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Existence results for a coupled system of multi-term Katugampola fractional differential equations with integral conditions

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Abstract: This paper investigates a coupled system of nonlinear multi-term Katugampola fractional differential equations. Under sufficient conditions, it establishes the existence and uniqueness results of the solution by using standard fixed point theorems. Additionally, the paper includes some illustrative examples to strengthen the presented main results.

Keywords: Coupled system; Katugampola fractional derivative; Existence and uniqueness; Integral conditions; Fixed point theorems.

2010 Mathematics Subject Classification. 26A33; 34A08; 34A12; 34A34; 47N20.

1 Introduction

We consider the following coupled system of nonlinear multi-term fractional differential equations:

$$\begin{cases} {}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t) = f_1\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right), \\ {}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t) = f_2\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right), \end{cases} \qquad t\in[0,\ell], \tag{1}$$

with the integral conditions

$$\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi\right)(0^+) = \left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi\right)(0^+) = 0, \tag{2}$$

where $\rho,\ell>0,\ 0<\beta_{ij}<\alpha_i<1$ and $f_i:[0,\ell]\times\mathbb{R}^4\to\mathbb{R}$ are continuous functions for every $i,j\in\{1,2\}$. The operator ${}^{\rho}\mathscr{D}_{0^+}^{\alpha}$ and ${}^{\rho}\mathscr{I}_{0^+}^{1-\alpha}$ represents the Katugampola fractional derivative and integral of order $\alpha>0$, respectively.

The initial value problems are a vast and significant area of research, as these problems have applications in various scientific fields. Recently, so-called fractional initial value problems have appeared and become widespread, allowing the modeling of many real-world phenomena, as well as giving an understanding of some mathematical problems such as the Abel equation [22],

$$\int_a^t y(s)(t-s)^{\alpha-1}\,ds = f(t), \qquad 0<\alpha<1.$$

Recently, the resolvability of fractional differential equations with different kinds of initial or boundary conditions has witnessed a remarkable trend, which has led to the publication of many works in this regard, for example, but not limited to, see [2,4,5,6,7,8,9,10,11,12,13,14,16,21,23] and references cited therein.

The existence and uniqueness result of the coupled system of fractional differential equations (1) with integral boundary condition has been investigated in [3], but the functions f_1 dependent on time t, unknown functions φ and

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$\mathscr{D}_{0^+}^{\beta_{12}}\psi$ while f_2 dependent on time t, unknown functions ψ and $\mathscr{D}_{0^+}^{\beta_{21}}\varphi$. The authors in [20], studied the existence and uniqueness of the solution for system (1) with integral conditions where the functions f_1 and f_2 dependent only on time t and unknown functions φ and ψ. A similar result was found in [23], where the function f_1 dependent only on time t and unknown function φ and f_2 dependent only on time t and unknown function ψ.

The main contribution of this paper can be summarized in obtaining the existence and uniqueness result of a coupled system, with some conditions on the functions of second member f_1 and f_2.

The organization of this paper is as follows: In Section 2, we describe some preliminary concepts related to the proposed study; in Section 3, we give some existence and uniqueness results for the problem (1)–(2). The results are based on Schauder's and contraction mapping principle fixed point theorems in a special Banach space. In Section 4, two examples are presented to explain the application of our main results. Finally, we present some conclusions in Section 5.

2 Preliminaries

Here, as in [19], we will look at the Katugampola's fractional integral, derivative and some of their properties. Let $r\in\mathbb{R}$, $p\in[1,\infty]$ and

$$X_r^p([0,\ell],\mathbb{R})=\left\{\varphi:[0,\ell]\longrightarrow\mathbb{R}\text{ Lebesgue measurable and }\|\varphi\|_{X_r^p}<\infty\right\},$$

with the norm

$$\|\varphi\|_{X_r^p}=\begin{cases}\left(\int_0^\ell\frac{|t^r\varphi(t)|^p}{t}dt\right)^{1/p}, & \text{for } 1\leq p<\infty,\\ \operatorname*{ess\,sup}_{0\leq t\leq\ell}\{t^r|\varphi(t)|\}, & \text{for } p=\infty.\end{cases}$$

Let $C([0,\ell],\mathbb{R})$ be the collection of continuous functions from $[0,\ell]$ into $\mathbb{R}$ with the norm

$$\|\varphi\|_\infty=\sup_{0\leq t\leq\ell}|\varphi(t)|.$$

Then $C([0,\ell],\mathbb{R})$ is Banach space.

Definition 1([17]). *The Katugampola's fractional integral of order $\alpha\in\mathbb{R}_+$ of a function $g\in X_r^p([0,\ell],\mathbb{R})$ is defined as*

$$^{\rho}\mathscr{I}_{0^+}^{\alpha}g(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}g(s)\,ds,\qquad t\in[0,\ell],\tag{3}$$

for $\rho>0$. This is a left-sided integral.

Similarly, for the right-sided integrals definition. From Definition 1 we can infer

$$\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathscr{I}_{0^+}^{\alpha+1}g(t)={}^{\rho}\mathscr{I}_{0^+}^{\alpha}g(t).\tag{4}$$

Definition 2([18]). *The generalized fractional derivative of order $\alpha\in\mathbb{R}_+$, corresponding to the Katugampola's fractional integral (3) is defined for any $t\in[0,\ell]$ as*

$$\begin{aligned}{}^{\rho}\mathscr{D}_{0^+}^{\alpha}g(t)&=\left(t^{1-\rho}\frac{d}{dt}\right)^n\left({}^{\rho}\mathscr{I}_{0^+}^{n-\alpha}g\right)(t)\\ &\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n\int_0^t s^{\rho-1}(t^\rho-s^\rho)^{n-\alpha-1}g(s)\,ds,\end{aligned}\tag{5}$$

if the integral exists. Here $\rho>0$ and $n=[\alpha]+1$, with $[\cdot]$ denotes the integer part.

Lemma 1([7]). *Let $\alpha,\rho>0$ and $g\in C([0,\ell],\mathbb{R})$. Then:*

1. The equation ${}^{\rho}\mathscr{D}_{0^+}^{\alpha}g(t)=0$ has a unique solution

$$g(t)=\sum_{i=1}^n c_i t^{\rho(\alpha-n)},\qquad n=[\alpha]+1,\ c_i\in\mathbb{R}_+.$$

2. If ${}^{\rho}\mathscr{D}^{\alpha}_{0^+}g(t) \in C([0,\ell],\mathbb{R})$ *and* $0 < \alpha \leq 1$, *then*

$$ {}^{\rho}\mathscr{I}^{\alpha}_{0^+}\, {}^{\rho}\mathscr{D}^{\alpha}_{0^+}g(t) = g(t) + ct^{\rho(\alpha-1)}, \tag{6} $$

for some constant $c \in \mathbb{R}_+$.
3. Let $0 < \beta < \alpha \leq 1$ *be such that* ${}^{\rho}\mathscr{D}^{\alpha}_{0^+}g(t) \in C([0,\ell],\mathbb{R})$ *then*

$$ {}^{\rho}\mathscr{I}^{\alpha-\beta}_{0^+}\, {}^{\rho}\mathscr{D}^{\alpha}_{0^+}g(t) = {}^{\rho}\mathscr{D}^{\beta}_{0^+}g(t) - \frac{\rho^{1-\alpha+\beta}\left({}^{\rho}\mathscr{I}^{1-\alpha}_{0^+}g\right)(0^+)}{\Gamma(\alpha-\beta)}t^{\rho(\alpha-\beta-1)}. \tag{7} $$

Moreover, if $\left({}^{\rho}\mathscr{I}^{1-\alpha}_{0^+}g\right)(0^+) = 0$, *we have*

$$ \left|{}^{\rho}\mathscr{D}^{\beta}_{0^+}g(t)\right| \leq \lambda^{\rho}_{\alpha-\beta}\left\|{}^{\rho}\mathscr{D}^{\alpha}_{0^+}g(t)\right\|_{\infty}, \tag{8} $$

where $\lambda^{\rho}_{\alpha-\beta} = \frac{\ell^{\rho(\alpha-\beta)}}{\rho^{\alpha-\beta}\Gamma(1+\alpha-\beta)}$.

3 Main results

Below, we prepare some important lemmas to illustrate our main results.

Lemma 2. *Let* $(\varphi,\psi), \left({}^{\rho}\mathscr{D}^{\alpha_1}_{0^+}\varphi, {}^{\rho}\mathscr{D}^{\alpha_2}_{0^+}\psi\right) \in C([0,\ell],\mathbb{R}) \times C([0,\ell],\mathbb{R})$. *Then the problem* (1)–(2) *is equivalent to the fractional integral equations:*

$$ \begin{cases} \varphi(t) = \int_0^t G_{\alpha_1}(t,s) f_1\left(s, \varphi(s), \psi(s), {}^{\rho}\mathscr{D}^{\beta_{11}}_{0^+}\varphi(s), {}^{\rho}\mathscr{D}^{\beta_{12}}_{0^+}\psi(s)\right) ds, \\ \psi(t) = \int_0^t G_{\alpha_2}(t,s) f_2\left(s, \varphi(s), \psi(s), {}^{\rho}\mathscr{D}^{\beta_{21}}_{0^+}\varphi(s), {}^{\rho}\mathscr{D}^{\beta_{22}}_{0^+}\psi(s)\right) ds, \end{cases} \tag{9} $$

where $G_{\alpha_i}(t,s) = \frac{\rho^{1-\alpha_i}s^{\rho-1}}{\Gamma(\alpha_i)}\left(t^{\rho} - s^{\rho}\right)^{\alpha_i-1}$.

Proof. Applying ${}^{\rho}\mathscr{I}^{\alpha_1}_{0^+}$ and ${}^{\rho}\mathscr{I}^{\alpha_2}_{0^+}$ to the first and second equations in (1), respectively, we get

$$ \begin{cases} {}^{\rho}\mathscr{I}^{\alpha_1}_{0^+}\, {}^{\rho}\mathscr{D}^{\alpha_1}_{0^+}\varphi(t) = {}^{\rho}\mathscr{I}^{\alpha_1}_{0^+} f_1\left(t, \varphi(t), \psi(t), {}^{\rho}\mathscr{D}^{\beta_{11}}_{0^+}\varphi(t), {}^{\rho}\mathscr{D}^{\beta_{12}}_{0^+}\psi(t)\right), \\ {}^{\rho}\mathscr{I}^{\alpha_2}_{0^+}\, {}^{\rho}\mathscr{D}^{\alpha_2}_{0^+}\psi(t) = {}^{\rho}\mathscr{I}^{\alpha_2}_{0^+} f_2\left(t, \varphi(t), \psi(t), {}^{\rho}\mathscr{D}^{\beta_{21}}_{0^+}\varphi(t), {}^{\rho}\mathscr{D}^{\beta_{22}}_{0^+}\psi(t)\right). \end{cases} \tag{10} $$

By using the relation (6), we obtain

$$ \begin{cases} \varphi(t) = {}^{\rho}\mathscr{I}^{\alpha_1}_{0^+} f_1\left(t, \varphi(t), \psi(t), {}^{\rho}\mathscr{D}^{\beta_{11}}_{0^+}\varphi(t), {}^{\rho}\mathscr{D}^{\beta_{12}}_{0^+}\psi(t)\right) - c_1 t^{\rho(\alpha_1-1)}, \\ \psi(t) = {}^{\rho}\mathscr{I}^{\alpha_2}_{0^+} f_2\left(t, \varphi(t), \psi(t), {}^{\rho}\mathscr{D}^{\beta_{21}}_{0^+}\varphi(t), {}^{\rho}\mathscr{D}^{\beta_{22}}_{0^+}\psi(t)\right) - c_2 t^{\rho(\alpha_2-1)}, \end{cases} \tag{11} $$

for some $c_1, c_2 \in \mathbb{R}$. Taking into account the condition (2) and the fact that

$$ {}^{\rho}\mathscr{I}^{\alpha}_{0^+}t^{\rho(\alpha-1)} = \rho^{\alpha-1}\Gamma(\alpha), $$

we find

$$ 0 = \left({}^{\rho}\mathscr{I}^{1-\alpha_1}_{0^+}\varphi\right)(0^+) = -c_1\rho^{\alpha_1-1}\Gamma(\alpha_1) \quad \Longrightarrow c_1 = 0 \tag{12} $$

and

$$ 0 = \left({}^{\rho}\mathscr{I}^{1-\alpha_2}_{0^+}\psi\right)(0^+) = -c_2\rho^{\alpha_2-1}\Gamma(\alpha_2) \quad \Longrightarrow c_2 = 0. \tag{13} $$

Combining the results (11), (12) and (13), we obtain (9).

Let us define the following Banach spaces [7],

$$E=\left\{\varphi\in C([0,\ell],\mathbb{R})/\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi\right)(0^+)=0\right\},$$

with the norm

$$\|\varphi\|_E=\sup_{0\le t\le\ell}|\varphi(t)|$$

and

$$F=\left\{\psi\in C([0,\ell],\mathbb{R})/\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi\right)(0^+)=0\right\},$$

with the norm

$$\|\psi\|_F=\sup_{0\le t\le\ell}|\psi(t)|.$$

Again the product space $(\Omega,\|\cdot\|_\Omega)$ is a Banach space with norm $\|(\varphi,\psi)\|_\Omega=\|\varphi\|_E+\|\psi\|_F$ for any $(\varphi,\psi)\in\Omega=E\times F$.

Now, we define an operator $\mathscr{T}:\Omega\to C([0,\ell],\mathbb{R})\times C([0,\ell],\mathbb{R})$ by

$$\mathscr{T}(\varphi,\psi)(t)=\left(\mathscr{T}_\varphi(\varphi,\psi)(t),\mathscr{T}_\psi(\varphi,\psi)(t)\right),\tag{14}$$

where

$$\mathscr{T}_\varphi(\varphi,\psi)(t)=\int_0^t G_{\alpha_1}(t,s)f_1\left(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\right)ds,$$
$$\mathscr{T}_\psi(\varphi,\psi)(t)=\int_0^t G_{\alpha_2}(t,s)f_2\left(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(s)\right)ds,$$

and $G_{\alpha_i}(t,s)=\frac{\rho^{1-\alpha_i}s^{\rho-1}}{\Gamma(\alpha_i)}(t^\rho-s^\rho)^{\alpha_i-1}$.

Lemma 3. *Let the integral operator $\mathscr{T}:\Omega\to C([0,\ell],\mathbb{R})\times C([0,\ell],\mathbb{R})$ given in (14), equipped with the norm*

$$\|\mathscr{T}(\varphi,\psi)\|_\infty=\sup_{0\le t\le\ell}\left|\mathscr{T}_\varphi(\varphi,\psi)\right|+\sup_{0\le t\le\ell}\left|\mathscr{T}_\psi(\varphi,\psi)\right|.$$

Then $\mathscr{T}(\Omega)\subset\Omega$.

Proof. Let $(\varphi,\psi)\in\Omega$. From (14), we have

$$\begin{aligned}\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\mathscr{T}_\varphi(\varphi,\psi)\right)(t)&={}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\,{}^{\rho}\mathscr{I}_{0^+}^{\alpha_1}f_1\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right)\\&={}^{\rho}\mathscr{I}_{0^+}^{1}\,f_1\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right)\end{aligned}$$

and

$$\begin{aligned}\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\mathscr{T}_\psi(\varphi,\psi)\right)(t)&={}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\,{}^{\rho}\mathscr{I}_{0^+}^{\alpha_2}f_2\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right)\\&={}^{\rho}\mathscr{I}_{0^+}^{1}f_2\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right).\end{aligned}$$

Using Definition 2 and relation (4), we get

$$\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\mathscr{T}_\varphi(\varphi,\psi)\right)(t)={}^{\rho}\mathscr{I}_{0^+}^{1}\,{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)={}^{\rho}\mathscr{I}_{0^+}^{1}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi(t)={}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\varphi(t)$$

and

$$\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\mathscr{T}_\psi(\varphi,\psi)\right)(t)={}^{\rho}\mathscr{I}_{0^+}^{1}\,{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)={}^{\rho}\mathscr{I}_{0^+}^{1}\left(t^{1-\rho}\frac{d}{dt}\right){}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi(t)={}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\psi(t).$$

Thus

$$\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_1}\mathscr{T}_\varphi(\varphi,\psi)\right)(0^+)=\left({}^{\rho}\mathscr{I}_{0^+}^{1-\alpha_2}\mathscr{T}_\psi(\varphi,\psi)\right)(0^+)=0.$$

As a result $\mathscr{T}(\Omega)\subset\Omega$.

Getting ready to present our results, we propose the following hypotheses:

Hyp.1. Let $f_1, f_2 : [0,\ell]\times\mathbb{R}^4 \to \mathbb{R}$ are continuous functions and there are two strictly positive constants k_1 and k_2 such that

$$|f_i(t,\varphi_1,\varphi_2,\varphi_3,\varphi_4) - f_i(t,\psi_1,\psi_2,\psi_3,\psi_4)| \le k_i \sum_{j=1}^{4} \left|\varphi_j - \psi_j\right|, \qquad i = 1,2,$$

for all $t \in [0,\ell]$ and $\varphi_i, \psi_i \in \mathbb{R},\ i = 1,2,3,4$.

Hyp.2. There exist a positive functions $a_i, b_i \in C([0,\ell],\mathbb{R})$, $i = 1,2,\ldots,5$ such that

$$|f_1(t,\varphi_1,\varphi_2,\varphi_3,\varphi_4)| \le a_1(t) + \sum_{i=2}^{5} a_i(t)|\varphi_i|$$

and

$$|f_2(t,\varphi_1,\varphi_2,\varphi_3,\varphi_4)| \le b_1(t) + \sum_{i=2}^{5} b_i(t)|\varphi_i|,$$

for any $\varphi_i \in \mathbb{R},\ i = 1,2,3,4$ and $t \in [0,\ell]$.

To simplify the computation, we adopt the notation:

$$\lambda_{ij}^{\rho} = \lambda_{\alpha_i-\beta_{ji}}^{\rho} = \frac{\ell^{\rho(\alpha_i-\beta_{ji})}}{\rho^{\alpha-\beta}\Gamma(1+\alpha_i-\beta_{ji})}, \quad i,j = 1,2,$$

$$\bar{a}_i = \max_{0\le t\le \ell}|a_i(t)|, \quad \bar{b}_i = \max_{0\le t\le \ell}|b_i(t)|, \quad i = 1,2,\ldots,5,$$

$$\bar{G}_{\alpha} = \frac{\rho^{-\alpha}\ell^{\rho\alpha}}{\Gamma(\alpha+1)}, \quad \bar{G} = \max\left\{\bar{G}_{\alpha_1}, \bar{G}_{\alpha_2}\right\},$$

$$d_1 = \frac{\bar{a}_1 + \bar{b}_1}{\min\left\{1 - \bar{a}_4\lambda_{11}^{\rho} - \bar{b}_4\lambda_{12}^{\rho}, 1 - \bar{a}_5\lambda_{21}^{\rho} - \bar{b}_5\lambda_{22}^{\rho}\right\}},$$

$$d_2 = \frac{\max\left\{\bar{a}_2 + \bar{b}_2, \bar{a}_3 + \bar{b}_3\right\}}{\min\left\{1 - \bar{a}_4\lambda_{11}^{\rho} - \bar{b}_4\lambda_{12}^{\rho}, 1 - \bar{a}_5\lambda_{21}^{\rho} - \bar{b}_5\lambda_{22}^{\rho}\right\}},$$

with

$$\max_{i\in\{1,2\}}\left\{\bar{a}_{3+i}\lambda_{i1}^{\rho} + \bar{b}_{3+i}\lambda_{i2}^{\rho}, k_1\lambda_{i1}^{\rho} + k_2\lambda_{i2}^{\rho}, \frac{\bar{G}_{\alpha_1}k_1\lambda_{i1}^{\rho} + \bar{G}_{\alpha_2}k_2\lambda_{i2}^{\rho}}{\bar{G}_{\alpha_i}}\right\} < 1. \tag{15}$$

Now, we present the principal theorems

Theorem 1. *Assume (Hyp.1) holds. If*

$$k_G = \frac{\left(k_1\bar{G}_{\alpha_1} + k_2\bar{G}_{\alpha_2}\right)\bar{G}}{\min\limits_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i} - \left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho} + k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}} < 1, \tag{16}$$

then the problem (1)–(2) *has a unique solution on* $[0,\ell]$.

