{p+4})^{n+p+4-i}\end{bmatrix}.$$

Then we obtain the Binet formula for the Fibonacci-Padovan p-numbers with the following Theorem.

Theorem 2.2. *Let p be a positive integer such that $p\geq 4$ and let $(B_p)^n=\left[b_{i,j}^{(p,n)}\right]$ for $n\geq 2p+1$, then*

$$b_{i,j}^{(p,n)}=\frac{\det V_p(i,j)}{\det V_p}.$$

Proof. Since the equation $x^{p+4}-x^{p+3}-x^{p+2}+x^{p}-x^{3}+x+1=0$ does not have multiple roots for $p\geq 4$, the eigenvalues of the Fibonacci-Padovan p-matrix B_p are distinct. Then, it is clear that B_p is diagonalizable. Let $D_p=diag(\lambda_1,\lambda_2,\ldots,\lambda_{p+4})$, then we may write $B_pV_p = V_pD_p$. Since the matrix V_p is invertible, we obtain the equation $(V_p)^{-1} B_pV_p = D_p$. Therefore, B_p is similar to D_p; hence, $(B_p)^n V_p = V_p (D_p)^n$ for $n \geq 2p + 1$. So we have the following linear system of equations:

$$\begin{cases} b_{i,1}^{(p,n)} (\lambda_1)^{p+3} + b_{i,2}^{(p,n)} (\lambda_1)^{p+2} + \cdots + b_{i,p+4}^{(p,n)} = (\lambda_1)^{n+p+4-i} \\ b_{i,1}^{(p,n)} (\lambda_2)^{p+3} + b_{i,2}^{(p,n)} (\lambda_2)^{p+2} + \cdots + b_{i,p+4}^{(p,n)} = (\lambda_2)^{n+p+4-i} \\ \vdots \\ b_{i,1}^{(p,n)} (\lambda_{p+4})^{p+3} + b_{i,2}^{(p,n)} (\lambda_{p+4})^{p+2} + \cdots + b_{i,p+4}^{(p,n)} = (\lambda_{p+4})^{n+p+4-i}. \end{cases}$$

Then we conclude that

$$b_{i,j}^{(p,n)} = \frac{\det V_p(i,j)}{\det V_p}$$

for each $i, j = 1, 2, \ldots, p + 4$. So the proof is complete. □

Thus by Theorem 2.2 and the matrix $(B_p)^n$, we have the following useful result for the Fibonacci-Padovan p-numbers.

Corollary 2.1. *Let p be a positive integer such that $p \geq 4$ and let $F_n^{Pa,p}$ be the nth element of Fibonacci-Padovan p-sequence, then*

$$F_n^{Pa,p} = \frac{\det V_p(p+4,1)}{\det V_p}$$

and

$$F_n^{Pa,p} = -\frac{\det V_p(p+3,p+4)}{\det V_p}$$

for $n \geq 2p + 1$.

Now we give the generating function of the Fibonacci-Padovan p-numbers:

Let

$$g(x) = F_{p+3}^{Pa,p} + F_{p+4}^{Pa,p}x + F_{p+5}^{Pa,p}x^2 + \cdots + F_{n+p+3}^{Pa,p}x^n + F_{n+p+4}^{Pa,p}x^{n+1} + \cdots.$$

By the definition of the Fibonacci-Padovan p-numbers, we can write

$$g(x) - xg(x) - x^2g(x) + x^4g(x) - x^{p+1}g(x) + x^{p+3}g(x) + x^{p+4}g(x) = x^{p+3}.$$

So we get

$$g(x) = \frac{x^{p+3}}{1 - x - x^2 + x^4 - x^{p+1} + x^{p+3} + x^{p+4}},$$

for $0 \leq x + x^2 - x^4 + x^{p+1} - x^{p+3} - x^{p+4} < 1$.

Then we can give an exponential representation for the Fibonacci-Padovan p-numbers by the aid of the generating function with the following Theorem.

Theorem 2.3. *The Fibonacci-Padovan p-sequence $\left\{F_n^{Pa,p}\right\}$ have the following exponential representation:*

$$g(x) = x^{p+3} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i}\left(1 + x - x^3 + x^p - x^{p+2} - x^{p+3}\right)^i\right),$$

where $p \geq 4$.