Proof. First, we define the fixed point problem, which is equivalent to the one problem (1)–(2) by

$$\mathscr{T}(\varphi,\psi)(t) = (\varphi,\psi)(t). \tag{17}$$

Let $(\varphi,\psi), (\bar{\varphi},\bar{\psi}) \in \Omega$, then we have

$$\begin{aligned}
&\left|\mathscr{T}_{\varphi}(\varphi,\psi)(t) - \mathscr{T}_{\varphi}(\bar{\varphi},\bar{\psi})(t)\right| \\
=&\left|\int_0^t G_{\alpha_1}(t,s)\left[f_1\left(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\right)\right.\right. \\
&\left.\left.-f_1\left(s,\bar{\varphi}(s),\bar{\psi}(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\bar{\varphi}(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\bar{\psi}(s)\right)\right]ds\right| \\
=&\left|\int_0^t G_{\alpha_1}(t,s)\left[{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s) - {}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(s)\right]ds\right| \\
\le& \int_0^t G_{\alpha_1}(t,s)\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s) - {}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(s)\right|ds
\end{aligned}$$

Using Hölder inequality and the fact that

$$\sup_{0\le t\le \ell}\int_0^t G_{\alpha_1}(t,s)\,ds=\frac{\rho^{-\alpha_1}\ell^{\rho\alpha_1}}{\Gamma(\alpha_1+1)},$$

we get

$$\begin{aligned}\left\|\mathscr{T}_\varphi(\varphi,\psi)(t)-\mathscr{T}_\varphi(\bar\varphi,\bar\psi)(t)\right\|_\infty &\le \int_0^t G_{\alpha_1}(t,s)\,ds\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty\\ &\le \frac{\rho^{-\alpha_1}\ell^{\rho\alpha_1}}{\Gamma(\alpha_1+1)}\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty.\end{aligned}\tag{18}$$

And in the same way, we obtain

$$\left\|\mathscr{T}_\psi(\varphi,\psi)(t)-\mathscr{T}_\psi(\bar\varphi,\bar\psi)(t)\right\|_\infty \le \frac{\rho^{-\alpha_2}\ell^{\rho\alpha_2}}{\Gamma(\alpha_2+1)}\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty.\tag{19}$$

Also, we have

$$\begin{aligned}\left\|\mathscr{T}(\varphi,\psi)(t)-\mathscr{T}(\bar\varphi,\bar\psi)(t)\right\|_\infty &\le \bar G_{\alpha_1}\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty+\bar G_{\alpha_2}\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty\\ &\le \bar G\left(\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty+\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty\right).\end{aligned}\tag{20}$$

By taking into account the hypothesis (Hyp.1), we obtain

$$\begin{aligned}\frac{1}{k_1}\left|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right| &\le |\varphi(t)-\bar\varphi(t)|+|\psi(t)-\bar\psi(t)|+\left|{}^\rho\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\beta_{11}}\bar\varphi(t)\right|\\ &+\left|{}^\rho\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\beta_{12}}\bar\psi(t)\right|.\end{aligned}$$

Using the equality (8), we get

$$\begin{aligned}\frac{1}{k_1}\left|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right| &\le |\varphi(t)-\bar\varphi(t)|+\lambda_{11}^\rho\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty\\ &+|\psi(t)-\bar\psi(t)|+\lambda_{21}^\rho\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty,\end{aligned}$$

Consequently

$$\begin{aligned}\frac{1}{k_1}\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty &\le \|\varphi(t)-\bar\varphi(t)\|_\infty+\lambda_{11}^\rho\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty\\ &+\|\psi(t)-\bar\psi(t)\|_\infty+\lambda_{21}^\rho\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty.\end{aligned}\tag{21}$$

In the same way, we can get

$$\begin{aligned}\frac{1}{k_2}\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty &\le \|\varphi(t)-\bar\varphi(t)\|_\infty+\lambda_{12}^\rho\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty\\ &+\|\psi(t)-\bar\psi(t)\|_\infty+\lambda_{22}^\rho\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty.\end{aligned}\tag{22}$$

Multiplying (21) by $k_1\bar G_{\alpha_1}$ and (22) by $k_2\bar G_{\alpha_2}$, then take the sum, we obtain

$$\begin{aligned}&\bar G_{\alpha_1}\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty+\bar G_{\alpha_2}\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty\\ &\le \left(k_1\bar G_{\alpha_1}+k_2\bar G_{\alpha_2}\right)\{\|\varphi(t)-\bar\varphi(t)\|_\infty+\|\psi(t)-\bar\psi(t)\|_\infty\}\\ &+\left(k_1\bar G_{\alpha_1}\lambda_{11}^\rho+k_2\bar G_{\alpha_2}\lambda_{12}^\rho\right)\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_1}\bar\varphi(t)\right\|_\infty\\ &+\left(k_1\bar G_{\alpha_1}\lambda_{21}^\rho+k_2\bar G_{\alpha_2}\lambda_{22}^\rho\right)\left\|{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^\rho\mathscr{D}_{0^+}^{\alpha_2}\bar\psi(t)\right\|_\infty,\end{aligned}\tag{23}$$

thus

$$
\begin{aligned}
&\min_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}\left[\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_\infty+\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_\infty\right]\\
&\le \bar{G}_{\alpha_1}-\left(k_1\bar{G}_{\alpha_1}\lambda_{11}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{12}^{\rho}\right)\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_\infty\\
&+\bar{G}_{\alpha_2}-\left(k_1\bar{G}_{\alpha_1}\lambda_{21}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{22}^{\rho}\right)\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_\infty\\
&\le \left(k_1\bar{G}_{\alpha_1}+k_2\bar{G}_{\alpha_2}\right)\|(\varphi(t),\psi(t))-(\bar{\varphi}(t),\bar{\psi}(t))\|_\Omega, \qquad (24)
\end{aligned}
$$

relation (15) guarantees that $\min_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}>0$, then

$$
\begin{aligned}
&\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\bar{\varphi}(t)\right\|_\infty+\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\bar{\psi}(t)\right\|_\infty\\
&\le \frac{k_1\bar{G}_{\alpha_1}+k_2\bar{G}_{\alpha_2}}{\min_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}}\|(\varphi(t),\psi(t))-(\bar{\varphi}(t),\bar{\psi}(t))\|_\Omega. \qquad (25)
\end{aligned}
$$

Combining (20) and (25), we get

$$
\|\mathscr{T}(\varphi,\psi)(t)-\mathscr{T}(\bar{\varphi},\bar{\psi})(t)\|_\Omega\le k_G\|(\varphi(t),\psi(t))-(\bar{\varphi}(t),\bar{\psi}(t))\|_\Omega,
$$

where

$$
k_G=\frac{\left(k_1\bar{G}_{\alpha_1}+k_2\bar{G}_{\alpha_2}\right)\bar{G}}{\min_{i\in\{1,2\}}\left\{\bar{G}_{\alpha_i}-\left(k_1\bar{G}_{\alpha_1}\lambda_{i1}^{\rho}+k_2\bar{G}_{\alpha_2}\lambda_{i2}^{\rho}\right)\right\}}.
$$

Since $k_G<1$ according to (16), then $\mathscr{T}$ is a contraction operator and has unique fixed point following the Banach's contraction principle [15]. Which means that the problem (1)–(2) has a unique solution on $[0,\ell]$.

Theorem 2. *Assume that hypotheses (Hyp.1) and (Hyp.2) hold. If we put*

$$
\bar{G}d_2<1, \qquad (26)
$$

then the problem (1)–(2) *has at least one solution on* $[0,\ell]$.

Proof. As in the previous proof, we will prove that the operator (17) has a fixed point using Schauder's theorem [15]. This is done through three steps:

Step 1: A is a continuous operator. Let $(\varphi_n,\psi_n)_{n\in\mathbb{N}}$ be real sequences such that $(\varphi_n,\psi_n)\to(\varphi,\psi)$ in Ω.
Using the same techniques used to prove theorem 1, then by replacing $(\bar{\varphi},\bar{\psi})$ by (φ_n,ψ_n), the relations (21) and (22) became

$$
\begin{aligned}
\frac{1}{k_1}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_\infty &\le \|\varphi_n(t)-\varphi(t)\|_\infty+\lambda_{11}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_\infty\\
&+\|\psi_n(t)-\psi(t)\|_\infty+\lambda_{21}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_\infty \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{k_2}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_\infty &\le \|\varphi_n(t)-\varphi(t)\|_\infty+\lambda_{12}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_\infty\\
&+\|\psi_n(t)-\psi(t)\|_\infty+\lambda_{22}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_\infty. \qquad (28)
\end{aligned}
$$

By combining (27) and (28), we obtain

$$
\begin{aligned}
&\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_\infty+\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_\infty\\
&\le \frac{(k_1+k_2)}{\min_{i\in\{1,2\}}\left\{1-k_1\lambda_{i1}^{\rho}-k_2\lambda_{i2}^{\rho}\right\}}\|(\varphi_n(t),\psi_n(t))-(\varphi(t),\psi(t))\|_\Omega,
\end{aligned}
$$

and from (15), we answer that $\min_{i\in\{1,2\}}\left\{1-k_1\lambda_{i1}^{\rho}-k_2\lambda_{i2}^{\rho}\right\}>0$. As $(\varphi_n,\psi_n)\underset{n\to\infty}{\longrightarrow}(\varphi,\psi)$ in Ω, then $\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n,{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n\right)\underset{n\to\infty}{\longrightarrow}\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi,{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi\right)$, for all $t\in[0,\ell]$.

Now, let $\delta > 0$ be such that for each $t \in [0,\ell]$, we have

$$\sup\left\{\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(t)\right|, \left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(t)\right|, \left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right|, \left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right|\right\} \leq \delta.$$

Then, we have

$$\begin{aligned}
&\Big|G_{\alpha_1}(t,s)\Big[f_1\Big(s,\varphi_n(s),\psi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi_n(s)\Big)\\
&-f_1\Big(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\Big)\Big]\Big|\\
&=\left|G_{\alpha_1}(t,s)\left({}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(s)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s)\right)\right|\\
&\leq G_{\alpha_1}(t,s)\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(s)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s)\right|\\
&\leq G_{\alpha_1}(t,s)\left(\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi_n(s)\right|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(s)\right|\right)\\
&\leq 2\delta G_{\alpha_1}(t,s)
\end{aligned}$$

and in the same way we find

$$G_{\alpha_2}(t,s)\left(\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi_n(s)-{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(s)\right|\right)\leq 2\delta G_{\alpha_2}(t,s)\,ds.$$

Which means that the functions $s\to\delta G_{\alpha_i}(t,s)$, $i=1,2$ are integrable for all $t\in[0,\ell]$.
Then Lebesgue dominated convergence theorem is applicable to the following

$$\begin{aligned}
&\Big|\int_0^t G_{\alpha_1}(t,s)\Big[f_1\Big(s,\varphi_n(s),\psi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi_n(s)\Big)\\
&-f_1\Big(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\Big)\Big]ds\Big|\\
&=\left|\mathscr{T}_{\varphi}(\varphi_n,\psi_n)(t)-\mathscr{T}_{\varphi}(\varphi,\psi)(t)\right|\underset{n\to\infty}{\longrightarrow}0
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_0^t G_{\alpha_2}(t,s)\Big[f_2\Big(s,\varphi_n(s),\psi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi_n(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi_n(s)\Big)\\
&-f_2\Big(s,\varphi(s),\psi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(s),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(s)\Big)\Big]ds\Big|\\
&=\left|\mathscr{T}_{\psi}(\varphi_n,\psi_n)(t)-\mathscr{T}_{\psi}(\varphi,\psi)(t)\right|\underset{n\to\infty}{\longrightarrow}0.
\end{aligned}$$

Therefore

$$\|\mathscr{T}(\varphi,\psi)(t)-\mathscr{T}(\bar{\varphi},\bar{\psi})(t)\|_{\Omega}\underset{n\to\infty}{\longrightarrow}0.$$

Hence the continuity of the operator $\mathscr{T}$.

Step 2: $A(B_\tau)\subset B_\tau$. Let B_τ be bounded, closed and convex subset of Ω, define by

$$B_\tau=\{(\varphi,\psi)\in\Omega\ /\ \|(\varphi,\psi)\|_{\Omega}\leq\tau\},$$

where $\tau\geq\frac{d_1}{(1/\bar{G}-d_2)}$.

Let $\mathscr{T}:B_\tau\to\Omega$ be the operator defined in (14). Then by applying the inequality (8) and hypothses (Hyp.2) for all $t\in[0,\ell]$, we have

$$\begin{aligned}
\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right| &=\left|f_1\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right)\right| \\
&\leq a_1(t)+a_2(t)|\varphi(t)|+a_3(t)|\psi(t)|+a_4(t)\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{11}}\varphi(t)\right|+a_5(t)\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{12}}\psi(t)\right|\\
&\leq\bar{a}_1+\bar{a}_2\|\varphi(t)\|_\infty+\bar{a}_3\|\psi(t)\|_\infty+\bar{a}_4\lambda_{11}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_\infty+\bar{a}_5\lambda_{21}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_\infty
\end{aligned}\tag{29}$$

and

$$\begin{aligned}
\left|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\varphi(t)\right| &=\left|f_2\left(t,\varphi(t),\psi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t),{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right)\right| \\
&\leq b_1(t)+b_2(t)|\varphi(t)|+b_3(t)|\psi(t)|+b_4(t)\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{21}}\varphi(t)\right|+b_5(t)\left|{}^{\rho}\mathscr{D}_{0^+}^{\beta_{22}}\psi(t)\right|\\
&\leq\bar{b}_1+\bar{b}_2\|\varphi(t)\|_\infty+\bar{b}_3\|\psi(t)\|_\infty+\bar{b}_4\lambda_{12}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_1}\varphi(t)\right\|_\infty+\bar{b}_5\lambda_{22}^{\rho}\left\|{}^{\rho}\mathscr{D}_{0^+}^{\alpha_2}\psi(t)\right\|_\infty,
\end{aligned}\tag{30}$$

Combining the results (29) and (30). Similarly to (20), for all $(\varphi,\psi)\in B_\tau$ we get

$$\begin{aligned}\left\|\mathscr{T}(\varphi,\psi)(t)\right\|_\Omega &\le \bar{G}d_1+\bar{G}d_2\tau\\ &=\bar{G}\left[\left(1/\bar{G}-d_2\right)\frac{d_1}{\left(1/\bar{G}-d_2\right)}+d_2\tau\right]\\ &\le\tau.\end{aligned}$$

Then, we conclude that $\mathscr{T}(B_\tau)\subset B_\tau$.

Step 3: $A(B_\tau)$ is relatively compact. Let $t_1,t_2\in[0,\ell]$, $t_1<t_2$ and $(\varphi,\psi)\in B_\tau$. Then, we get

$$\begin{aligned}&\left|\mathscr{T}_\varphi(\varphi,\psi)(t_2)-\mathscr{T}_\varphi(\varphi,\psi)(t_1)\right|+\left|\mathscr{T}_\psi(\varphi,\psi)(t_2)-\mathscr{T}_\psi(\varphi,\psi)(t_1)\right|\\ &\le(d_1+d_2\tau)\left[\max_{i\in\{1,2\}}\int_0^{t_1}\left|G_{\alpha_i}(t_2,s)-G_{\alpha_i}(t_1,s)\right|ds+\max_{i\in\{1,2\}}\int_{t_1}^{t_2}G_{\alpha_i}(t_2,s)\,ds\right].\end{aligned}\tag{31}$$

On the other hand

$$\begin{aligned}\int_0^{t_1}\left|G_{\alpha_i}(t_2,s)-G_{\alpha_i}(t_1,s)\right|ds&=\frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)}\int_0^{t_1}s^{\rho-1}\left|\left(t_2^\rho-s^\rho\right)^{\alpha_i-1}-\left(t_1^\rho-s^\rho\right)^{\alpha_i-1}\right|ds\\ &\le\frac{1}{\alpha_i\rho^{\alpha_i}\Gamma(\alpha_i)}\left[\left(t_2^\rho-t_1^\rho\right)^{\alpha_i}+\left(t_2^{\rho\alpha_i}-t_1^{\rho\alpha_i}\right)\right]\end{aligned}\tag{32}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\left|G_{\alpha_i}(t_2,s)\right|ds&=\frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)}\int_{t_1}^{t_2}s^{\rho-1}\left(t_2^\rho-s^\rho\right)^{\alpha_i-1}ds\\ &=\frac{1}{\alpha_i\rho^{\alpha_i}\Gamma(\alpha_i)}\left(t_2^\rho-t_1^\rho\right)^{\alpha_i}.\end{aligned}\tag{33}$$

Applying (32) and (33), then (31) becomes

$$\begin{aligned}&\left|\mathscr{T}_\varphi(\varphi,\psi)(t_2)-\mathscr{T}_\varphi(\varphi,\psi)(t_1)\right|+\left|\mathscr{T}_\psi(\varphi,\psi)(t_2)-\mathscr{T}_\psi(\varphi,\psi)(t_1)\right|\\ &\le(d_1+d_2\tau)\left[\max_{i\in\{1,2\}}\left\{\frac{1}{\alpha_i\rho^{\alpha_i}\Gamma(\alpha_i)}\left[\left(t_2^\rho-t_1^\rho\right)^{\alpha_i}+\left(t_2^{\rho\alpha_i}-t_1^{\rho\alpha_i}\right)\right]\right\}\right.\\ &\quad\left.+\max_{i\in\{1,2\}}\left\{\frac{1}{\alpha_i\rho^{\alpha_i}\Gamma(\alpha_i)}\left(t_2^\rho-t_1^\rho\right)^{\alpha_i}\right\}\right].\end{aligned}$$

Hence, we conclude that for all $(\varphi,\psi)\in B_\tau$, $\left\|\mathscr{T}(\varphi,\psi)(t_2)-\mathscr{T}(\varphi,\psi)(t_1)\right\|_\Omega\underset{t_1\to t_2}{\longrightarrow}0$.

From step 1-3 and Ascoli-Arzelà Theorem [1], we show that $\mathscr{T}:B_\tau\to B_\tau$ is continuous, compact and so by Schauder's fixed point, the operator $\mathscr{T}$ has at least one fixed point which corresponds to the solution of the problem (1)–(2) on $[0,\ell]$.

4 Examples

Example 1. Consider the following problem

$$\begin{cases}{}^\rho\mathscr{D}_{0^+}^{\frac{1}{2}}\varphi(t)=\dfrac{1/2}{\sqrt{2}\cos\left(\frac{\pi t}{4}\right)+|\varphi(t)|+|\psi(t)|}+\dfrac{1/11}{\cosh t+\left|{}^\rho\mathscr{D}_{0^+}^{\frac{1}{4}}\varphi(t)\right|+\left|{}^\rho\mathscr{D}_{0^+}^{\frac{1}{8}}\psi(t)\right|}, & t\in[0,1],\\ {}^\rho\mathscr{D}_{0^+}^{\frac{2}{3}}\psi(t)=\dfrac{1/4}{1+t+|\varphi(t)|+|\psi(t)|+\left|{}^\rho\mathscr{D}_{0^+}^{\frac{1}{2}}\varphi(t)\right|+\left|{}^\rho\mathscr{D}_{0^+}^{\frac{1}{3}}\psi(t)\right|}, & t\in[0,1],\\ \left({}^\rho\mathscr{I}_{0^+}^{\frac{1}{2}}\varphi\right)(0^+)=\left({}^\rho\mathscr{I}_{0^+}^{\frac{1}{3}}\psi\right)(0^+)=0.\end{cases}\tag{34}$$

Obviously, the condition (Hyp.1) is satisfied with $k_1=1/11$ and $k_2=1/4$. Then;

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
k_G	10.76	3.917	2.050	1.458	1.148	0.963	0.838	0.745	0.674	0.628
	Theorem 1 is not applicable in this example.					from Theorem 1, the problem (34) has a unique solution.				

Example 2. Consider the following problem

$$\begin{cases} {}^{\rho}\mathscr{D}_{0^+}^{\frac{1}{4}}\varphi(t) = \frac{10^{-2}\sin t}{1+|\varphi(t)|+|\psi(t)|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\frac{1}{9}}\varphi(t)\right|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\frac{1}{7}}\psi(t)\right|}, & t\in[0,2], \\ {}^{\rho}\mathscr{D}_{0^+}^{\frac{4}{5}}\psi(t) = \frac{3e^{t-2}}{5}\frac{|\varphi(t)|}{1+|\varphi(t)|} + \frac{10^{-2}\cos t}{1+t+|\psi(t)|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\frac{1}{5}}\varphi(t)\right|+\left|{}^{\rho}\mathscr{D}_{0^+}^{\frac{2}{9}}\psi(t)\right|}, & t\in[0,2], \\ \left({}^{\rho}\mathscr{I}_{0^+}^{\frac{3}{4}}\varphi\right)(0^+) = \left({}^{\rho}\mathscr{I}_{0^+}^{\frac{1}{5}}\psi\right)(0^+) = 0. \end{cases} \tag{35}$$

Obviously, the hypotheses (Hyp.1) and (Hyp.2) are satisfied with $k_1 = 10^{-2}$, $k_2 = 3/5$, $\bar{a}_1 = \bar{b}_1 = 1$, $\bar{a}_2 = 0$, $\bar{b}_2 = 3/5$ and $\bar{a}_i = \bar{b}_i = 0$ for $i = 2,3,4,5$. Then, $d_1 = 2$, $d_2 = 0.6$; Thus

ρ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\bar{G}d_2$	4.065	2.334	1.688	1.341	1.122	0.969	0.857	0.770	0.701	0.662
	Theorem 2 is not applicable in this example.					from Theorem 2, the problem (35) has at least a solution.				

5 Conclusion

Using the Banach contraction principle and Schauder's fixed point theorem, this paper explores the existence and main properties of at least one solution and its uniqueness for a class of new coupled systems of nonlinear multi-term fractional differential equations with integral conditions. Katugampola's fractional derivative is used as the differential operator, which is crucial to generalizing Hadamard and Riemann-Liouville's fractional derivatives into a single form.

Declarations

Competing interests: The authors declare no competing interests.
Authors' contributions:
Billal Lekdim: Formal analysis; Investigation; Resources; Software; Visualization; Writing-original draft.
Bilal Basti: Methodology; Supervision; Validation; Writing-review and editing; Project administration.
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Relations between $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q}$ and $T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$ and their applications

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Abstract: Let $\mathscr{B}^+(\mathscr{H})$ represent the cone comprising all positive invertible operators on a complex separable Hilbert space $\mathscr{H}$. When T and S belong to $\mathscr{B}^+(\mathscr{H})$, it holds true that for any $\gamma \geq 0$, $\delta \geq 0$, and $0 < q \leq 1$, the following two inequalities are equivalent:

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q} \quad \text{and} \quad T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$$

In this article, we will explore the connections between these inequalities and provide some applications of this discovery to operator class theory. Furthermore, we will provide a positive response to the question posed in [16].

Keywords: class p-$wA(\alpha,\beta)$; Löwner-Heinz theorem; Normal operator; Aluthge transformation.

2010 Mathematics Subject Classification. 47B20; 47A11; 47A63.

1 Introduction

Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra encompassing all bounded linear operators acting on a complex, separable Hilbert space referred to as $\mathscr{H}$. Within this context, we use the symbol I to represent the identity operator. An operator, denoted as T, is characterized as positive, denoted as $T \geq 0$, if it satisfies the condition $\langle Tx, x\rangle \geq 0$ for every vector x in the Hilbert space $\mathscr{H}$. Additionally, an operator T is regarded as strictly positive, symbolized as $T > 0$, if it fulfills two criteria: firstly, it must be positive, and secondly, it must be invertible, meaning that $\langle Tx, x\rangle > 0$ for all nonzero vectors x within $\mathscr{H}$. To clarify further, when we express $T \geq S \geq 0$, it indicates that the operator $T - S$ is positive, or in other words, $\langle (T-S)x, x\rangle \geq 0$ for all vectors x within the Hilbert space $\mathscr{H}$.

The following result, which is crucial to understanding non-normal operators, is the first in this section.

Theorem 1(Furuta's inequality[10]). *If $T \geq S \geq 0$, then for each $t \geq 0$,*

(i)$(S^{\frac{t}{2}}T^pS^{\frac{t}{2}})^{\frac{1}{q}} \geq S^{\frac{t+p}{q}}$ and
(ii)$T^{\frac{t+p}{q}} \geq (T^{\frac{p}{2}}S^tT^{\frac{p}{2}})^{\frac{1}{q}}$
hold for $p \geq 0$ and $q \geq 1$ with $(1+t)q \geq p+t$.

It's worth mentioning that if we substitute $t = 0$ into either condition (i) or (ii) from the previously mentioned theorems, we obtain the well-known Löwener-Heinz theorem, which asserts that "$T \geq S \geq 0$ guarantees $T^{\alpha} \geq S^{\alpha}$ for any $\alpha \in [0,1]$." The subsequent results were established as applications of Theorem 1 in the references [7] and [11]. For positive invertible operators T and S, the order relation $\log T \geq \log S$ (referred to as chaotic order) holds if and only if $(S^{\frac{r}{2}}T^pS^{\frac{r}{2}})^{\frac{r}{p+r}} \geq S^r$

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for all $p \geq 0$ and $r \geq 0$, and this equivalence also extends to $T^p \geq (T^{\frac{p}{2}}S^rT^{\frac{p}{2}})^{\frac{p}{p+r}}$ for all $p \geq 0$ and $r \geq 0$. It's worth noting that when $p = r$, this conclusion serves as an extension of the results presented in [2]. The following assertions are well-established concerning these operator inequalities: Let T and S be strictly positive operators. Then, we have

(a)$T \geq S \Rightarrow \log T \geq \log S$.

(b)$\log T \geq \log S \Rightarrow (S^{\frac{\alpha}{2}}T^{\beta}S^{\frac{\alpha}{2}})^{\frac{\alpha}{\beta+\alpha}} \geq S^{\alpha}$ and $T^{\beta} \geq (T^{\frac{\beta}{2}}S^{\alpha}T^{\frac{\beta}{2}})^{\frac{\beta}{\beta+\alpha}}$ for all $\beta \geq 0$ and $\alpha \geq 0$.

(c)For each $\beta \geq 0$ and $\alpha \geq 0$, $(S^{\frac{\alpha}{2}}T^{\beta}S^{\frac{\alpha}{2}})^{\frac{\alpha}{\beta+\alpha}} \geq S^{\alpha} \Leftrightarrow T^{\beta} \geq (T^{\frac{\beta}{2}}S^{\alpha}T^{\frac{\beta}{2}})^{\frac{\beta}{\beta+\alpha}}$ [11].

Regarding these findings, the requirement for invertibility in conditions (a) and (b) can be substituted with the condition $\ker(T) = \ker(S) = 0$. This condition implies that (a) and (b) remain valid even for specific non-invertible operators T and S, as established in [24]. The authors of [15] delved into the relationships between the following inequalities:

$$(S^{\frac{\alpha}{2}}T^{\beta}S^{\frac{\alpha}{2}})^{\frac{\alpha}{\beta+\alpha}} \geq S^{\alpha} \quad \text{and} \quad T^{\beta} \geq (T^{\frac{\beta}{2}}S^{\alpha}T^{\frac{\beta}{2}})^{\frac{\beta}{\beta+\alpha}}$$

when it is not possible to invert operators T and S.

An operator $T \in \mathscr{B}(\mathscr{H})$ is referred to as hyponormal when it satisfies the inequality $T^*T \geq TT^*$. The Aluthge transformation, denoted as $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, was introduced by Aluthge in [1]. It is a key component of the polar decomposition of $T \in \mathscr{B}(\mathscr{H})$, which can be represented as $T = U|T|$. Furthermore, the formula $\tilde{T}_{s,t} = |T|^sU|T|^t$ describes the generalized Aluthge transformation $\tilde{T}_{s,t}$ with $0 < s,t$. It's important to note that an operator $T \in \mathscr{B}(\mathscr{H})$ is defined as p-hyponormal if $(T^*T)^p \geq (TT^*)^p$. Additionally, it falls into class $wA(s,t)$ if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$ ([14]). The class $A(k)$, which encompasses p-hyponormal and log-hyponormal operators, was introduced by Furuta et al. in their study [9], where $A(1)$ corresponds to the class A operator. Furthermore, if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$, we assert that an operator T belongs to class $A(k)$, where $k > 0$. In this paper, we aim to establish the relationships between the following inequalities:

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q} \quad \text{and} \quad T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}} \tag{1}$$

These relationships will be explored in cases where operators T and S are not invertible. We will also demonstrate the normality of the class p-$A(\alpha,\beta)$ for $\alpha > 0, \beta > 0$, and $0 < p \leq 1$. Furthermore, we will prove that if either T or T belongs to class p-$A(\alpha,\beta)$ for some $\alpha > 0, \beta > 0$, with $0 < p \leq 1$, and S is an operator such that $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is a self-adjoint operator.

2 Relations between $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q}$ and $T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$

In this section, we will present the following outcome:

Theorem 2. *Let $T,S \in \mathscr{B}^+(\mathscr{H})$. Then for each $\gamma \geq 0$, $\delta \geq 0$ and $0 < q \leq 1$, the following assertions hold:*

(i)If $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{\delta q}$, then $T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$.

(ii)If $T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}$ and $\ker(T) \subset \ker(S)$, then $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{q\delta}$.

We would like to note that Theorem 2 serves as an extension of Theorem 1 in [15]. The following results are organized to provide a proof and illustration of Theorem 2.

Lemma 1. *[13, Löwner-Heinz inequality] Let $T,S \in \mathscr{B}^+(\mathscr{H})$. If $T \geq S \geq 0$, then $T^{\gamma} \geq S^{\gamma}$ for every $\gamma \in [0,1]$.*

Lemma 2. *[8] Let $T,S \in \mathscr{B}(\mathscr{H})$. Assume that T is positive ($T > 0$), and that S is an invertible operator. Under these conditions, the following holds for any real number λ:*

$$(STS^*)^{\lambda} = ST^{\frac{1}{2}}(T^{\frac{1}{2}}S^*ST^{\frac{1}{2}})^{\lambda-1}T^{\frac{1}{2}}S^*.$$

Proof. For the sake of convenience, we provide a proof of this self-evident result. Let's start with the polar decomposition of the invertible operator $ST^{\frac{1}{2}}$ as $ST^{\frac{1}{2}} = UQ$, where U is a unitary operator and $Q = |ST^{\frac{1}{2}}|$. Then,

$$\begin{aligned}(STS^*)^{\lambda} &= (UQ^2U)^{\lambda} = UQ^{2\lambda}U^* \\ &= ST^{\frac{1}{2}}Q^{-1}Q^{2\lambda}Q^{-1}T^{\frac{1}{2}}S^* = ST^{\frac{1}{2}}(Q^2)^{\lambda-1}T^{\frac{1}{2}}S^* \\ &= ST^{\frac{1}{2}}(T^{\frac{1}{2}}S^*ST^{\frac{1}{2}})^{\lambda-1}T^{\frac{1}{2}}S^*.\end{aligned}$$

Proposition 1.*[21] Let $T,S \in \mathscr{B}^+(\mathscr{H})$. Consequently, the following statements are true:*

(i)If $(S^{\frac{\delta_0}{2}}T^{\gamma_0}S^{\frac{\delta_0}{2}})^{\frac{\delta_0 p}{\gamma_0+\delta_0}} \geq S^{\delta_0 p}$ *maintains for fixed* $\gamma_0 > 0$, $\delta_0 > 0$ *and* $0 < p \leq 1$, *then*

$$(S^{\frac{\delta}{2}}T^{\gamma_0}S^{\frac{\delta}{2}})^{\frac{\delta p_1}{\gamma_0+\delta}} \geq S^{\delta p_1} \tag{2}$$

holds for any $\delta \geq \delta_0$ *and* $0 < p_1 \leq p \leq 1$. *Moreover, for each fixed* $\gamma \geq -\gamma_0$,

$$f_{\gamma_0,\gamma}(\delta) = (T^{\frac{\gamma_0}{2}}S^{\delta}T^{\frac{\gamma_0}{2}})^{\frac{(\gamma_0+\gamma)p_1}{\gamma_0+\delta}}$$

is a decreasing function for $\delta \geq \max\{\delta_0,\gamma\}$. *Hence the inequality*

$$(T^{\frac{\gamma_0}{2}}S^{\delta_1}T^{\frac{\gamma_0}{2}})^{p_1} \geq (T^{\frac{\gamma_0}{2}}S^{\delta_2}T^{\frac{\gamma_0}{2}})^{\frac{p_1(\gamma_0+\delta_1)}{\gamma_0+\delta_2}} \tag{3}$$

holds for any δ_1 *and* δ_2 *such that* $\delta_2 \geq \delta_1 \geq \delta_0$ *and* $0 < p_1 \leq p$.

(ii)If $T^{\gamma_0 p} \geq (T^{\frac{\gamma_0}{2}}S^{\delta_0}T^{\frac{\gamma_0}{2}})^{\frac{\gamma_0 p}{\gamma_0+\delta_0}}$ *holds for fixed* $\gamma_0 > 0$, $\delta_0 > 0$ *and* $0 < p \leq 1$, *then*

$$T^{\gamma p_1} \geq (T^{\frac{\gamma}{2}}S^{\delta_0}T^{\frac{\gamma}{2}})^{\frac{\gamma p_1}{\gamma+\delta_0}} \tag{4}$$

holds for any $\gamma \geq \gamma_0$ *and* $0 < p_1 \leq p \leq 1$. *Furthermore, for each fixed* $\delta \geq -\delta_0$,

$$g_{\delta_0,\delta}(\gamma) = (S^{\frac{\delta_0}{2}}T^{\gamma}S^{\frac{\delta_0}{2}})^{\frac{(\delta+\delta_0)p_1}{\gamma+\delta_0}}$$

is an increasing function for $\gamma \geq \max\{\gamma_0,\delta\}$. *Therefore the inequality*

$$(S^{\frac{\delta_0}{2}}T^{\gamma_2}S^{\frac{\delta_0}{2}})^{\frac{p_1(\gamma_1+\delta_0)}{\gamma_2+\delta_0}} \geq (S^{\frac{\delta_0}{2}}T^{\gamma_1}S^{\frac{\delta_0}{2}})^{p_1} \tag{5}$$

holds for any γ_1 *and* γ_2 *such that* $\gamma_2 \geq \gamma_1 \geq \gamma_0$ *and* $0 < p_1 \leq p$.

By applying the Furuta inequality, we derive Theorem 2. Our approach relies on the utilization of the subsequent expression, which constitutes a pivotal element of the Furuta inequality presented in Theorem 1.

Lemma 3.*Let $T,S \in \mathscr{B}(\mathscr{H})$. If $T \geq S \geq 0$, then*

(i)$(S^{x/2}T^yS^{x/2})^{\frac{1+x}{x+y}} \geq S^{1+x}$ *and*

(ii)$T^{1+x} \geq (T^{x/2}S^yT^{x/2})^{\frac{1+x}{x+y}}$

hold for $x \geq 0$ *and* $y \geq 1$.

Proof(Proof of Theorem 2). (i) Suppose that the following relation

$$(S^{\delta_0/2}T^{\gamma_0}S^{\delta_0/2})^{\frac{q\delta_0}{\gamma_0+\delta_0}} \geq S^{q\delta_0} \tag{6}$$

holds for fixed $\gamma_0 > 0$ and $\delta_0 > 0$ and $0 < q \leq 1$. Applying Lemma 3 to (6), we have

$$\{S^{\frac{q\delta_0 r_1}{2}}(S^{\delta_0/2}T^{\gamma_0}S^{\delta_0/2})^{\frac{p_1 q\delta_0}{\gamma_0+\delta_0}}S^{\frac{q\delta_0 r_1}{2}}\}^{\frac{1+r_1}{p_1+r_1}} \geq S^{q\delta_0(1+r_1)} \tag{7}$$

for any $p_1 \geq 1$ and $r_1 \geq 0$. Putting $p_1 = \frac{\gamma_0+\delta_0}{q\delta_0}$ in (7), we have

$$(S^{\frac{\delta_0(1+qr_1)}{2}}T^{\gamma_0}S^{\frac{\delta_0(1+qr_1)}{2}})^{\frac{q\delta_0(1+r_1)}{\gamma_0+\delta_0+r_1 q\delta_0}} \geq S^{q\delta_0(1+r_1)} \tag{8}$$

for any $r_1 \geq 0$. Put $\delta = \delta_0(1+qr_1) \geq \delta_0$ in (8). Then we have

$$(S^{\frac{\delta}{2}}T^{\gamma_0}S^{\frac{\delta}{2}})^{\frac{\delta-(1-q)\delta_0}{\gamma_0+\delta}} \geq S^{\delta-(1-q)\delta_0}. \tag{9}$$

Hence we have

$$(S^{\frac{\delta}{2}}T^{\gamma_0}S^{\frac{\delta}{2}})^{\frac{\mu}{\gamma_0+\delta}} \geq S^{\mu} \text{ for } 0<\mu\leq\delta-(1-q)\delta_0. \tag{10}$$

Next, we demonstrate $f(\delta)=(T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta}}$ is decreasing for $\delta\geq\delta_0$. By Löwner-Heinz theorem, (10) ensures the following (11)

$$(S^{\frac{\delta}{2}}T^{\gamma_0}S^{\frac{\delta}{2}})^{\frac{\mu}{\gamma_0+\delta}} \geq S^{\mu} \text{ for } 0<\mu\leq\delta-(1-q)\delta_0. \tag{11}$$

Next, we have

$$\begin{aligned}
f(\delta) &= (T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta}}\\
&= \{(T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{\gamma_0+\delta+\mu}{\gamma_0+\delta}}\}^{\frac{q\gamma_0}{\gamma_0+\delta+\mu}}\\
&= \{T^{\gamma_0/2}S^{\delta/2}(S^{\delta/2}T^{\gamma_0}S^{\delta/2})^{\frac{\mu}{\gamma_0+\delta}}S^{\delta/2}T^{\gamma_0/2}\}^{\frac{q\gamma_0}{\gamma_0+\delta+\mu}} \text{ (by Lemma 2)}\\
&\geq (T^{\gamma_0/2}S^{\delta+\mu}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta+\mu}}\\
&= f(\delta+\mu).
\end{aligned}$$

Hence $f(\delta)$ is decreasing for $\delta\geq\delta_0$. Consequently,

$$T^{q\gamma_0} \geq (T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta}} \text{ for } \delta\geq\delta_0 \tag{12}$$

holds since

$$T^{q\gamma_0} \geq (T^{\gamma_0/2}S^{\delta_0}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta_0}} = f(\delta_0)\geq f(\delta) = (T^{\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta}}.$$

Again applying Theorem 1 to (12), we have

$$T^{q\gamma_0(1+r_2)} \geq (T^{\frac{qr_2\gamma_0}{2}}(T^{qr_2\gamma_0/2}S^{\delta}T^{\gamma_0/2})^{\frac{p_2q\gamma_0}{\gamma_0+\delta}}T^{\frac{qr_2\gamma_0}{2}})^{\frac{1+r_2}{p_2+r_2}} \tag{13}$$

for any $p_2\geq 1$ and $r_2\geq 0$. Putting $p_2=\frac{\gamma_0+\delta}{q\gamma_0}\geq 1$ in (13), we have

$$T^{q\gamma_0(1+r_2)} \geq (T^{\frac{\gamma_0(1+qr_2)}{2}}S^{\delta}T^{\frac{\gamma_0(1+qr_2)}{2}})^{\frac{q\gamma_0(1+r_2)}{\gamma_0+\delta+qr_2\gamma_0}} \tag{14}$$

for any $r_2\geq 0$. Put $\gamma=\gamma_0(1+qr_2)\geq\gamma_0$ in (14). Then we have

$$T^{\gamma+\gamma_0(q-1)} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{\gamma+\gamma_0(q-1)}{\delta+\gamma}} \tag{15}$$

for all $\gamma\geq\gamma_0$ and $\delta\geq\delta_0$. Now, since $0<\frac{q_1\gamma}{\gamma+\gamma_0(q-1)}\leq 1$, making use of Löwner-Heinz theorem to (15), we have

$$T^{q_1\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q_1\gamma}{\delta+\gamma}}$$

for all $\gamma\geq\gamma_0$, $\delta\geq\delta_0$ and $0<q_1\leq q$.
(ii) Suppose that $\ker(T)\subset\ker(S)$ and

$$T^{q\gamma_0} \geq (T^{\gamma_0/2}S^{\delta_0}T^{\gamma_0/2})^{\frac{q\gamma_0}{\gamma_0+\delta_0}} \tag{16}$$

holds for fixed $\gamma_0>0$ and $\delta_0>0$ and $0<q\leq 1$. Applying Lemma 3 to (16), we have

$$T^{q\gamma_0(1+r_3)} \geq (T^{\frac{qr_3\gamma_0}{2}}(T^{\gamma_0/2}S^{\delta_0}T^{\gamma_0/2})^{\frac{p_3q\gamma_0}{\gamma_0+\delta_0}}T^{\frac{qr_3\gamma_0}{2}})^{\frac{1+r_3}{p_3+r_3}} \tag{17}$$

for any $p_3\geq 1$ and $r_3\geq 0$. Putting $p_3=\frac{\gamma_0+\delta_0}{q\gamma_0}\geq 1$ in (17), we have

$$T^{q\gamma_0(1+r_3)} \geq (T^{\frac{\gamma_0(1+qr_3)}{2}}S^{\delta_0}T^{\frac{\gamma_0(1+qr_3)}{2}})^{\frac{q\gamma_0(1+r_3)}{\gamma_0+\delta_0+qr_3\gamma_0}} \tag{18}$$

for any $r_3\geq 0$. Put $\gamma=\gamma_0(1+qr_3)\geq\gamma_0$ in (18). Then we have

$$T^{\gamma+\gamma_0(q-1)} \geq (T^{\frac{\gamma}{2}}S^{\delta_0}T^{\frac{\gamma}{2}})^{\frac{\gamma+\gamma_0(q-1)}{\delta_0+\gamma}} \text{ for } \gamma\geq\gamma_0. \tag{19}$$