Proof. Since

$$\ln g(x) = \ln x^{p+3} - \ln\left(1 - x - x^2 + x^4 - x^{p+1} + x^{p+3} + x^{p+4}\right)$$

and

$$\begin{aligned} -\ln\left(1 - x - x^2 + x^4 - x^{p+1} + x^{p+3} + x^{p+4}\right) = & -[-x\left(1 + x - x^3 + x^p - x^{p+2} - x^{p+3}\right) - \\ & \frac{1}{2}x^2\left(1 + x - x^3 + x^p - x^{p+2} - x^{p+3}\right)^2 - \cdots \\ & -\frac{1}{i}x^i\left(1 + x - x^3 + x^p - x^{p+2} - x^{p+3}\right)^i - \cdots] \end{aligned}$$

it is clear that

$$g(x) = x^{p+3} \exp\left(\sum_{i=1}^{\infty} \frac{(x)^i}{i}\left(1 + x - x^3 + x^p - x^{p+2} - x^{p+3}\right)^i\right)$$

by a simple calculation, we obtain the conclusion. □

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows:

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_v \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.4. *(Chen and Louck [4]) The (i,j) entry $k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v)$ in the matrix $K^n(k_1, k_2, \ldots, k_v)$ is given by the following formula:*

$$\text{(2.2)} \quad k_{i,j}^{(n)}(k_1, k_2, \ldots, k_v) = \sum_{(t_1, t_2, \ldots, t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \cdots + t_v} \times \binom{t_1 + \cdots + t_v}{t_1, \ldots, t_v} k_1^{t_1} \cdots k_v^{t_v}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v} = \frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ *is a multinomial coefficient, and the coefficients in (2.2) are defined to be* 1 *if* $n = i - j$.

Then we can give combinatorial representations for the Fibonacci-Padovan p-numbers by the following Corollary.

Corollary 2.2. *Let* $F_n^{Pa,p}$ *be the nth Fibonacci-Padovan p-number for* $n \geq 2p+1$. *Then*

i.

$$F_n^{Pa,p} = \sum_{(t_1,t_2,\ldots,t_{p+4})} \binom{t_1+t_2+\cdots+t_{p+4}}{t_1,t_2,\cdots,t_{p+4}} (-1)^{t_4+t_{p+3}+t_{p+4}}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+4)\,t_{p+4} = n-p-3$.

ii.

$$F_n^{Pa,p} = -\sum_{(t_1,t_2,\ldots,t_4)} \frac{t_{p+4}}{t_1+t_2+\cdots+t_{p+4}} \times \binom{t_1+t_2+\cdots+t_{p+4}}{t_1,t_2,\cdots,t_{p+4}} (-1)^{t_4+t_{p+3}+t_{p+4}}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+(p+4)\,t_{p+4} = n+1$.

Proof. If we take $i = p+4$, $j = 1$ for the case i. and $i = p+3$, $j = p+4$ for the case ii. in theorem 2.4, then we can directly see the conclusions from $(B_p)^n$. □

Now we consider the relationship between the Fibonacci-Padovan p-numbers and the permanent of a certain matrix which is obtained using the Fibonacci-Padovan p-matrix $(B_p)^n$.

Definition 2.1. A $u \times v$ real matrix $M = [m_{i,j}]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.

Suppose that x_1, x_2, $\ldots$,x_u are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In [3], Brualdi and Gibson obtained that $per\,(M) = per\,(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Now we concentrate on finding relationships among the Fibonacci-Padovan p-numbers and the permanents of certain matrices which are obtained by using the generating matrix of the Fibonacci-Padovan p-numbers. Let $E_{m,p}^{F,Pa} = \left[e_{i,j}^{(p)}\right]$ be the $m \times m$ super-diagonal matrix, defined by