Next we show that $g(\gamma)=(S^{\delta_0/2}A^{\gamma}S^{\delta_0/2})^{\frac{q\delta_0}{\gamma_0+\delta_0}}$ is increasing for $\gamma\geq\gamma_0$. Löwner-Heinz theorem , when applied to (19), guarantees the following:

$$T^{u}\geq(T^{\frac{\gamma}{2}}S^{\delta_0}T^{\frac{\gamma}{2}})^{\frac{u}{\delta_0+\gamma}}\text{ for }0\leq u\leq\gamma+\gamma_0(q-1). \tag{20}$$

Then we have

$$\begin{aligned}
g(\gamma)&=(S^{\delta_0/2}T^{\gamma}S^{\delta_0/2})^{\frac{q\delta_0}{\gamma+\delta_0}}\\
&=\{(S^{\delta_0/2}T^{\gamma}S^{\delta_0/2})^{\frac{\gamma+\delta_0+u}{\gamma+\delta_0}}\}^{\frac{q\delta_0}{u+\delta_0+\gamma}}\\
&=\{S^{\delta_0/2}T^{\gamma/2}(T^{\gamma/2}S^{\delta_0}T^{\gamma/2})^{\frac{u}{\gamma+\delta_0}}T^{\gamma/2}S^{\delta_0/2}\}^{\frac{q\delta_0}{u+\delta_0+\gamma}}\\
&\leq(S^{\delta_0/2}T^{\gamma+u}S^{\delta_0/2})^{\frac{q\delta_0}{u+\delta_0+\gamma}}\\
&=g(\gamma+u).
\end{aligned}$$

Hence $g(\gamma)$ is increasing for $\gamma\geq\gamma_0$. Therefore

$$(S^{\delta_0/2}T^{\gamma}S^{\delta_0/2})^{\frac{q\delta_0}{\gamma+\delta_0}}\geq S^{q\delta_0}\text{ for }\gamma\geq\gamma_0 \tag{21}$$

holds since

$$(S^{\delta_0/2}T^{\gamma}S^{\delta_0/2})^{\frac{q\delta_0}{\gamma+\delta_0}}=g(\gamma)\geq g(\gamma_0)=(S^{\delta_0/2}T^{\gamma_0}S^{\delta_0/2})^{\frac{q\delta_0}{\gamma_0+\delta_0}}\geq S^{q\delta_0}.$$

Again applying Theorem 1 to (21), we have

$$\{S^{\frac{qr_4\delta_0}{2}}(S^{\delta_0/2}T^{\gamma}S^{\delta_0/2})^{\frac{p_4q\delta_0}{\gamma+\delta_0}}S^{\frac{qr_4\delta_0}{2}}\}^{\frac{1+r_4}{p_4+r_4}}\geq S^{q\delta_0(1+r_4)} \tag{22}$$

for any $p_4\geq1$ and $r_4\geq0$. Putting $p_4=\frac{\gamma+\delta_0}{q\delta_0}\geq1$ in (22), we have

$$(S^{\frac{\delta_0(1+qr_4)}{2}}T^{\gamma}S^{\frac{\delta_0(1+qr_4)}{2}})^{\frac{q\delta_0(1+r_4)}{\gamma+\delta_0+q\delta_0r_4}}\geq S^{q\delta_0(1+r_4)} \tag{23}$$

for any $r_4\geq0$. Put $\delta=\delta_0(1+qr_4)\geq\delta_0$ in (23). Then we have

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{\delta+\delta_0(q-1)}{\gamma+\delta}}\geq S^{\delta+\delta_0(q-1)}\text{ for }\gamma\geq\gamma_0\text{ and }\delta\geq\delta_0. \tag{24}$$

Applying the Löwner-Heinz theorem to (24), we now obtain since $0<\frac{q_1\delta}{\delta+\delta_0(q-1)}\leq1$,

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q_1\delta}{\gamma+\delta}}\geq S^{q_1\delta}$$

for all $\gamma\geq\gamma_0$, $\delta\geq\delta_0$ and $0<q_1\leq q$, consequently, the proof is conclusive.

Proposition 2.*Let $T,S\in\mathscr{B}^{+}(\mathscr{H})$ and let $\gamma_0>0$, $\delta_0>0$ and $0<q\leq1$. Suppose that*

$$(S^{\frac{\delta_0}{2}}T^{\gamma_0}S^{\frac{\delta_0}{2}})^{\frac{q\delta_0}{\gamma_0+\delta_0}}\geq S^{q\delta_0} \tag{25}$$

and

$$T^{q\gamma_0}\geq(T^{\frac{\gamma_0}{2}}S^{\delta_0}T^{\frac{\gamma_0}{2}})^{\frac{q\gamma_0}{\gamma_0+\delta_0}}. \tag{26}$$

Consequently, the following statements are true:

(i)For every $\gamma\geq\gamma_0$, $\delta\geq\delta_0$ and $0<q_1\leq q$

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q_1\delta}{\gamma+\delta}}\geq S^{q_1\delta}.$$

Moreover, for each fixed $\gamma\geq-\gamma_0$,

$$f_{\gamma_0,\gamma}(\delta)=(T^{\frac{\gamma_0}{2}}S^{\delta}T^{\frac{\gamma_0}{2}})^{\frac{(\gamma_0+\gamma)p_1}{\gamma_0+\delta}}$$

is a decreasing function for $\delta\geq\max\{\delta_0,\gamma\}$. Hence the inequality

$$(T^{\frac{\gamma_0}{2}}S^{\delta_1}T^{\frac{\gamma_0}{2}})^{p_1}\geq(T^{\frac{\gamma_0}{2}}S^{\delta_2}T^{\frac{\gamma_0}{2}})^{\frac{p_1(\gamma_0+\delta_1)}{\gamma_0+\delta_2}} \tag{27}$$

holds for any δ_1 and δ_2 such that $\delta_2\geq\delta_1\geq\delta_0$ and $0<p_1\leq p$.

(ii)For each $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$

$$T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}}.$$

Additionally, for every fixed $\delta \geq -\delta_0$,

$$g_{\delta_0,\delta}(\gamma) = (S^{\frac{\delta_0}{2}}T^{\gamma}S^{\frac{\delta_0}{2}})^{\frac{(\delta+\delta_0)p_1}{\gamma+\delta_0}}$$

is an increasing function for $\gamma \geq \max\{\gamma_0,\delta\}$. Hence the inequality

$$(S^{\frac{\delta_0}{2}}T^{\gamma_2}S^{\frac{\delta_0}{2}})^{\frac{p_1(\gamma_1+\delta_0)}{\gamma_2+\delta_0}} \geq (S^{\frac{\delta_0}{2}}T^{\gamma_1}S^{\frac{\delta_0}{2}})^{p_1} \tag{28}$$

holds for any γ_1 and γ_2 such that $\gamma_2 \geq \gamma_1 \geq \gamma_0$ and $0 < p_1 \leq p$.

*Proof.*We will provide the proof for part (ii), noting that the proof for part (i) follows a similar pattern. We begin by observing that inequality (2) implies inequality (4), as established in Proposition 1. Therefore, we have:

$$T^{q\gamma_0} \geq (T^{\frac{\gamma_0}{2}}S^{\delta_0}T^{\frac{\gamma_0}{2}})^{\frac{q\gamma_0}{\gamma_0+\delta_0}} \geq (T^{\frac{\gamma_0}{2}}S^{\delta}T^{\frac{\gamma_0}{2}})^{\frac{q\gamma_0}{\gamma_0+\delta}}$$

This inequality holds for all $\beta \geq \beta_0$ based on inequality (4) and the Löwner-Heinz inequality. Consequently, we can conclude part (ii) by invoking Proposition 1 (ii).

In Proposition 2, when considering $\gamma > 0$, $\delta > 0$, and $0 < q \leq 1$, one might naturally anticipate that the inequality $T^{q\gamma} \geq \left(T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}}\right)^{\frac{q\gamma}{\gamma+\delta}}$ is equivalent to $\left(S^{\frac{\delta}{2}}T^{\gamma}T^{\frac{\delta}{2}}\right)^{\frac{q\delta}{\gamma+\delta}} \geq S^{q\delta}$, even in cases where T and S are not invertible. However, this assumption is disproven by the following example.

*Example 1.*There exists positive bounded linear operators T and S such that $T^{q\gamma} \geq \left(T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}}\right)^{\frac{q\gamma}{\gamma+\delta}}$ and $\left(S^{\frac{\delta}{2}}T^{\gamma}T^{\frac{\delta}{2}}\right)^{\frac{q\delta}{\gamma+\delta}} \ngeq S^{q\delta}$. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$T^{q\gamma} - \left(T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}}\right)^{\frac{q\gamma}{\gamma+\delta}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

and

$$\left(S^{\frac{\delta}{2}}T^{\gamma}T^{\frac{\delta}{2}}\right)^{\frac{q\delta}{\gamma+\delta}} - S^{q\delta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \ngeq 0$$

for $\gamma > 0$, $\delta > 0$, and $0 < q \leq 1$. Therefore $T^{q\gamma} \geq \left(T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}}\right)^{\frac{q\gamma}{\gamma+\delta}}$ and $\left(S^{\frac{\delta}{2}}T^{\gamma}T^{\frac{\delta}{2}}\right)^{\frac{q\delta}{\gamma+\delta}} \ngeq S^{q\delta}$ for $\gamma > 0$, $\delta > 0$, and $0 < q \leq 1$.

Corollary 1.*Let $T,S \in \mathscr{B}^{+}(\mathscr{H})$ and let $\gamma_0 > 0$, $\delta_0 > 0$. Then, the following claims are true:*

(i)If $0 < q \leq 1$, then

$$(S^{\frac{\delta_0}{2}}T^{\gamma_0}S^{\frac{\delta_0}{2}})^{\frac{q\delta_0}{\gamma_0+\delta_0}} \geq S^{q\delta_0} \Longrightarrow (S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q_1\delta}{\gamma+\delta}} \geq S^{q_1\delta} \tag{29}$$

holds for any $\gamma \geq \gamma_0$ and $\delta \geq \delta_0$, thus $T^{q_1\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q_1\gamma}{\gamma+\delta}}$ holds for any $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$.

(ii)If $0 < q \leq 1$ and $\ker(T) \subset \ker(S)$, then

$$T^{q\gamma_0} \geq (T^{\frac{\gamma_0}{2}}S^{\delta_0}T^{\frac{\gamma_0}{2}})^{\frac{q\gamma_0}{\gamma_0+\delta_0}} \Longrightarrow T^{q\gamma} \geq (T^{\frac{\gamma}{2}}S^{\delta}T^{\frac{\gamma}{2}})^{\frac{q\gamma}{\gamma+\delta}} \tag{30}$$

holds for any $\gamma \geq \gamma_0$ and $\delta \geq \delta_0$, thus $(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q_1\delta}{\gamma+\delta}} \geq S^{q_1\delta}$ holds for any $\gamma \geq \gamma_0$, $\delta \geq \delta_0$ and $0 < q_1 \leq q$.

*Proof.*We present the proof for part (i), and it's worth noting that the proof for part (ii) follows a similar line of reasoning.

Based on the provided hypothesis, the Löwner-Heinz theorem, and Proposition 2, we can establish the following inequality for all $\delta \geq \delta_0$, $\gamma \geq \gamma_0$, and $0 < q \leq 1$:

$$(S^{\frac{\delta}{2}}T^{\gamma}S^{\frac{\delta}{2}})^{\frac{q\delta}{\gamma+\delta}} \geq S^{q\delta}$$

This inequality, derived from the hypothesis and known theorems, validates Corollary 1 (i). The application of the Löwner-Heinz theorem and Theorem 2 further supports this conclusion.

*Remark.*We need to keep in mind the assumptions (i) and (ii) of Theorem 2. In the context of Theorem 2, we consider the scenario where $\gamma = \delta = 1$ and $0 < q \leq 1$. The following conditions are relevant:

(a)$(S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} \geq S^q$.
(b)$T^q \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}}$ and $\ker(T) \subset \ker(S)$.

We have shown that in Theorem 2, condition (a) implies $T^q \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}}$, and condition (b) ensures condition (a). Consequently, one might expect that conditions (a) and (b) are analogous. However, we have a counterexample to demonstrate otherwise.

Example 2.$(S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} \geq S^q$ and $T^q \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}}$, but $\ker(T) \nsubseteq \ker(S)$.
Let $T = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}, S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $T^{\frac{1}{2}} = \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, S^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = S$. Hence

$$2^{\frac{q}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} \geq S^q$$

and

$$T^q = \begin{pmatrix} 2^q \cdot 5^{q-1} & 2^{q+1} \cdot 5^{q-1} \\ 5^{q-1} \cdot 2^{q+1} & 5^{q-1} \cdot 2^{q+2} \end{pmatrix} \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}} = \begin{pmatrix} \frac{2^{\frac{q}{2}}}{5} & \frac{2^{\frac{q+2}{2}}}{5} \\ \frac{2^{\frac{q+2}{2}}}{5} & \frac{2^{\frac{q+4}{2}}}{5} \end{pmatrix}.$$

But $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \in \ker(A)$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \notin \ker(S)$, so that $\ker(T) \nsubseteq \ker(S)$.

Moreover, we have the following example.

*Example 3.*We have $T^q \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}}$, but $(S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} \ngeq S^q$ and $\ker(T) \nsubseteq \ker(S)$.
Set $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $T^q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq (T^{\frac{1}{2}}ST^{\frac{1}{2}})^{\frac{q}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $(S^{\frac{1}{2}}TS^{\frac{1}{2}})^{\frac{q}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ngeq S^q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\ker(T) \nsubseteq \ker(S)$.

3 Applications

In this section, we will illustrate the application of Theorem 2 to various operator classes.

Definition 1.*Consider the following operator classes defined in terms of $\alpha > 0$, $\beta > 0$, $0 < p \leq 1$, the polar decomposition of T as $T = U|T|$, and the generalized Aluthge transformation $\tilde{T}_{\alpha,\beta} = |T|^\alpha U|T|^\beta$:*

(i)T is classified as belonging to the p-$A(\alpha,\beta)$ class if it satisfies the inequality $(|T^*|^\beta|T|^{2\alpha}|T^*|^\beta)^{\frac{p\beta}{\alpha+\beta}} \geq |T^*|^{2p\beta}$ [16].
(ii)T is categorized as part of the p-w$A(\alpha,\beta)$ class if it meets the criteria:

$$(|T^*|^\beta|T|^{2\alpha}|T^*|^\beta)^{\frac{p\beta}{\alpha+\beta}} \geq |T^*|^{2p\beta} \quad \text{and} \quad |T|^{2p\alpha} \geq (|T|^\alpha|T^*|^{2\beta}|T|^\alpha)^{\frac{p\alpha}{\alpha+\beta}}$$

or equivalently, $|\tilde{T}_{\alpha,\beta}|^{\frac{2p\beta}{\beta+\alpha}} \geq |T|^{2p\beta}$ and $|T|^{2p\alpha} \geq |(\tilde{T}_{\alpha,\beta})^*|^{\frac{2p\alpha}{\alpha+\beta}}$ as defined in [16].
(iii)T is classified as a member of the p-A class if $|T^2|^p \geq |T|^{2p}$, which is equivalent to T being part of the p-$A(1,1)$ class, as stated in [16].
(iv)T is considered p-w-hyponormal if and only if it satisfies the inequalities: $|\tilde{T}|^{\frac{p}{2}} \geq |T|^p \geq |(\tilde{T})^*|^{\frac{p}{2}}$. This classification corresponds to T belonging to the p-w$A(\frac{1}{2},\frac{1}{2})$ class, where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation, as outlined in [3].
(v)T is termed (α,p)-w-hyponormal if and only if it satisfies the following inequalities: $|\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha} \geq |(\tilde{T}_{\alpha,\alpha})^*|^{\frac{p}{2}}$. This characterization corresponds to T belonging to the p-w$A(\alpha,\alpha)$ class, where $\tilde{T}_{\alpha,\alpha} = |T|^\alpha U|T|^\alpha$ is the generalized Aluthge transformation, as discussed in [12] and [19].

Operators classified as p-$wA(\alpha,\beta)$ exhibit several significant properties typical of hyponormal operators. These properties encompass the Fuglede-Putnam type theorem, Weyl type theorem, subscalarity, and Putnam's inequality, as documented in [4], [5], [17], [18], and [23]. It's important to note that the Aluthge transformation has garnered considerable attention from various authors, including [1], [4], [6], and [25]. These classes are categorized as normaloid operators, denoted by $\|T\| = r(T)$, where $r(T)$ represents the spectral radius of T, as discussed in [17], [3], and [12]. For α, β, and $0 < p \leq 1$, it has been established that class p-$A(\alpha,\beta)$ includes class p-$A(\alpha,\beta)$ based on the definition in 1 (i) and (ii). Furthermore, as demonstrated in [16], both class p-$wA(\alpha,\beta)$ and class p-$wA(\alpha,\beta)$ are invertible for any $\alpha > 0$, $\beta > 0$, and $0 < p \leq 1$. Previous research has also provided more precise inclusion relations among class p-$wA(\alpha,\beta)$.

Lemma 4.*[4] If $T \in B(\mathscr{H})$ is class p-$wA(s,t)$ and $0 < s \leq \gamma, 0 < t \leq \delta, 0 < p_1 \leq p \leq 1$, then T is class p_1-$wA(\gamma,\delta)$.*

In their study [16], the authors posed the following question:
Question: Does the class p-$A(s,t)$ imply p-$wA(s,t)$ for $0 < p < 1$?
The subsequent theorem provides an affirmative answer to this question.

Theorem 3.*For each $\alpha > 0, \beta > 0$ and $0 < p \leq 1$, the following assertions hold:*

(i)class p-$A(\alpha,\beta)$ and class p-$wA(\alpha,\beta)$ are equivalent.
(ii)class p-A and class p-wA are equivalent.
(iii)class p-$A(\frac{1}{2},\frac{1}{2})$ and the class of p-w-hyponormal operators are equivalent, i.e., class p-$wA(\frac{1}{2},\frac{1}{2})$.
(iv)class p-$A(\alpha,\alpha)$ and class (α,p)-w-hyponormal operators are equivalent, i.e., class p-$wA(\alpha,\alpha)$.

*Proof.*We choose not to provide a proof here, as we can easily establish Theorem 3 by applying Theorem 2 to the definitions of these classes.

Notice that Theorem 3 in reference [15] corresponds to a specific case where $q = 1$, and therefore, Theorem 3 can be seen as an extension or generalization of it.

*Remark.*By (iv) of Theorem 3, we have

$$\begin{aligned} |\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha} &\Leftrightarrow (|T^*|^{\alpha}|T|^{2\alpha}|T^*|^{\alpha})^{\frac{p}{2}} \geq |T^*|^{2p\alpha} \Leftrightarrow T : \text{class } p - A(\alpha,\alpha) \\ &\Leftrightarrow T : (\alpha,p) - w - \text{hyponormal} \Leftrightarrow |\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha} \geq |(\tilde{T}_{\alpha,\alpha})^*|^{\frac{p}{2}}. \end{aligned}$$

Hence

$$|\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha} \Rightarrow |T|^{2p\alpha} \geq |(\tilde{T}_{\alpha,\alpha})^*|^{\frac{p}{2}},$$

that is, we may as will define (α,p)-w-hyponormal by only $|\tilde{T}_{\alpha,\alpha}|^{\frac{p}{2}} \geq |T|^{2p\alpha}$.