$$e_{i,j}^{(p)} = \begin{cases} 1 & \begin{array}{c} \text{if } i = \varepsilon \text{ and } j = \varepsilon \text{ for } 1 \le \varepsilon \le m, \\ i = \varepsilon \text{ and } j = \varepsilon + 1 \text{ for } 1 \le \varepsilon \le m-1, \\ i = \varepsilon \text{ and } j = \varepsilon + p \text{ for } 1 \le \varepsilon \le m-p \\ \text{and} \\ i = \varepsilon + 1 \text{ and } j = \varepsilon \text{ for } 1 \le \varepsilon \le m-1, \end{array} \\ -1 & \begin{array}{c} \text{if } i = \varepsilon \text{ and } j = \varepsilon + 3 \text{ for } 1 \le \varepsilon \le m-3, \\ i = \varepsilon \text{ and } j = \varepsilon + p + 2 \text{ for } 1 \le \varepsilon \le m-p-2 \\ \text{and} \\ i = \varepsilon \text{ and } j = \varepsilon + p + 3 \text{ for } 1 \le \varepsilon \le m-p-3, \end{array} \\ 0 & \text{otherwise.} \end{cases} \quad \text{, for } m \ge p+4.$$

Then we have the following Theorem.

Theorem 2.5. *For $m \ge p + 4$,*

$$per E_{m,p}^{F,Pa} = F_{m+p+3}^{Pa,p}.$$

Proof. Let us consider matrix $E_{m,p}^{F,Pa}$ and let the equation be hold for $m \ge p + 4$. Then we show that the equation holds for $m + 1$. If we expand the $per E_{m,p}^{F,Pa}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$per E_{m+1,p}^{F,Pa} = per E_{m,p}^{F,Pa} + per E_{m-1,p}^{F,Pa} - per E_{m-3,p}^{F,Pa} + per E_{m-p,p}^{F,Pa} - per E_{m-p-2,p}^{F,Pa} - per E_{m-p-3,p}^{F,Pa}.$$

Since

$$per E_{m,p}^{F,Pa} = F_{m+p+3}^{Pa,p},$$

$$per E_{m-1,p}^{F,Pa} = F_{m+p+2}^{Pa,p},$$

$$per E_{m-3,p}^{F,Pa} = F_{m+p}^{Pa,p},$$

$$perE_{m-p,p}^{F,Pa} = F_{m+3}^{Pa,p},$$

$$perE_{m-p-2,p}^{F,Pa} = F_{m+1}^{Pa,p}$$

and

$$perE_{m-p-3,p}^{F,Pa} = F_{m}^{Pa,p},$$

we easily obtain that $perE_{m+1,p}^{F,Pa} = F_{m+p+4}^{Pa,p}$. So the proof is complete. □

Let $G_{m,p}^{F,Pa} = \left[g_{i,j}^{(p)}\right]$ be the $m \times m$ matrix, defined by

$$g_{i,j}^{(p)} = \begin{cases} 1 & \begin{array}{c} \text{if } i = \varepsilon \text{ and } j = \varepsilon \text{ for } 1 \leq \varepsilon \leq m-2, \\ i = \varepsilon \text{ and } j = \varepsilon + 1 \text{ for } 1 \leq \varepsilon \leq m-2, \\ i = \varepsilon \text{ and } j = \varepsilon + p \text{ for } 1 \leq \varepsilon \leq m-p-1 \\ \text{and} \\ i = \varepsilon + 1 \text{ and } j = \varepsilon \text{ for } 1 \leq \varepsilon \leq m-1, \end{array} \\ -1 & \begin{array}{c} \text{if } i = \varepsilon \text{ and } j = \varepsilon + 3 \text{ for } 1 \leq \varepsilon \leq m-p-1, \\ i = \varepsilon \text{ and } j = \varepsilon + p + 2 \text{ for } 1 \leq \varepsilon \leq m-p-3 \\ \text{and} \\ i = \varepsilon \text{ and } j = \varepsilon + p + 3 \text{ for } 1 \leq \varepsilon \leq m-p-3, \end{array} \\ 0 & \text{otherwise.} \end{cases}, \text{ for } m \geq p+4.$$

Then we have the following Theorem.