Next, we shall show some properties of class p-$A(s,t)$.

Theorem 4.*If $T \in B(\mathscr{H})$ is class p-$A(s,t)$ and $0 < s \leq \gamma, 0 < t \leq \delta, 0 < p_1 \leq p \leq 1$, then T is class p_1-$A(\gamma,\delta)$.*

*Proof.*We skip the proof because it can be accomplished easily using (i) of Theorem 3 and Theorem 5.

We will show that certain non-normal operators can be proven to be normal. It is established that an operator T is normal if both T and T^* belong to the class A. However, the situation becomes less clear when T and T^* belong to classes weaker than class A. Thanks to the research efforts of various authors on this topic, the following results were previously unknown until now.

Lemma 5.*[21] Let $\alpha_i,\beta_i > 0$ and $0 < p_i \leq 1$, where $i = 1,2$. If T is a class p_1-$wA(\alpha_1,\beta_1)$ operator and T^* is a class p_2-$wA(\alpha_2,\beta_2)$ operator, then T is normal.*

Corollary 2.*Let $\alpha_i,\beta_i > 0$ and $0 < p_i \leq 1$, where $i = 1,2$. If T is a class p_1-$A(\alpha_1,\beta_1)$ operator and T^* is a class p_2-$A(\alpha_2,\beta_2)$ operator, then T is normal.*

*Proof.*Theorem 3 and Lemma 5 lead directly to the proof.

Lemma 6.*[21] Let $p,r > 0$, $0 < q \leq 1$, $s \geq p$ and $t \geq r$. If T is a class q-$wA(p,r)$ operator and $\tilde{T}_{s,t}$ is normal, then T is normal.*

Corollary 3.*Let $p,r > 0$, $0 < q \leq 1$, $s \geq p$ and $t \geq r$. If T is a class q-$A(p,r)$ operator and $\tilde{T}_{s,t}$ is normal, then T is normal.*

*Proof.*Theorem 3 and Lemma 6 are prerequisites for the proof.

*Remark.*Please take note that Corollaries 2 and 3, along with Lemmas 5 and 6, offer generalizations of several findings found in the existing literature. Notable examples include the extension of Theorem 6 in reference [15], as well as other results in papers such as [19] and [3].

The numerical range of an operator M, represented as $W(M)$, is defined as the set given by:

$$W(M) = \{\langle Mx, x\rangle : \|x\| = 1\}.$$

In a general context, it's important to note that neither the condition $N^{-1}MN = M^*$ nor the statement $0 \notin \overline{W(M)}$ guarantees that the operator M is normal. This is exemplified when considering the case of $M = NB$, where N is positive and invertible, B is self-adjoint, and N and B do not commute. In this scenario, $N^{-1}MN = M^*$ and $0 \notin \overline{W(N)}$, but the operator M is not normal. This naturally leads to the following question:
Question: Under what conditions does an operator M become normal when both $N^{-1}MN = M^*$ and $0 \notin \overline{W(N)}$ hold true?
In 1966, Sheth demonstrated in [22] that if M is a hyponormal operator and $N^{-1}MN = M^*$ for certain operators N, where $0 \notin \overline{W(N)}$, then M is self-adjoint. Rashid later extended Sheth's result to encompass the class $A(k)$ operators for $k > 0$ in [20]. This work further expands upon Sheth's result, demonstrating that it holds true for the class p-$A(\alpha,\beta)$ operators, as detailed below.

Corollary 4.*Let $M \in \mathscr{B}(\mathscr{H})$. If M or M^* belongs to class p-$A(\alpha,\beta)$ for every $\alpha > 0, \beta > 0$ and $0 < p \le 1$ and N is an operator for which $0 \notin \overline{W(N)}$ and $NM = M^*N$, then M is self-adjoint.*

*Proof.*The conclusion drawn is a result of Theorem 3 and the findings presented in [21, Theorem 2.14].

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SD-Separability in Topological Spaces

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Abstract: Our aim is to introduce a generalization of dense set in topological space, namely *SD*-dense set, when we used the notion of somewhere dense closure operator. We provide the characterization of this class of sets, and their implications with dense sets and with somewhere dense sets, and study their union and intersection properties, moreover we discuss their behavior as subspaces in some special cases, additionally, we investigate their properties in some particular spaces, and then we prove that *SD*-dense sets, dense sets, somewhere dense sets and open sets are equivalent in strongly hyperconnected space, after that we illustrate the image of *SD*-dense sets by particular maps; as *SD*-irresolute map and *SD*-continuous map. Finally, we define a new axiom of separability, namely *SD*-separable space using the notion of *SD*-dense sets, then we state that *SD*-separable space is stronger than separable space, and in submaximal space these notions become equivalent, moreover we study the subspaces and the images of *SD*-separable spaces.

Keywords: Separability; topological space and generalizations; continuous and generalization; somewhere dense set; strongly hyperconnected space.

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1 Introduction

Researchers have mentioned various forms of generalized open sets; for instance α-open set, preopen set, semi open set, b-open set, β-open set and somewhere dense set. The notion of somewhere dense set was due to Pugh [19], then in 2017 [1] Al-shami provided the topological properties of this class of sets, and he studied some operators as; somewhere dense interior, somewhere dense closure and somewhere dense boundary and he used these notions to defined the axiom of ST_1space, and with Noiri [2,3] they investigated some particular maps as; *SD*-irresolute maps and *SD*-continuous maps, then they introduced the concepts of Lindelofness and compactness using somewhere dense sets, and recently, Arwini et al. [4] stated that somewhere dense sets and open sets are coinciding if and only if a space is strongly hyperconnected, moreover, Sarbout et al. [20] defined somewhere dense connected space, and they showed that this space is stronger than hyperconnected space but weaker than strongly hyperconneted space.
In 1906 [15] Frechet defined separable space, which is a space that contains a countable dense subset, since then different types of separability were defined, as d-separable space, D-separable space, weakly separable space, b-separable space, dense-separable space, r-separable space, pre-separable space etc. Kurepa [17] in 1936 introduced a generalized form of separable and metrizable spaces, namely d-separable space, and then in 1981 Arhangelskii [5] studied some properties of d-separable spaces and proved that any product of d-separable spaces is d-separable. D-Separable spaces were due to Bella et al. [8], and in 2012 Aurihi et al. [7] investigated the properties of D-separable space and showed their implication with d-separable spaces. Weakly separable spaces were defined by Beshimov in 1994 [9] when he proved that any weakly separable Hausdorff compact space is separable, moreover he studied its separable compactifications, see more in [10,11]. In 2013 Selvarani [21] defined b-dense sets and b-separable spaces using b-open sets, then in 2021 Arwini et al. [6,16] introduced two different types of separability, the first type is called dense separable space, and they showed that dense separable space, dense second countable space and separable space are equivalent, while in the second type they used the notion of regular open sets to defined r-separable space, then they illustrated that r-separable space is weaker than separable space, but they became equivalent in regular space. Recently, Elbhilil et al. [14] introduced pre-separable spaces

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using pre-dense set notion, and they showed that pre-separable space is placed between b-separable space and separable space, and in submaximal spaces, the axioms of separability and pre-separability became equivalent.
In this work we use the notion of somewhere dense set to introduce *SD*-dense set, which is a generalization of dense set, we provide its behavior with some operations as union and intersection, then we discuss the characterization of this class of sets in some particular spaces and study their images by some particular maps. After that, we define the axiom of *SD*-separable space using the notion of *SD*-dense sets, then we illustrate the implication between this space and separable space, and study their subspaces and images.
We divided this article into five sections; as follows: section two concludes the basic concepts concerning somewhere dense sets, section three presents the definition of *SD*-dense set, including some of its union and intersection properties and its behavior as a subspace, then we present its characterization in some spaces, after that we examine their images by some particular maps, and section four includes the basic studies of *SD*-separable space, and finally section five gives an overview of our results in the conclusion.
Throughout this paper, a topological space (Z,τ) denotes by Z and $D(\tau)$ denotes the collection of all dense sets in Z, and if E and F are subsets of a space Z; $\overline{E}$, E°, $P(E)$, E^c and E/F denote the closure of E, the interior of E, the power set of E, the complement of E and the difference of E and F; respectively. Additionally, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{K}$, $\mathbb{R}^+$, $\mathbb{R}^-$ are the sets of real numbers, integer numbers, rational numbers, irrational numbers, positive real numbers, negative real numbers; respectively.

2 Preliminaries

In the present section, we provide the basic properties and characterizations of somewhere dense set, and their behavior in subspaces and in strongly hyperconnected spaces.

Definition 1.*[1] In a topological space (Z,τ), a subset A is called somewhere dense (namely s-dense) if $\overline{A}^{\circ} \neq \phi$. The complement of s-dense set is called closed somewhere dense (namely cs-dense) set, and the collection of all s-dense sets in Z is denoted by $S(\tau)$. Clearly any non-empty open (dense) set is s-dense.*

Theorem 1.*[1] If A is s-dense subset in a space Z and $A \subseteq B$, then B is s-dense.*

Theorem 2.*[1] Any subset of a space Z is s-dense or cs-dense.*

Definition 2.*[20] A subset B of a space Z is called SD-clopen if B is s-dense and cs-dense. Clearly any clopen set is SD-clopen.*

Definition 3.*[20] In a space Z if any open set is closed, then Z is called partition space. Clearly any non-empty subset of partition space is s-dense.*

Definition 4.*[14] A space (Z,τ) is called S-space if any subset of Z that contains a non-empty open set is also open.*

Definition 5.*[1, 13, 18] A space Z is called:*

i. submaximal if any dense set is open.
ii. hyperconnected if any non-empty open set is dense.
iii. strongly hyperconnected if Z is submaximal and hyperconnected.

Theorem 3.*[4] In a space Z, the following conditions are equivalent:*
1. Z is strongly hyperconnected space.
2. dense sets are equivalent with non-empty open sets.
3 s-dense sets are equivalent with non-empty open sets; where Z is a non-discrete space.

Definition 6.*[10] If Z is a space, then a subset B is called regular closed (namely r-closed) if $B = \overline{B^{\circ}}$, while the complement of regular closed set is called regular open (r-open). Clearly any r-closed set is closed.*

Theorem 4.*[20] Any proper non-empty r-closed set is SD-clopen.*

Theorem 5.*[4, 12] Let Z be a space, W be a subspace of Z and $B \subseteq W$ then:*

1) $\overline{B}^W = \overline{B} \cap W$, where $\overline{B}^W$ is the closure of B with respect to the relative topology on W.
2) where W is r-closed in Z, then B is s-dense in W if and only if B is s-dense in Z.

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Definition 7.[1] *Let Z be a space and B be a subset of Z, then:*

i. the intersection of all cs-dense sets in Z containing B is denoted by $\overline{B}^S$*. Clearly* $\overline{B}^S$ *is cs-dense set.*
ii. the union of all s-dense sets contained in B is denoted by $B^{\circ S}$*. Clearly* $B^{\circ S}$ *is s-dense set.*

Theorem 6.[1] *Let A and B be two subsets of a space Z, then:*

1) $A \subseteq \overline{A}^S \subseteq \overline{A}$ *and* $A^\circ \subseteq A^{\circ S} \subseteq A$.
2) If $A \subseteq B$*, then* $\overline{A}^S \subseteq \overline{B}^S$ *and* $A^{\circ S} \subseteq B^{\circ S}$.
3) A is cs-dense if and only if $\overline{A}^S = A$*, while A is s-dense if and only if* $A^{\circ S} = A$.

Definition 8.[1] *In a topological space* (Z,τ)*, a subset A of Z is called:*

i. α*-open if* $A \subseteq \overline{A^\circ}^\circ$.
iii. pre-open if $A \subseteq \overline{A}^\circ$.
iv. b-open if $A \subseteq \overline{A^\circ} \cup \overline{A}^\circ$.
v. β*-open if* $A \subseteq \overline{\overline{A}^\circ}$.

Theorem 7.[1] *The implications between the class of generalization open sets are given in the following diagram:*

open set $\Rightarrow$ α-open set $\Rightarrow$ pre-open set $\Rightarrow$ *b*-open set $\Rightarrow$ β-open set $\Rightarrow$ *s*-dense set

Definition 9.[2] *A map* $f : (Z,\tau) \to (X,\sigma)$ *is called SD-irresolute if the inverse image of any s-dense in X is empty or s-dense in Z, while f is called SD-continuous if the inverse image of any open set in X is empty or s-dense in Z.*

Definition 10.[12] *A space is called separable if it contains a countable dense set.*

3 SD-Density

This section consists the definition of a new generalization of dense set using the concept of somewhere dense set, namely *SD*-dense set. Union and intersection properties of *SD*-dense sets and their implication with dense sets are given, additionally, the behavior of *SD*-dense sets as subspaces in particular conditions are shown, after that, we investigate the characterization of *SD*-dense sets in some spaces. Finally, we study the images of *SD*-dense sets by some particular maps; such as *SD*-continuous map and *SD*-irresolute map.

3.1 SD-Dense Sets

Here we provide some basic properties of the class of *SD*-dense sets.

Definition 11.*A subset F of a space* (Z,τ) *is called SD-dense if* $\overline{F}^S = Z$*. The collection of all SD-dense sets in Z is denoted by* $SD(\tau)$*.*

*Example 1.*In a space (Z,τ) where $Z = \{1,2,3\}$ and $\tau = \{Z,\phi,\{1,2\}\}$, we have $S(\tau) = \{Z,\{1\},\{2\},\{1,2\},\{1,3\},\{2,3\}\}$. Therefore Z and $\{1,2\}$ are the only *SD*-dense sets in Z.

Corollary 1.*Any SD-dense set is dense.*

*Proof.*Obvious, since $\overline{E}^S \subseteq \overline{E}$ for any subset E of a space (Z,τ), i.e., $SD(\tau) \subseteq D(\tau)$; where $D(\tau)$ is the collection of all dense sets in Z.

SD-dense $\Rightarrow$ dense $\Rightarrow$ pre-open set $\Rightarrow$ *b*-open set $\Rightarrow$ β-open set $\Rightarrow$ *s*-dense

*Remark.*No general relations between *SD*-dense set and open set, for instance:

1. In the usual topology, $(0,1)$ is open (s-dense) set but not *SD*-dense nor dense, since $(0,1)^c$ is s-dense but disjoint from $(0,1)$. Moreover, the set $\mathbb{Q}$ is dense but not *SD*-dense, since the set $\mathbb{K}$ is also *s*-dense but disjoint from $\mathbb{Q}$.
2. In the space $(\mathbb{R},\tau)$, where $\tau = P(\mathbb{K}) \cup \mathbb{R}$ we have $S(\tau) = P(\mathbb{K}) \cup \{A \subseteq \mathbb{R} : A \cap \mathbb{Q}, A \cap \mathbb{K} \neq \phi\}$. Then the set $\mathbb{K} \cup B$ where $B \subseteq \mathbb{Q}$ is *SD*-dense, so if $B \neq \phi$ the set $\mathbb{K} \cup B$ is *SD*-dense but not open.

Theorem 8. *If E is a subset of a space Z, then these statements are equivalent:*

1) E is SD-dense.
2) E intersects all s-dense sets in Z.
3) $(E^c)^{oS} = \phi$.
4) E^c is not s-dense.
5) E^o is dense.
6) E contains an open dense subset F in Z.

Proof. 1) $\Rightarrow$ 2) Let F be a SD-dense in Z, and suppose that A is a s-dense set which is disjoint from F, then A^c is cs-dense that contained F, hence $\overline{F}^S \neq \mathbb{Z}$, which is a contradiction.
2) $\Rightarrow$ 3) Suppose that $(E^c)^{oS}$ is a non-empty set, then there exists a s-dense set A containing in E^c, hence a s-dense set A does not intersect E, which is a contradiction.
3) $\Rightarrow$ 4) Suppose that $(E^c)^{oS} = \phi$, then from theorem (6) the set E^c is not s-dense.
4) $\Rightarrow$ 5) Suppose E^c is not s-dense, then $\overline{E^c}^{\circ} = \phi$, i.e., $\overline{E^c}$ does not contains any non-empty open set, therefore $(\overline{E^c})^c = E^o$ is dense.
5) $\Rightarrow$ 6) Direct since E^o is open dense set.
6) $\Rightarrow$ 1) Let F be an open dense subset in Z and $F \subseteq E$, and suppose that $\overline{E}^S \neq Z$, then there exists a s-dense set A which does not intersect E, and since F^c is closed set, then $A \subseteq \overline{A} \subseteq \overline{E^c} \subseteq F^c$, but A is a s-dense, then $\overline{A}$ contains a non-empty open set which is disjoint from the dense set F, which is a contradiction, therefore $\overline{E}^S = Z$, thus E is SD-dense.

Corollary 2. *In a space (Z,τ) we have $\tau \cap D(\tau) \subseteq SD(\tau)$, moreover $SD(\tau) = \{A \subseteq Z : B \subseteq A$ for some open dense set $B\}$.*

Proof. According to the previous theorem number (5) we obtain $\tau \cap D(\tau) \subseteq SD(\tau)$, and by using number (6) clearly any subset that contains an open dense set is SD-dense.

Remark. Let (Z,τ) be a space, then Z is the only SD-dense set if and only if Z is the only open dense set in Z.

Example 2. In the usual topology, the set $\mathbb{Z}$ of integer numbers is not s-dense set, hence $\mathbb{R}/\mathbb{Z}$ is SD-dense. Note that $\mathbb{R}/\mathbb{Z}$ is open dense set.

3.2 Operation on SD-Dense Sets

Union and intersection operations of the class of SD-dense sets are investigated.

Corollary 3. *1) Any set contains SD-dense set is SD-dense.*
2) Union of SD-dense sets is SD-dense.

Proof. Obvious using theorem (8).

Theorem 9. *Any two SD-dense sets have non-empty intersection.*

Proof. Suppose E and F are two SD-dense sets in a space Z, then from corollary (1) we get E is s-dense and F is SD-dense, and according to theorem (8) we obtain $E \cap F \neq \phi$.

Remark. The infinite intersection of SD-dense sets can be empty set; for example in the usual topology, $\{\{x\}^c\}_{x\in\mathbb{R}}$ is a collection of SD-dense sets in $\mathbb{R}$, but $\cap_{x\in\mathbb{R}}\{x\}^c = \phi$.

Lemma 1. *If A and B are not s-dense sets in a space Z, then $A \cup B$ is also not s-dense.*

Proof. If at least one of the sets A and B are empty, then the prove is obvious. Now let A and B be non-empty not s-dense sets, and suppose that $A \cup B$ is s-dense, so there exists a non-empty open set V such that $V \subseteq \overline{A \cup B} = \overline{A} \cup \overline{B}$, since A is not dense we have $\overline{A}^c$ is a non-empty open set, and $V \cap \overline{A}^c \subseteq (\overline{A} \cup \overline{B}) \cap \overline{A}^c \subseteq \overline{B}$, therefore $V \cap \overline{A}^c$ is a non-empty open set contained in $\overline{B}$, so $\overline{B}^{\circ}$ is a non-empty set, hence B is s-dense, which contradicts the assumption. Therefore $A \cup B$ is not s-dense.

Remark. Infinite union of non s-dense sets can be s-dense; for example in the usual topology any singleton is not s-dense set, but $\cup_{x\in\mathbb{R}}\{x\} = \mathbb{R}$ is s-dense.

Theorem 10. *The intersection of SD-dense sets is SD-dense.*

*Proof.*Suppose E and F are two SD-dense sets in a space Z, then from theorem (9) we have $E\cap F\neq\phi$, and by theorem (8) the sets E^c and F^c are not s-dense, and by the previous lemma we obtain $E^c\cup F^c=(E\cap F)^c$ is not s-dense, hence $E\cap F$ is SD-dense.

Corollary 4.*Any finite intersection of SD-dense sets is SD-dense.*

*Proof.*Direct from the mathematical induction.

3.3 SD-Dense Sets as Subspaces

Here we study the characterizations of SD-dense sets as subspaces, and states some conditions that make subspaces SD-dense.

Theorem 11.*In a space Z, if W is a subspace of Z and $E\subseteq W$ where E is SD-dense in Z, then E is SD-dense in W.*

*Proof.*Since E is SD-dense Z, then E contains an open dense set F, then F is open dense set in W, therefore E is SD-dense in W.

*Example 3.*Let (Z,τ) is a space, where $Z=\mathbb{R}$ and $\tau=\{\phi\}\cup\{U\subseteq Z:0\in U\}$, then $S(Z)=\tau/\{\phi\}$. If $W=\{0\}^c$, then W is SD-dense in W, but W is not SD-dense in Z.

Lemma 2.*In a space Z, if W is an open (dense) subspace of Z and $D\subseteq Z$ where D is an open dense in Z, then $D\cap W$ is an open dense in W.*

Theorem 12.*In a space Z, if W is an open (dense) subspace of Z and E is SD-dense in Z, then $E\cap W$ is SD-dense in W.*

*Proof.*Suppose E is SD-dense in Z, then E contains an open dense subset F, and by the previous lemma we obtain $F\cap W$ is open dense subset in W, which contained in $E\cap W$, thus $E\cap W$ is SD-dense in W.

*Example 4.*In the previous example if $W=\{0\}^c$, then the singleton $\{0\}$ is SD-dense in Z but does not intersect W, additionally, the set $\{0,1\}$ is also SD-dense in Z, but $\{0,1\}\cap W=\{1\}$ is not SD-dense in W. Note that the subspace W is closed but not open nor dense in Z.

Theorem 13.*If Z is a space, W is an r-closed subspace of Z and E is SD-dense subset in Z, then $E\cap W$ is SD-dense in W.*

*Proof.*Suppose E is SD-dense in Z, then by theorem (4) the set W is s-dense in Z, hence we have $E\cap W$ is non-empty set. Now suppose A is s-dense subset in W, and by theorem (5) since W is r-closed we obtain A is s-dense in Z, so we have $A\cap E\neq\phi$, therefore $A\cap(E\cap W)=A\cap E\neq\phi$, thus $E\cap W$ is SD-dense in W.

3.4 SD-Dense Sets in Some Special Spaces

In the present subsection, we study the characterizations of SD-dense sets in some spaces, as partition space, S-space, submaximal space, hyperconnected space and strongly hyperconnected space

Theorem 14.*A space (Z,τ) is partition if and only if the only SD-dense set is Z.*

*Proof.*If a space Z is partition, then we have $S(\tau)=P(Z)/\{\phi\}$, therefore Z is the only SD-dense set. Conversely, suppose Z is not partition space, then there is an open set V which is not closed, so V^c is not open, hence $V^{c\circ}=\phi$ or $V^{c\circ}\neq\phi$. In the case where $V^{c\circ}=\phi$ we obtain V is open dense set, hence it is SD-dense, while in the second case, we obtain $V\cup V^{c\circ}$ is open dense set, so it is SD-dense. Thus complete the prove.

Corollary 5.*In S-space (Z,τ), we have $SD(\tau)=\tau\cap D(\tau)$.*

*Proof.*Direct since any subset of Z that contains an open dense set is also open dense set, so it is SD-dense.

*Example 5.*A space (Z,τ) that satisfies $SD(\tau)=\tau\cap D(\tau)$ can be not S-space; for example the usual topological space $(\mathbb{R},\tau)$ is not S-space but $SD(\tau)=\tau\cap D(\tau)$.

Theorem 15.*If F is a subset of a submaximal space Z, then these statements are equivalents:*

1) F is dense in Z.
2) F is SD-dense in Z.

Proof. 1) $\Rightarrow$ 2) Let F be a dense set in Z, then F is open dense set, and by using theorem (8) the set F is SD-dense.
2) $\Rightarrow$ 1) Obvious using corollary (1).

Remark. In submaximal space (Z,τ), we have $SD(\tau) = D(\tau) \subseteq \tau$.

SD-dense $\equiv$ dense $\Rightarrow$ open set $\equiv$ pre-open set $\Rightarrow$ b-open set $\Rightarrow$ β-open set $\Rightarrow$ s-dense

Example 6. A space (Z,τ) that satisfies $SD(\tau) = D(\tau)$ may not be submaximal space; for example in $(\mathbb{R},\tau)$ given in remark (3.1), we have $SD(\tau) = D(\tau) = \{\mathbb{K} \cup A \subseteq \mathbb{R} : A \cap \mathbb{Q} \neq \phi\} \cup \{\mathbb{K}\}$, but $\mathbb{R}$ is not submaximal, since $\mathbb{K} \cup \{0\}$ is dense but not open.

Theorem 16. *A space (Z,τ) is hyperconnected if and only if $\tau/\{\phi\} \subseteq SD(\tau)$.*

Proof. If V is a non-empty open set, then it is open dense, hence it is SD-dense. Conversely, suppose that V is a non-empty open set, then V is SD-dense, i.e., V contains an open dense set, therefore it is dense, thus complete the prove.

Non-empty open set $\Rightarrow$ SD-dense $\Rightarrow$ dense open set $\Rightarrow$ pre-open set $\Rightarrow$ b-open set $\Rightarrow$ α-open set $\Rightarrow$ s-dense

Remark. In hyperconnected space (Z,τ), we have: $SD(\tau) = \{A \subseteq Z : V \subseteq A \text{ for some non-empty open set } V\}$.

Proof. Direct since any non-empty open set is dense, so it is SD-dense. Moreover, any set that contains a non-empty open set is SD-dense.

Example 7. Hyperconnected space can contains a dense subset which is not SD-dense, for instance: If (Z,τ) is the trivial space where Z has more than one element, then Z is hyperconnected space and $S(\tau) = P(Z)/\{\phi\}$, so any singleton $\{a\}$ is dense in Z but not SD-dense. Clearly, the only SD-dense in Z is Z.

Corollary 6. *In hyperconnected S-space (Z,τ), non-empty open sets and SD-dense sets are equivalent, i.e., $SD(\tau) = \tau/\{\phi\}$.*

Proof. Direct using the previous remark and definition (4).

Theorem 17. *If F is a non-empty subset of a strongly hyperconnected space Z, then these statements are equivalents:*

1) F is SD-dense in Z.
2) F is dense in Z.
3) F is s-dense set in Z.
4) F is open set in Z.

Proof. 1) $\Rightarrow$ 2) Obvious.
2) $\Rightarrow$ 3) Obvious.
3) $\Rightarrow$ 4) Obvious using theorem (3).
4) $\Rightarrow$ 1) Obvious using the submaximality in theorem (15).

Remark.1. A space that satisfy any dense set is SD-dense may not by strongly hyperconnected space, for example, In a space $\mathbb{R}$ with $\tau = \{U \subseteq \mathbb{R} : 0 \notin U\} \cup \{\mathbb{R}\}$, we have $S(\tau) = P(\mathbb{R})/\{\phi, \{0\}\}$, so $SD(\tau) = \{\mathbb{R}, \{0\}^c\} = D(\tau)$. Therefore, SD-dense set and dense set are coinciding, but the space $(\mathbb{R},\tau)$ is not strongly hyperconnected space since the singleton $\{1\}$ is open but not dense.
2. A space that satisfy any s-dense set is SD-dense may not by strongly hyperconnected space, for example, In a space (Z,τ), where $Z = \{1,2,3\}$, with $\tau = \{Z, \phi, \{1\}, \{1,2\}\}$ we have $S(\tau) = \{Z, \{1\}, \{1,2\}, \{1,3\}\} = SD(\tau)$. Therefore, SD-dense set and dense set are coinciding, but the space Z is not strongly hyperconnected space, since it is not submaximal, since $\{1,3\}$ is dense but not open.
3. A space that satisfy $SD(\tau) = \tau/\{\phi\}$ may not be strongly hyperconnected space, for example in the space $(\mathbb{R},\tau)$, where $\tau = \{\phi\} \cup \{U \subseteq \mathbb{R} : \{0\} \cup \mathbb{R}^+ \subseteq U\}$, we obtain $SD(\tau) = \tau/\{\phi\}$, but the singleton $\{0\}$ is dense but not open, so $\mathbb{R}$ is not submaximal, therefore is not strongly hyperconnected.

Corollary 7. *In a strongly hyperconnected space Z if $E \subseteq Z$, these conditions are equivalents:*

1) E is a non-empty open set.
2) E is a non-empty α-open set.

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3) E is a non-empty pre-open set.
5) E is a non-empty b-open set.
6) E is a non-empty β-open set.
7) E is a s-dense set.
8) E is a dense set.
9) E is a SD-dense set.

*Proof.*According to theorems (3,17).

3.5 Images of SD-Dense Sets

Here we show that *SD*-irresolute map preserves *SD*-dense sets, while *SD*-continuous map dose not, moreover, the image of *SD*-dense set by *SD*-continuous map is dense.

Theorem 18.*If $f : (Z,\tau) \to (X,\sigma)$ is surjective SD-irresolute map, then the image of any SD-dense set in Z is SD-dense in X.*

*Proof.*Suppose A is *SD*-dense in Z, but $f(A)$ is not *SD*-dense in X, then there is a s-dense set B in X such that $f(A)\cap B = \phi$. Since f is surjective and *SD*-irresolute, then $A\cap f^{-1}(B) = \phi$, where $f^{-1}(B)$ is s-dense in Z, which contradict that A is *SD*-dense.

*Example 8.*The image of *SD*-dense set in Z by *SD*-irresolute map need not be *SD*-dense set in X, for example: Let $\tau = \{U \subseteq \mathbb{R} : 0 \in U\} \cup \{\phi\}$ and $\sigma = \{U \subseteq \mathbb{R} : 0 \notin U\} \cup \{\mathbb{R}\}$ be two topologies on $\mathbb{R}$, then $SD(\tau) = \tau/\{\phi\}$ while $SD(\sigma) = \{R, \{0\}^c\} = D(\sigma)$. Therefore the map $f : (\mathbb{R},\sigma) \to (\mathbb{R},\sigma)$ which defined by: $f(r) = \begin{cases} 1, & r \in \mathbb{Q} \\ 0, & r \in \mathbb{K} \end{cases}$ is *SD*-irresolute map but not surjective, while the singleton $\{0\}$ is *SD*-dense in τ but $f(\{0\}) = \{1\}$ is not *SD*-dense in σ.

Theorem 19.*If $f : (Z,\tau) \to (X,\sigma)$ is surjective SD-continuous map, then the image of any SD-dense set in Z is dense in X.*

*Proof.*Suppose A is *SD*-dense in Z but $f(A)$ is not dense in X, so there exists an open set B in X such that $f(A)\cap B = \phi$. Since f is surjective and *SD*-continuous, then $A\cap f^{-1}(B) = \phi$, where $f^{-1}(B)$ is s-dense in Z, which contradict that A is *SD*-dense.

*Example 9.*The image of *SD*-dense in Z by *SD*-continuous (continuous) map need not be *SD*-dense in X, for instance: If $\tau = \{U \subseteq \mathbb{R} : 0 \in U\} \cup \{\phi\}$ while σ is the trivial topology on R, hence the identity map $I : (R,\tau) \to (R,\sigma)$ is surjective *SD*-continuous (also is continuous) from; while the singleton $\{0\}$ is *SD*-dense in τ but $I\{0\} = \{0\}$ is not *SD*-dense in σ, since the only *SD*-dense in σ is $\mathbb{R}$.

4 SD-Separability

In this section, we define *SD*-separable space, which is stronger than separable space and then we study its properties; as subspaces and images.

4.1 SD-Separable Spaces

Definition 12.*A space that contains a countable SD-dense set is called SD-separable space.*

Corollary 8.*Every SD-separable space is separable space.*

*Proof.*Obvious since any *SD*-dense is dense.

Example 10.
1) If (Z,τ) is the trivial topological space where Z is uncountable set, then Z is separable space but not *SD*-separable, because the only *SD*-dense set is Z.
2) If $(\mathbb{R},\mu)$ is the usual space, then $\mathbb{R}$ is separable but not *SD*-separable, because if F is a non-empty countable subset of R then F^c is also s-dense and $F\cap F^c = \phi$, so F is not *SD*-dense, i.e., any *SD*-dense in $\mathbb{R}$ is uncountable.

3) If $(\mathbb{R},\tau)$ is a space where $\tau = P(Q)\cup\{\mathbb{R}\}$, then $S(\tau) = P(\mathbb{Q})\cup\{A\subseteq\mathbb{R} : A\cap\mathbb{Q}\neq\phi, A\cap\mathbb{K}\neq\phi\}$. Hence the set $\mathbb{Q}\cup B$ where $B\subseteq\mathbb{K}$ is *SD*-dense, so $\mathbb{Q}\cup\{\sqrt{2}\}$ is a countable *SD*-dense in $\mathbb{R}$, therefore $(\mathbb{R},\tau)$ is *SD*-separable space.

Definition 13.*A space (Z,τ) is called SD-countable if the collection $S(\tau)$ is countable.*

Corollary 9.*Any SD-countable space is countable.*

*Proof.*Obvious since $\tau/\{\phi\}\subseteq S(\tau)$.

*Example 11.*Countable space need not be *SD*-countable, for instance if (Z,τ) is the trivial space on infinite set Z, then τ is countable space but not *SD*-countable, since $S(\tau) = P(Z)/\{\phi\}$.

Corollary 10.*Every SD-countable space is SD-separable space.*

*Proof.*Since the collection $S(\tau)$ is countable, now we can choose a point from each *s*-dense set from $S(\tau)$, the set F of all such point is clearly countable and *SD*-dense, therefore Z is *SD*-separable.

Example 12.SD-separable space need not be *SD*-countable, for instance the space $\tau = \{\phi\}\cup\{U\subseteq\mathbb{R} : 0\in U\}$ on $\mathbb{R}$, we obtain $S(\tau) = \tau/\{\phi\}$, then $\mathbb{R}$ is SD-separable space, since $\{0\}$ is a countable *SD*-dense in $\mathbb{R}$, but $S(\tau)$ is uncountable, therefore τ is not *SD*-countable.

Corollary 11.*If Z is a SD-separable space where Z is uncountable, then there exists an uncountable set which is not s-dense.*

*Proof.*Since Z contains a countable *SD*-dense subset E, then E^c is uncountable and not *s*-dense.

*Remark.*The inverse of the previous corollary is not true in general, for example: in the space $(\mathbb{R},\tau)$ given in remark (3.4) number (1), the set of negative real numbers $\mathbb{R}^-$ is uncountable and it is not *s*-dense, while $\mathbb{R}$ it is not *SD*-separable space, since all *SD*-dense sets are uncountable.

Corollary 12.*A space Z is SD-separable if and only if there exists a set A which is not s-dense, where A, A^c are uncountable and countable sets; respectively.*

Corollary 13.*Submaximal separable space is SD-separable space.*

*Proof.*Obvious since dense sets and *SD*-dense sets are equivalent from theorem (15).

4.2 Subspaces and Images of SD-Separable Spaces

*Example 13.*Subspace of *SD*-separable space need not *SD*-separable space in general, for instance: where Z is uncountable set with topology $\tau = \{U\subseteq Z : a\in U\}\cup\{\phi\}$, where a is a fixed point in Z, so $S(\tau) = \tau/\{\phi\}$. The singleton $\{a\}$ is *SD*-dense, hence Z is *SD*-separable space but the subspace $\{a\}^c$ is not *SD*-separable space, since it is the discrete space. Note that the subspace $\{a\}^c$ is not open subspace nor *r*-closed.

Corollary 14.*Any open (dense or r-closed) subspace of SD-separable space is SD-separable space.*

*Proof.*Direct using theorem (12) (theorem (13)).

Corollary 15.*If a map $f : (Z,\tau)\rightarrow(X,\sigma)$ is surjective and SD-irresolute from SD-separable space Z, then X is SD-separable.*

*Proof.*Direct using theorem (18).

*Example 14.*The image of *SD*-separable space by *SD*-continuous map need not be *SD*-separable, for instance if $(\mathbb{R},\tau)$ a space given in example (3) while $(\mathbb{R},\sigma)$ is the trivial topological space, hence the identity map $I : (\mathbb{R},\tau)\Rightarrow(\mathbb{R},\sigma)$ is *SD*-continuous (also is continuous) from *SD*-separable space $(\mathbb{R},\tau)$; since $\{0\}$ is countable *SD*-dense in $(\mathbb{R},\tau)$, while the space $(\mathbb{R},\sigma)$ is not *SD*-separable, since the only *SD*-dense in $(\mathbb{R},\sigma)$ is $\mathbb{R}$.

Corollary 16.*If a map $f : (Z,\tau)\rightarrow(X,\sigma)$ is surjective SD-continuous from SD-separable space Z, then X is separable.*

*Proof.*Suppose E is a countable *SD*-dense subset in Z, according to theorem (19) we obtain $f(E)$ is a countable dense in X, therefore X is separable space.

5 Conclusion

Using the concept of somewhere dense closure operator, we define a generalization of dense set; namely *SD*-dense set, then we introduce a new type of separability; namely *SD*-separable space. Here we outline the results that summarized the properties of *SD*-dense sets and *SD*-separable space:

A. *SD*-Dense Set $\Rightarrow$ Dense Set.
B. A subset F of a space Z is *SD*-dense if and only if F intersect all s-dense sets; equivalently if $(F^c)^{oS} = \varphi$; equivalently if F^c is not s-dense; equivalently if F^o is open dense set; equivalently if F contains an open dense set.
C. The union of two non s-dense sets is also non s-dense.
D. The intersection of two *SD*-dense sets is *SD*-dense.
E. If W is open (dense or regular closed) subspace of a space Z, and F is *SD*-dense subset in Z, then $F \cap W$ is *SD*-dense in W.
F. A space is partition if and only if it has no proper *SD*-dense set.
G. *SD*-Dense Set $\xrightarrow{S\text{-Space}}$ Open Dense Set.
H. *SD*-Dense Set $\xLeftrightarrow{\text{Submaximal Space}}$ Dense Set.
I. Non-empty Open Set $\xRightarrow{\text{Hyperconnected Space}}$ *SD*-Dense Set.
J. *SD*-Dense Set $\xLeftrightarrow{\text{Hyperconnected } S\text{-Space}}$ Non-empty Open Set
K. In strongly hyperconnected space, all these statements are equivalent: *SD*-dense set, dense set, s-dense set, β-open set, b-open set, preopen set and open set.
L. *SD*-Separable Space $\Rightarrow$ Separable Space.
M. *SD*-Separable Space $\xLeftrightarrow{\text{Submaximal Space}}$ Separable Space.
N. *SD*-separable space satisfy the open (dense or regular closed) hereditary property.
O. *SD*-irresolute map preserves *SD*-separable space (*SD*-dense set).
P. *SD*-continuous map does not preserve *SD*-separable space (*SD*-dense set), but the image of *SD*-separable space (*SD*-dense set) is separable space (dense set).

Note that some properties of *SD*-dense sets are different from dense sets, as in A, B (the third part), C and D.

Declarations

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Optimal Conjugate Gradient with Spline Scheme for Solving Bagley-Torvik Fractional Differential Problems

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Abstract: In this work, a non-polynomial spline function is constructed to solve the Bagley-Torvik Fractional Differential Problems involving derivatives in the Caputo sense. This method transforms the fractional differential equation into a system of linear equations using a spline scheme. The conjugate gradient method is employed for the iterative solution of the linear system. To validate the accuracy of the method, numerical examples with known analytical solutions are tested. The numerical experiments demonstrate satisfactory agreement with the exact solution.

Keywords: Fractional calculus; Bagley-Torvik Fractional Differential equation; Caputo fractional derivatives; Non-polynomial Spline; Conjugate gradient method.

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1 Introduction

Fractional differential equations have received significant attention nowadays in several fields of science and engineering due to its applications such as : electrical engineering[1],economic[2],Modelling of Viscoelastic Systems[3],diffusion processes[4],medicine[5]. It is difficult to find an exact analytical solution of all fractional differential equations therefore several methods and techniques have been invented to solve fractional differential equation for instance: fractional finite difference method[6], Adomain decomposition method[7],spectral method[8],Bessel collocation method[9].

Spline technique has been investigated by many researchers for solving fractional differential equations due to its accurate and efficiency for example: W. K. ZAHRA and et al proposed cubic spline solution of fractional Bagley-Torvik equation[10], semiorthogonal B-spline collection is applied for solving the fractional differential equations[11], NonPolynomial Spline discussed by Faraidun K. Hamasalh and et al to solve FDE[12], Faraidun K. Hamasalh and Karzan A. Hamza, used Quintic B-spline polynomial for Solving Bagely-Torvik Fractional Differential Problems[13], fourth order homogeneous parabolic partial differential equations solved using non-polynomial cubic spline technique[14].