Theorem 2.6. *For $m \geq p+4$,*

$$perG_{m,p}^{F,Pa} = -F_{m-1}^{Pa,p}.$$

Proof. Let us consider matrix $G_{m,p}^{F,Pa}$ and let the equation be hold for $m \geq p+4$. Then we show that the equation holds for $m+1$. If we expand the $perG_{m,p}^{F,Pa}$ by the Laplace expansion of permanent with respect to the first row, then we obtain

$$perG_{m+1,p}^{F,Pa} = perG_{m,p}^{F,Pa} + perG_{m-1,p}^{F,Pa} - perG_{m-3,p}^{F,Pa} + perG_{m-p,p}^{F,Pa} - perG_{m-p-2,p}^{F,Pa} - perG_{m-p-3,p}^{F,Pa}.$$

Since

$$perG_{m,p}^{F,Pa} = -F_{m-1}^{Pa,p},$$

$$perG_{m-1,p}^{F,Pa} = -F_{m-2}^{Pa,p},$$

$$perG_{m-3,p}^{F,Pa} = -F_{m-4}^{Pa,p},$$

$$perG^{F,Pa}_{m-p,p} = -F^{Pa,p}_{m-p-1},$$

$$perG^{F,Pa}_{m-p-2,p} = -F^{Pa,p}_{m-p-3}$$

and

$$perG^{F,Pa}_{m-p-3,p} = -F^{Pa,p}_{m-p-4},$$

we easily obtain that $perG^{F,Pa}_{m+1,p} = -F^{Pa,p}_{m}$. So the proof is complete. □

Assume that $H^{F,Pa}_{m,p} = \left[h^{(p)}_{i,j}\right]$ be the $m \times m$ matrix, defined by

$$H^{F,Pa}_{m,p} = \begin{bmatrix} & & (m-p-4)\,\text{th} & & & \\ & & \downarrow & & & \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & G^{F,Pa}_{m-1,p} & & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}, \text{ for } m > p+4,$$

then we have the following results:

Theorem 2.7. *For $m > p+4$,*

$$perH^{F,Pa}_{m,p} = -\sum_{i=0}^{m-2} F^{Pa,p}_{i}.$$

Proof. If we extend $per\ H^{F,Pa}_{m,p}$ with respect to the first row, we write

$$perH^{F,Pa}_{m,p} = perH^{F,Pa}_{m-1,p} + perG^{F,Pa}_{m-1,p}.$$

Thus, by the results and an inductive argument, the proof is easily seen. □

A matrix M is called convertible if there is an $n \times n$ $(1,-1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Now we give relationships among the Fibonacci-Padovan p-numbers and the determinants of certain matrices which are obtained by using the matrix $E^{F,Pa}_{m,p}$, $G^{F,Pa}_{m,p}$

and $H_{m,p}^{F,Pa}$. Let $m > p+4$ and let R be the $m \times m$ matrix, defined by

$$R = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 2.3. *For $m > p+4$,*

$$\det\left(E_{m,p}^{F,Pa} \circ R\right) = F_{m+p+3}^{Pa,p},$$

$$\det\left(G_{m,p}^{F,Pa} \circ R\right) = -F_{m-1}^{Pa,p}$$

and

$$\det\left(H_{m,p}^{F,Pa} \circ R\right) = -\sum_{i=0}^{m-2} F_i^{Pa,p}.$$

Proof. Since $perE_{m,p}^{F,Pa} = \det\left(E_{m,p}^{F,Pa} \circ R\right)$, $perG_{m,p}^{F,Pa} = \det\left(G_{m,p}^{F,Pa} \circ R\right)$ and $perH_{m,p}^{F,Pa}$ $= \det\left(H_{m,p}^{F,Pa} \circ R\right)$ for $m > p+4$, by Theorem 2.5, Theorem 2.6 and Theorem 2.7, we have the conclusion. □

Now we consider the sums of the Fibonacci-Padovan p-numbers. Let

$$S_\alpha = \sum_{i=0}^{\alpha} F_i^{Pa,p}$$

for $n \geq 2p+1$ and $p \geq 4$, and let $A_p^{F,Pa}$ and $\left(A_p^{F,Pa}\right)^n$ be the $(p+5) \times (p+5)$ matrix such that

$$A_p^{F,Pa} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & B_p & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}.$$

If we use induction on n, then we obtain

$$\left(A_p^{F,Pa}\right)^n = \begin{bmatrix} 1 & 0 \; 0 \; \cdots \; 0 \; 0 \\ S_{n+p+2} & \\ S_{n+p+1} & \\ \vdots & (B_p)^n \\ S_{n-1} & \\ S_n & \end{bmatrix}.$$

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