Conjugate gradient method is an appropriate and efficient method for solving a system of equations. The linear conjugate gradient method was proposed in the 1950s by Hestenes and Stiefel to solve a linear system of equations with positive definite matrices as an alternative to Gauss elimination[15], Fletcher and Reeves were discussed the nonlinear conjugate gradient method in 1964[16]. Presently, conjugate gradient (CG) techniques are considered as a popular and efficient approach to solve engineering optimization problems. As recent examples, shape optimization with nonlinear conjugate gradient method proposed in[17], application in signal processing of decent hybrid nonlinear conjugate gradient method discussed by Zohre Aminifard and etal[18] Abubakar and et al investigated a modified a three-term

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conjugate gradient projection with application in signal recovery[19].

The rest of this paper is organized as follows: in section 2, we briefly review the main definitions of fractional calculus, some definitions and properties of the matrix. Mathematical formulation of the nonpolynomial spline function discussed in section 3. In section 4 numerical results are illustrated to present applicability of the method. Finally, the conclusion is presented in section 5.

2 Some basic definitions

Definition 1.[20] *The Riemann-Liouville fractional derivative of order* $\lambda > 0$ *is defined by* $D^{\lambda} f(t) = \frac{1}{\Gamma(m-\lambda)} \frac{d^m}{d\tau^m} \int_a^t (t-\tau)^{m-\lambda-1} f(\tau) d\tau, \quad m-1 < \lambda < m \in N$

Definition 2.[21] *The Caputo fractional derivative of order* $\lambda > 0$ *is defined by* $D^{\lambda} f(t) = \frac{1}{\Gamma(m-\lambda)} \int_a^t (t-\tau)^{m-\lambda-1} \frac{d^m}{d\tau^m} f(\tau) d\tau, \quad m-1 < \lambda < m \in N$

Definition 3.[20] *The Riemann-Liouville fractional integral of order* $\lambda > 0$ *is defined by* $I^{\lambda} f(t) = \frac{1}{\Gamma \lambda} \int_a^t (t-\tau)^{\lambda-1} f(\tau) d\tau, \quad m-1 < \lambda < m \in N$

Definition 4.[20] *The Caputo derivative of order* λ *of a polynomial function* x^d *is defined by* $D^{\lambda} x^d = \frac{\Gamma(d+1)}{\Gamma(d-\lambda+1)} x^{d-\lambda}$

Definition 5.[22] *The Spectral radius* $\mu(M)$ *where* M *is an* $n \times n$ *matrix is given by* $\mu(M) = max(|\lambda|)$ *where* λ *is an eigenvalue of* M.

Definition 6.[23] *A square matrix M is called diagonally dominate if* $|m_{ij}| < \Sigma_{i \neq j} |m_{ij}|$

Definition 7.[22] *An* $n \times n$ *matrix* M *is converges if* $\mu(M) < 1$.

3 Mathematical Formulation

In this study we consider the fractional differential equation of the form

$$y^{(\frac{3}{2})} + \phi(x) y'' + \psi(x) y = \tau(x), \qquad x \in [a,b] \tag{1}$$

with the boundary conditions

$$y(a) = B_1, \quad y(b) = B_2 \tag{2}$$

Where $\phi(x), \psi(x) and \tau(x)$ are functions of x, B_1 and B_2 are constants.Then the interval $[a,b]$ can be uniformly divide into j subintervals the length of uniform subintervals can be define as:$\Delta x = h = \frac{b-a}{j}, \quad n = j-1$. In this existing literature we can modify the model of nonpolynomial spline and the factional continuity by using Caputo type as follows:

$$S(x) = S_i(x), x \in [x_i, x_{i+1}], i = 0,1,2,...,n \tag{3}$$

Here the nonpolynomial spline function with fractional order defined by

$$S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 + e_i sin(k(x-x_i)) + f_i cos(k(x-x_i)) \tag{4}$$

where $a_i, b_i, c_i, d_i, e_i, f_i$ are constants for $i = 0,1,2,...,n$ and k is a free parameter .The function $S_i(x)$ interpolates $y(x)$ at the points x_i by depending on k. To find the value of constants in equation (4) we supposed the following conditions:

$$\begin{gathered} S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1}, S_i''(x_i) = y_i'', \\ S_{i+1}''(x_{i+1}) = y_{i+1}'', S_i^{(\frac{3}{2})}(x_i) = p_i, S_{i+1}^{(\frac{3}{2})}(x_{i+1}) = p_{i+1}. \end{gathered} \tag{5}$$

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Applying the conditions in equation (5) the value of all constants in equation (4) obtained as follows:

$$
\begin{aligned}
a_i &= (1-\tfrac{2}{3A_1}\sqrt{\tfrac{h}{\pi}})y_i'' + \tfrac{4}{3}\sqrt{\tfrac{h}{\pi}}y_{i+1}'' - \tfrac{1}{A_1}p_{i+1} + \tfrac{A_2}{A_1}p_i,\\
b_i &= \tfrac{y_{i+1}-y_i}{h} - \tfrac{1}{h}(\tfrac{\theta^2}{2A_1} + \tfrac{h^3\beta}{A_1} + \tfrac{sin\theta+cos\theta-1}{A_1})p_{i+1} - \tfrac{1}{h}((1-\theta^2)\tfrac{A_2}{A_1} + \tfrac{h^3}{A_5}\\
&+sin\theta(\tfrac{1}{k}\sqrt{\tfrac{2}{k}} - \tfrac{A_2}{A_1}) - \tfrac{A_2cos\theta}{A_1})P_i - \tfrac{1}{h}(\tfrac{4}{3}\sqrt{\tfrac{h}{\pi}} - \tfrac{2}{3}\theta^2\sqrt{\tfrac{h}{\pi}} + h^3A_4 - \tfrac{4}{3}\sqrt{\tfrac{h}{\pi}}sin\theta\\
&-\tfrac{\frac{4}{3}\sqrt{\frac{h}{\pi}}cos\theta}{A_1})y_{i+1}'' + \tfrac{1}{h}(\tfrac{h^2}{2} + \tfrac{\theta^2A_3}{2A_1} - h^3(\tfrac{\beta A_3}{A_1} - \tfrac{1}{6h}) + \tfrac{(sin\theta+cos\theta)A_3}{A_1})y_i'',\\
c_i &= \tfrac{k^2}{2A_1}P_{i+1} - \tfrac{k^2A_2}{2A_1}p_i - \tfrac{2}{3}k^2\sqrt{\tfrac{h}{\pi}}y_{i+1} + (\tfrac{1}{2} - \tfrac{\frac{1}{3}k^2\sqrt{\frac{h}{\pi}}}{A_1})y_i''\\
d_i &= (\tfrac{1}{6h} - \tfrac{4\beta}{3A_1}\sqrt{\tfrac{h}{\pi}})y_{i+1}'' - (\tfrac{2\beta}{3A_1}\sqrt{\tfrac{h}{\pi}} + \tfrac{1}{6h})y_i'' + \tfrac{\beta}{A_1}p_{i+1} +\\
&(\tfrac{\sqrt{2}k^2sin\theta}{6hk^{\frac{3}{2}}} - \tfrac{\beta A_2}{A_1})p_i,\\
e_i &= (\tfrac{\sqrt{2}}{k^{\frac{3}{2}}} - \tfrac{A_2}{A_1})p_i + \tfrac{1}{A_1}p_{i+1} - \tfrac{4}{3}\sqrt{\tfrac{h}{\pi}}y_{i+1}'' + \tfrac{4}{3A_1}\sqrt{\tfrac{h}{\pi}}y_i'',\\
f_i &= \tfrac{1}{A_1}p_{i+1} - \tfrac{A_2}{A_1}p_i - \tfrac{4}{3}\sqrt{\tfrac{h}{\pi}}y_{i+1}'' + \tfrac{4}{3A_1}\sqrt{\tfrac{h}{\pi}}y_i''.
\end{aligned}
\tag{6}
$$

where $\beta = \frac{k^2(sin\theta+cos\theta-1)}{6h}, A_1 = 2k^2\sqrt{\frac{h}{\pi}} + 8\beta h\sqrt{\frac{h}{\pi}} + k^{\frac{3}{2}}(cos(\theta+\frac{3\pi}{4}) + sin(\theta+\frac{3\pi}{4}))$, $A_2 = \frac{4\sqrt{2\theta sin\theta}}{3\sqrt{\pi}} - \sqrt{2}sin(\theta+\frac{3\pi}{4}), A_3 = \frac{-2}{3}\sqrt{\frac{h}{\pi}}, A_4 = \frac{1}{6h} - \frac{4}{3}\sqrt{\frac{h}{\pi}}, A_5 = \frac{\sqrt{2k}sin\theta}{6h} - \frac{\beta A_2}{A_1}, \theta = kh, i = 0,1,...,n$. Therefore we obtain the nonpolynomial spline function, it can be easily verified that the spline scheme approximation $S(x)$, is successfully uniquely determined using the equation (6) recurrence formula for all h which in the interval, see[24]. Substitute these values in equation (4) we obtain

$$
\begin{aligned}
S(x) &= (1-\tfrac{2}{3A_1}\sqrt{\tfrac{h}{\pi}})y_i'' + \tfrac{4}{3}\sqrt{\tfrac{h}{\pi}}y_{i+1}'' - \tfrac{1}{A_1}p_{i+1} + \tfrac{A_2}{A_1}p_i + (\tfrac{y_{i+1}-y_i}{h}\\
&-\tfrac{1}{h}(\tfrac{\theta^2}{2A_1} + \tfrac{h^3\beta}{A_1} + \tfrac{sin\theta+cos\theta-1}{A_1})p_{i+1} - \tfrac{1}{h}((1-\theta^2)\tfrac{A_2}{A_1} + \tfrac{h^3}{A_5} +\\
&sin\theta(\tfrac{1}{k}\sqrt{\tfrac{2}{k}} - \tfrac{A_2}{A_1}) - \tfrac{A_2cos\theta}{A_1})P_i - \tfrac{1}{h}(\tfrac{4}{3}\sqrt{\tfrac{h}{\pi}} - \tfrac{2}{3}\theta^2\sqrt{\tfrac{h}{\pi}}\\
&+h^3A_4 - \tfrac{4}{3}\sqrt{\tfrac{h}{\pi}}sin\theta - \tfrac{\frac{4}{3}\sqrt{\frac{h}{\pi}}cos\theta}{A_1})y_{i+1}'' + \tfrac{1}{h}(\tfrac{h^2}{2} + \tfrac{\theta^2A_3}{2A_1} - h^3(\tfrac{\beta A_3}{A_1}\\
&-\tfrac{1}{6h}) + \tfrac{(sin\theta+cos\theta)A_3}{A_1})y_i'')(x-x_i)(\tfrac{k^2}{2A_1}P_{i+1} - \tfrac{k^2A_2}{2A_1}p_i - \tfrac{2}{3}k^2\sqrt{\tfrac{h}{\pi}}y_{i+1} +\\
&(\tfrac{1}{2} - \tfrac{\frac{1}{3}k^2\sqrt{\frac{h}{\pi}}}{A_1})y_i'')(x-x_i)^2 + ((\tfrac{1}{6h} - \tfrac{4\beta}{3A_1}\sqrt{\tfrac{h}{\pi}})y_{i+1}'' - (\tfrac{2\beta}{3A_1}\sqrt{\tfrac{h}{\pi}} + \tfrac{1}{6h})y_i'' +\\
&\tfrac{\beta}{A_1}p_{i+1} + (\tfrac{\sqrt{2}k^2sin\theta}{6hk^{\frac{3}{2}}} - \tfrac{\beta A_2}{A_1})p_i)(x-x_i)^3 +\\
&((\tfrac{\sqrt{2}}{k^{\frac{3}{2}}} - \tfrac{A_2}{A_1})p_i + \tfrac{1}{A_1}p_{i+1} - \tfrac{4}{3}\sqrt{\tfrac{h}{\pi}}y_{i+1}'' + \tfrac{4}{3A_1}\sqrt{\tfrac{h}{\pi}}y_i'')sin(k(x-x_i)) +\\
&(\tfrac{1}{A_1}p_{i+1} - \tfrac{A_2}{A_1}p_i - \tfrac{4}{3}\sqrt{\tfrac{h}{\pi}}y_{i+1}'' + \tfrac{4}{3A_1}\sqrt{\tfrac{h}{\pi}}y_i'')cos(k(x-x_i)).
\end{aligned}
\tag{7}
$$

Now apply the fractional continuity conditions of the spline function $S_i(x)$ where the splines $S_{i-1}^m(x) = S_i^m(x), m = \frac{1}{2}, 1$ joined, we obtained the following equations:

$$
S_i^{(\frac{1}{2})}(x_i) = \frac{\sqrt{2k}}{A_1}p_{i+1} + \sqrt{\frac{k}{2}}(\frac{\sqrt{2}}{k^{\frac{3}{2}}} - \frac{2A_2}{A_1})p_i - \frac{4\sqrt{\theta}}{3\sqrt{2\pi}}(1+\frac{1}{A_1})y_{i+1}'' - \frac{4\sqrt{\theta}}{3\sqrt{2\pi}A_1}y_i'' \tag{8}
$$

, And

$$S_{i-1}^{(\frac{1}{2})}(x_i) = \frac{2}{\sqrt{h\pi}}(y_i - y_{i-1}) + (\frac{\sqrt{k}(cos(\theta+\frac{\pi}{4})+sin(\theta+\frac{\pi}{4}))}{A_1} + \frac{42\beta h^{\frac{5}{2}}}{15A_1\sqrt{\pi}} + \frac{4k^2h^{\frac{3}{2}}}{3A_1\sqrt{\pi}} - \frac{2L_1}{\sqrt{\pi h}})p_i + (\sqrt{k}sin(\theta+\frac{\pi}{4})(\frac{\sqrt{2}}{k^{\frac{3}{2}}} - \frac{A_2}{A_1}) - \frac{\sqrt{k}cos(\theta+\frac{\pi}{4})A_2}{A_1} + \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{\sqrt{2k}sin\theta}{6h} - \frac{A_2\beta}{A_1}) - \frac{4k^2h^{\frac{3}{2}}}{3\sqrt{\pi}A_1} - \frac{2L_2}{\sqrt{h\pi}})p_{i-1} + (\frac{42\beta h^{\frac{5}{2}}}{15A_1\sqrt{\pi}}(\frac{1}{6h} - \frac{4\beta\sqrt{h}}{3A_1\sqrt{\pi}}) - \frac{4\sqrt{\theta}sin(\theta+\frac{\pi}{4})}{3\sqrt{\pi}} - \frac{4\sqrt{\theta}cos(\theta+\frac{\pi}{4})}{3A_1\sqrt{\pi}} - \frac{16\theta^2}{9\pi} - \frac{2L_3}{\sqrt{h\pi}})y_i'' + (\frac{8h^{\frac{3}{2}}}{3\sqrt{\pi}}(\frac{1}{2} - \frac{k^2\sqrt{h}}{3A_1\sqrt{\pi}}) - \frac{2L_4}{\sqrt{\pi h}} - \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{2\beta\sqrt{h}}{3A_1\sqrt{\pi}} + \frac{1}{6h}) - \frac{2\sqrt{\theta}(sin(\theta+\frac{\pi}{4})+cos(\theta+\frac{\pi}{4}))}{3A_1\sqrt{\pi}})y_{i-1}'' \tag{9}$$

Such that,

$L_1 = \frac{\theta^2}{2A_1} + \frac{h^3\beta+cos\theta+sin\theta-1}{A_1}$,

$L_2 = \frac{A_2}{A_1}(1-\theta^2-sin\theta) + h^3A_5 + (\frac{\sqrt{2}}{k^{\frac{3}{2}}} - \frac{A_2}{A_1})$

$L_3 = h^3A_4 + (\frac{4}{3} - \frac{2\theta^2}{3} - \frac{4sin\theta}{3} - \frac{4cos\theta}{3A_1})\sqrt{\frac{h}{\pi}}$,

$L_4 = \frac{h^2}{2} + \frac{\theta^2A_3}{2A_1} + h^3(\frac{\beta A_3}{A_1} - \frac{1}{6h}) + \frac{A_3}{A_1}(sin\theta + cos\theta)$

Here by equating equation (8) and equation (9) we obtain

$$\frac{\sqrt{2}}{A_1}p_{i+1} + C_1p_i - \frac{4\sqrt{\theta}}{3\sqrt{2\pi}}(1+\frac{1}{A_1})y_{i+1}'' - C_2y_i'' - \frac{2}{\sqrt{\pi h}}(y_i - y_{i-1}) + C_3p_{i-1} + C_4y_{i-1}'' = 0 \tag{10}$$

Where, $C_1 = \frac{1}{k} - \frac{\sqrt{2k}A_2}{A_1} - \frac{4k^2h^{\frac{3}{2}}}{3A_1\sqrt{\pi}} - \frac{42h^{\frac{5}{2}}\beta}{15\sqrt{\pi}A_1} - \frac{\sqrt{k}(sin(\theta+\frac{\pi}{4})+cos(\theta+\frac{\pi}{4}))}{A_1}$

$C_2 = \frac{4\sqrt{\theta}}{3\sqrt{2\pi}A_1} - \frac{2L_3}{\sqrt{h\pi}} - \frac{16\theta^2}{9\pi} + \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{1}{6h} - \frac{4\beta\sqrt{h}}{3A_1\sqrt{\pi}}) - \frac{4\sqrt{\theta}sin(\theta+\frac{\pi}{4})}{3\sqrt{\pi}} - \frac{4\sqrt{\theta}cos(\theta+\frac{\pi}{4})}{3\sqrt{\pi}A_1}$,

$C_3 = \frac{2L_2}{\sqrt{h\pi}} + \frac{4k^2h^{\frac{3}{2}}}{3A_1\sqrt{\pi}} + \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{\sqrt{2k}sin\theta}{6h} - \frac{A_2\beta}{A_1}) - \sqrt{k}sin(\theta+\frac{\pi}{4})(\frac{\sqrt{2}}{k^{\frac{3}{2}}} - \frac{A_2}{A_1}) + \sqrt{k}cos(\theta+\frac{\pi}{4})\frac{A_2}{A_1}$,

$C_4 = \frac{2L_4}{\sqrt{h\pi}} - \frac{8h^{\frac{3}{2}}}{3\sqrt{\pi}}(\frac{1}{2} - \frac{k^2\sqrt{h}}{3A_1\sqrt{\pi}}) + \frac{42h^{\frac{5}{2}}}{15\sqrt{\pi}}(\frac{1}{6h} + \frac{2\beta\sqrt{h}}{3A_1\sqrt{\pi}}) + \frac{\sqrt{2\theta}(cos(\theta+\frac{\pi}{4})+sin(\theta+\frac{\pi}{4}))}{3A_1\sqrt{\pi}}$.

from equation (1), and using backward, central, and forward difference formula for $y_{i+1}'', y_i'', and y_{i-1}''$ respectively we have

$$\begin{gathered} p_{i+1} = -\phi_{i+1}(x)y_{i+1}'' - \psi_{i+1}(x)y_{i+1} + \tau_{i+1}(x) \\ p_i = -\phi_i(x)y_i'' - \psi_i(x)y_i + \tau_i(x), \\ p_{i-1} = -\phi_{i-1}(x)y_{i-1}'' - \psi_{i-1}(x)y_{i-1} + \tau_{i-1}(x) \\ y_{i+1}'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}, y_i'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2}, y_{i-1}'' = \frac{y_{i+1}-2y_i+y_{i-1}}{h^2} \end{gathered} \tag{11}$$

substitute equation (11) in equation (10) we obtain:

$$a_iy_{i-1} + b_iy_i + c_iy_{i+1} = F_i \tag{12}$$

Then, a system of linear equation is formulated using equation(12) as follows :

$$Ay = F \tag{13}$$

such that

$$A = \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix}$$

$y = [y_1 \ \ y_2 \ \ y_3 \ \ \cdots \ \ y_{n-1} \ \ y_n]^T$, and $F = [F_1 - a_1 y_0 \ \ F_2 \ \ \ \cdots \ \ F_{n-1} \ \ F_n - c_n y_{n+1}]$
Such that,
$a_i = \frac{-\sqrt{2k}\phi_{i+1}}{h^2 A_1} - \frac{4\sqrt{\theta}}{3\sqrt{2\pi}h^2}(1+\frac{1}{A_1}) - \frac{C_1\phi_i}{h^2} - \frac{1}{h^2}(C_4 - C_2) + \frac{2}{\sqrt{h\pi}} - \frac{C_3\phi_{i-1}}{h^2} - C_3\psi_{i-1},$

$b_i = \frac{2\sqrt{2k}\phi_{i+1}}{h^2 A_1} - \frac{8\sqrt{\theta}}{3\sqrt{2\pi}h^2}(1+\frac{1}{A_1}) + \frac{2C_1\phi_i}{h^2} - \frac{2C_3\phi_{i-1}}{h^2} - \frac{2}{h^2}(C_4 - C_2) - C_1\psi_i - \frac{2}{\sqrt{h\pi}},$
$c_i = \frac{-\sqrt{2k}\phi_{i+1}}{h^2 A_1} - \frac{\sqrt{2k}\psi_{i+1}}{A_1} - \frac{4\sqrt{\theta}}{3\sqrt{2\pi}h^2}(1+\frac{1}{A_1}) - \frac{C_1\phi_i}{h^2} - \frac{C_3\phi_{i-1}}{h^2} + \frac{1}{h^2}(C_4 - C_2)$
$F_i = \frac{\sqrt{2k}}{A_1}\tau_{i+1} - C_1\tau_i - C_3\tau_{i-1}, i = 1, 2, \cdots, n.$

4 Numerical experiments

In this section the method applied to several numerical examples of boundary fractional differential equations, the result compared with the exact analytical solution to show the methods efficiency. The computational programs were written in MatLab. Here the algorithms of the conjugate gradient method is presented.

Algorithm 1*suppose that we have the linear system (13) where A is symmetric positive definite matrix The conjugate gradient algorithm expressed as:*

–*chose* $y_0 \in R^n$*, and put* $d_0 = r_0 = F - Ay_0$ *for* $k = 0, 1, 2, \cdots$
–*If* $d_k = 0$*, stop and* y_k *is a solution of* $Ay = F$*.*
otherwise compute

–$\alpha_k = \frac{r_k^T r_k}{d_k^T A d_k}, y_{k+1} = y_k + \alpha_k d_k,$

–$r_{k+1} = r_k - \alpha_k A d_k, \beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$

–$d_{k+1} = r_{k+1} + \beta_k d_k.$

Example 1*.[20] Consider the fractional differential equation*

$$D^2 y(x) + D^{(\frac{3}{2})} y(x) + y(x) = 1 + x, x \in [0, 1]. \quad (14)$$

with the boundary conditions $y(0) = 1, y(1) = 2$
,The exact solution of (14) is given by $y(x) = 1 + x$*.*

The numerical results using conjugate gradient method with, $h = \frac{1}{32}$, and 31 iterations tabulated in Table1

x	Exact solution	proposed method	Absolute error
0.125	1.125	1.125927	9.27×10^{-4}
0.25	1.25	1.251416	1.41×10^{-3}
0.375	1.375	1.376678	1.67×10^{-3}
0.5	1.5	1.501712	1.71×10^{-3}
0.625	1.625	1.626516	1.51×10^{-3}
0.75	1.75	1.751092	1.09×10^{-3}
0.875	1.875	1.875442	4.42×10^{-4}

Table 1: Exact, approximation solution, absolute error of example 1

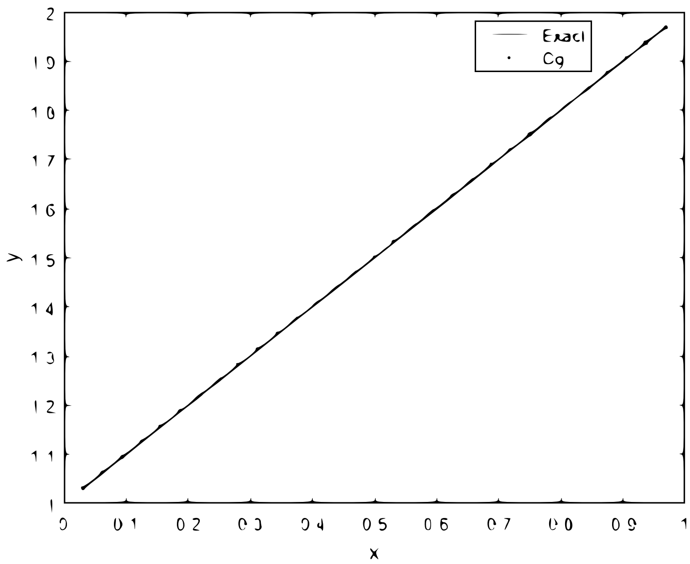

Fig. 1: Exact and approximate solution of example 1 with $h = \frac{1}{32}$

Example 2.*[25] Consider the fractional differential equation*

$$D^{(\frac{3}{2})}y(x) = cos(x + \frac{\pi}{4}), x \in [0, 1]. \quad (15)$$

with the boundary conditions $y(0) = 1, y(1) = 1.84147$
,The exact solution of (15) is given by $y(x) = sin(x) + 1$.

The numerical results using conjugate gradient method with, $h = 0.01$, and 99 iterations tabulated in Table 2 with comparison to reference [25].

x	Exact solution	proposed method	Absolute error	Absolute error [25]
0.1	1.09983	1.09051	9.32×10^{-3}	2.29×10^{-3}
0.2	1.19866	1.17982	1.884×10^{-2}	9.97×10^{-2}
0.3	1.29552	1.26783	2.768×10^{-2}	1.03×10^{-1}
0.4	1.38941	1.35445	3.496×10^{-2}	8.901×10^{-2}
0.5	1.47942	1.43959	3.982×10^{-2}	1.995×10^{-2}
0.6	1.56464	1.52320	4.144×10^{-2}	9.144×10^{-2}
0.7	1.64421	1.60521	3.900×10^{-2}	8.577×10^{-2}
0.8	1.71735	1.68560	3.174×10^{-2}	9.177×10^{-2}
0.9	1.7833	1.76435	1.896×10^{-2}	7.467×10^{-2}

Table 2: Exact, approximation solution, absolute error of example 2

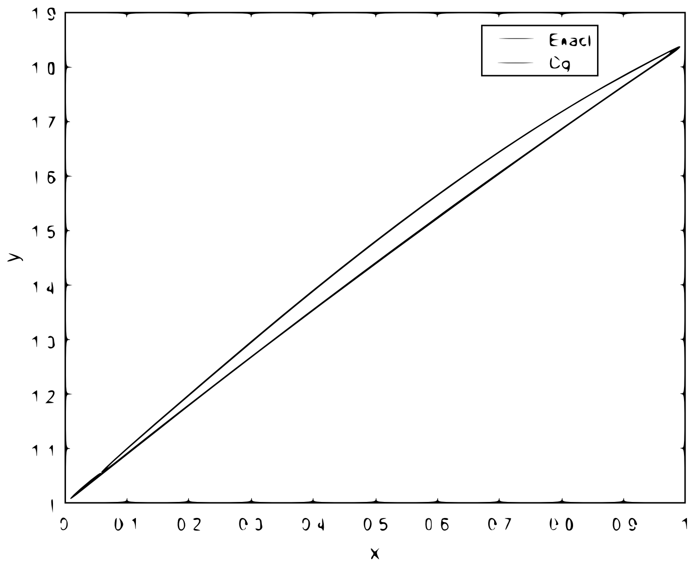

Fig. 2: Exact and approximate solution of example 2 with $h = 0.01$

Example 3.*[26] Consider the fractional differential equation*

$$D^2 y(x) + \sqrt{\pi} D^{(\frac{3}{2})} y(x) + y(x) = 0, x \in [0,1]. \quad (16)$$

with the boundary conditions $y(0) = 1, y(1) = 0.775989$.

The numerical results using conjugate gradient method with, $h = 0.125$, and 7 iterations tabulated in Table 3, with comparison to reference [26]

x	Exact solution	proposed method	Absolute error	Absolute error[26]
0.125	0.99437	0.98819	6.17×10^{-3}	1.24×10^{-3}
0.25	0.979919	0.971592	8.32×10^{-3}	5.11×10^{-3}
0.375	0.958424	0.95024	8.17×10^{-3}	1.387×10^{-2}
0.5	0.930957	0.92424	6.71×10^{-3}	2.614×10^{-2}
0.625	0.898335	0.89367	4.65×10^{-3}	4.039×10^{-2}
0.75	0.861241	0.85868	2.56×10^{-3}	5.579×10^{-2}
0.875	0.820277	0.81939	8.8×10^{-4}	7.148×10^{-2}

Table 3: Exact, approximation solution, absolute error of example 3

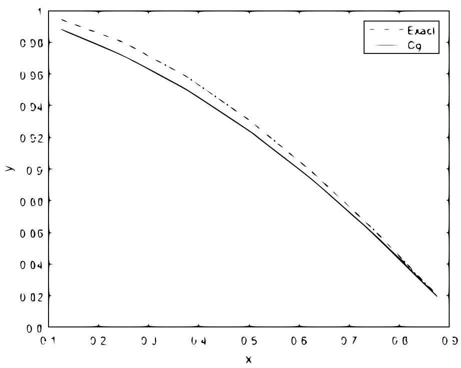

Fig. 3: Exact and approximate solution of example 3 with $h = 0.125$

5 Conclusion

This study constructs a non-polynomial spline function to approach the Bagley-Torvik Fractional Differential Problems with the conjugate gradient method. The numerical examples demonstrate that the non-polynomial spline and conjugate gradient techniques are more adaptable for approximating functions. The graphs of exact and approximate solutions for numerical examples show the superiority of our approach.

Declarations

Competing interests
The author declare that he has neither financial nor conflict interest.
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Automorphisms of semidirect products fixing the non-normal subgroup

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Abstract: In this paper, we describe the automorphism group of semidirect product of two groups that fixes the non-normal subgroup of it. We have computed these automorphisms for the non-abelian metacyclic p-group and non-abelian p-groups $(p \geq 5)$ of order p^4, where p is a prime.

Keywords: Automorphism Group; Semidirect product; p-group.

2010 Mathematics Subject Classification. 20D45; 20E36.

1 Introduction

Let H and K be two groups and $\phi : K \longrightarrow Aut(H)$ be a group homomorphism, where $Aut(H)$ is the group of automorphisms of the group H. Then $G = H \rtimes_{\phi} K$ is called the external semidirect product of groups H and K. On the other hand, let $G = HK$ be a group, where H and K are subgroups of G and K acts on H by conjugation defined as $h^k = khk^{-1}$ for all $h \in H$ and $k \in K$. Then $G = HK$ is called the internal semidirect product of subgroups H and K, where H is a normal subgroup of G and K is a non-normal subgroup of G.

Bidwell et. al. [1] studied the structure of automorphism group of direct product of two groups as the matrices of maps satisfying some certain conditions. The next interesting question was to study the structure of the automorphism group of semidirect product of two groups. The automorphism group of semidirect product of two groups was studied by Bidwell and Curran [2]. Later, M . J. Curran [5] and D. Jill [6] studied the automorphism group of the semidirect product of two groups that fixes the normal subgroup. In this paper, we study the structure of the automorphism group of the semidirect product of two groups that fixes the non-normal subgroup. We apply our main result to compute such automorphisms of non-abelian metacyclic p-groups and non-abelian p-groups $(p \geq 5)$ of order p^4, where p is a prime.

Let K be a non-normal subgroup of a group G. Let S be a right transversal to K in G with $1 \in S$. Then the group operation on G induces a binary operation on S with respect to it S becomes a right loop, a right action θ of K on S and two map $f : S \times S \longrightarrow K$ and $\sigma : S \times K \longrightarrow K$ (see [8] for details) . Let $Aut_K(G) = \{\Theta \in Aut(G) \mid \Theta(K) = K\}$. In [8, Theorem 2.6, p. 73], R. Lal obtained that $\Theta \in Aut_K(G)$ can be identified with the triple (α, γ, δ), where $\alpha \in Map(S,S)$, $\gamma \in Map(S,K)$ and $\delta \in Aut(K)$ satisfying the conditions in [8, Definition 2.5, p. 73] given below,

$(i)\alpha(xy) = (\alpha(x)\theta\gamma(y))\alpha(y)$
$(ii)\delta(f(x,y))\gamma(xy) = \gamma(x)\sigma_{\alpha(x)}(\gamma(y))f(\alpha(x)\theta\gamma(y), \alpha(y))$
$(iii)\alpha(x\theta k) = \alpha(x)\theta\delta(k)$
$(iv)\delta(\sigma_x(k))\gamma(x\theta k) = \gamma(x)\sigma_{\alpha(x)}(\delta(k))$

for all $x, y \in S$ and $k \in K$. In the case when there is a right transversal H to K in G which is a normal subgroup of G, the group G is the semidirect product of K and H. In this case, the conditions on α, γ and δ agree with the conditions given in [7, Lemma 1.1, p. 1000]. These conditions are given as follows.

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$(C1)\alpha(hh') = \alpha(h)\alpha(h')^{\gamma(h)}$,
$(C2)\gamma(h^k) = \gamma(h)^{\delta(k)}$,
$(C3)\alpha(h^k) = \alpha(h)^{\delta(k)}$,
$(C4)$For any $h'k' \in G$, there exists a unique $hk \in G$ such that $\alpha(h) = h'$ and $\gamma(h)\delta(k) = k'$.

*Remark.*In [8], the author put the non-normal subgroup in the left in the factorization of G. To match the terminology with that in [5], we put the non-normal subgroup K in the right, that is $G = HK$. Through out the paper, we will use the terminology used in [5]. We will identify the internal semidirect product $G = HK$ with the external semidirect product $H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is the corresponding homomorphism.

2 Structure of the automorphism group $Aut_K(G)$

In this section, we will give the structure of the group $Aut_K(G)$.Consider a set

$$\hat{\mathscr{M}}_K = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \mid \begin{matrix} \alpha \in Map(H,H), \; \gamma \in Hom(H,K), \\ \text{and} \quad \delta \in Aut(K) \end{matrix} \right\},$$

where the maps α, γ and δ satisfy the conditions $(C1)-(C4)$. Let us define a binary operation on the set $\hat{\mathscr{M}}_K$ as,

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha' + \delta\gamma' & \delta\delta' \end{pmatrix}, \tag{1}$$

where $\alpha\alpha', \delta\delta'$ are the usual composition of maps and $(\gamma\alpha' + \delta\gamma')(h) = \gamma(\alpha'(h))\delta(\gamma'(h))$, for all $h \in H$. Then using $(C1)-(C4)$, for all $h,\, h' \in H$, we have $(\gamma\alpha' + \delta\gamma')(hh') = \gamma(\alpha'(hh'))\delta(\gamma'(hh')) = \gamma(\alpha'(h)\alpha'(h')^{\gamma'(h)})\delta(\gamma'(h)\gamma'(h')) = \gamma(\alpha'(h))\gamma(\alpha'(h')^{\gamma'(h)})\delta(\gamma'(h))\ \delta(\gamma'(h')) = \gamma\alpha'(h)\gamma(\alpha'(h'))^{\delta(\gamma'(h))}\delta(\gamma'(h))\delta(\gamma'(h')) = \gamma\alpha'(h)\delta(\gamma'(h))\gamma\alpha'(h')\delta(\gamma'(h')) = (\gamma\alpha' + \delta\gamma')(h)(\gamma\alpha' + \delta\gamma')(h')$. Thus the map $\gamma\alpha' + \delta\gamma' \in Hom(H,K)$. Since $\alpha,\, \alpha' \in Map(H,H)$ and $\delta,\, \delta' \in Aut(K)$, $\alpha\alpha' \in Map(H,H)$ and $\delta\delta' \in Aut(K)$.

Now, $\alpha\alpha'(hh') = \alpha(\alpha'(hh')) = \alpha(\alpha'(h)\alpha'(h')^{\gamma'(h)}) = \alpha(\alpha'(h))\alpha(\alpha'(h')^{\gamma'(h)})^{\gamma(\alpha'(h))} = \alpha\alpha'(h)(\alpha(\alpha'(h'))^{\delta(\gamma'(h))})^{\gamma\alpha'(h)} = \alpha\alpha'(h)\alpha\alpha'(h')^{(\gamma\alpha'+\delta\gamma')(h)}$. Also, $(\gamma\alpha' + \delta\gamma')(h^k) = \gamma(\alpha'(h^k))\delta(\gamma'(h^k)) = \gamma(\alpha'(h)^{\delta'(k)})\delta(\gamma'(h)^{\delta'(k)}) = \gamma(\alpha'(h))^{\delta(\delta'(k))}\delta(\gamma'(h))^{\delta(\delta'(k))} = (\gamma(\alpha'(h))\delta(\gamma'(h)))^{\delta(\delta'(k))} = (\gamma\alpha' + \delta\gamma')(h)^{\delta\delta'(k)}$. Clearly, $\alpha\alpha'(h^k) = \alpha(\alpha'(h)^{\delta'(k)}) = \alpha(\alpha'(h))^{\delta(\delta'(k))}$. Hence, $\begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha' + \delta\gamma' & \delta\delta' \end{pmatrix}$ satisfies $(C1)-(C4)$. The inverse of an arbitrary element $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in \hat{\mathscr{M}}_K$ is given as

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} & 0 \\ -\delta^{-1}\gamma\alpha^{-1} & \delta^{-1} \end{pmatrix}$$

and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element, where 1 denotes the identity group homomorphism and 0 denotes the trivial group homomorphism. Hence $\hat{\mathscr{M}}_K$ is a group with the binary operation as given in the Equation (1).

Proposition 1.*$Aut_K(G)$ is a subgroup of $Aut(G)$.*

*Proof.*Let $\Theta_1, \Theta_2 \in Aut_K(G)$. Then $\Theta_1(K) = K$ and $\Theta_2(K) = K$. Then $\Theta_1\Theta_2(K) = \Theta_1(\Theta_2(K)) = \Theta_1(K) = K$. Also, since $\Theta_1, \Theta_2 \in Aut(G)$, $\Theta_1\Theta_2 \in Aut(G)$. Hence, $\Theta_1\Theta_2 \in Aut_K(G)$. Further, for all $\Theta \in Aut_K(G)$, $\Theta^{-1}(K) = K$. Thus, $\Theta^{-1} \in Aut_K(G)$. Hence, $Aut_K(G)$ is a subgroup of $Aut(G)$.

Proposition 2.*Let $G = H \rtimes K$ be the semidirect product of groups H and K. Let $\hat{\mathscr{M}}_K$ be defined as above. Then the group $Aut_K(G)$ is isomorphic to the group $\hat{\mathscr{M}}_K$.*

*Proof.*Let $\Theta \in Aut_K(G)$. Then define the maps α, γ and δ by means of $\Theta(h) = \alpha(h)\gamma(h)$ and $\Theta(k) = \delta(k)$ for all $h \in H$ and $k \in K$. Now, for all $h, h' \in H$, $\alpha(hh')\gamma(hh') = \Theta(hh') = \Theta(h)\Theta(h') = \alpha(h)\gamma(h)\alpha(h')\gamma(h') = \alpha(h)\alpha(h')^{\gamma(h)}\gamma(h)\gamma(h')$. Therefore, by the uniqueness of representation, $\gamma \in Hom(H,K)$ and $(C1)$ holds. Using a similar argument, we get $\delta \in Hom(K)$. Now, $\alpha(h^k)\gamma(h^k)\delta(k) = \Theta(h^k k) = \Theta(khk^{-1}k) = \Theta(kh) = \Theta(k)\Theta(h) = \delta(k)\alpha(h)\gamma(h) = \alpha(h)^{\delta(k)}\delta(k)\gamma(h)$. Then, by the

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uniqueness of representation, $(C2)$ and $(C3)$ hold. Since Θ is a bijection, $(C4)$ holds. As a result, we can assign to every $\Theta \in Aut_K(G)$ a unique element $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in \hat{\mathscr{M}}_K$. This defines a map $\psi : Aut_K(G) \longrightarrow \hat{\mathscr{M}}_K$ given by $\Theta \mapsto \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$.

On the other hand, let $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix} \in \hat{\mathscr{M}}_K$. Then we define a map $\Theta : G \longrightarrow G$ by $\Theta(hk) = \alpha(h)\gamma(h)\delta(k)$. Now, for all $h, h' \in H$ and $k, k' \in K$, using $(C1) - (C4)$, we get

$$\begin{aligned}
\Theta(hkh'k') &= \Theta(h(h')^k kk') \\
&= \alpha(h(h')^k)\gamma(h(h')^k)\delta(kk') \\
&= \alpha(h)\alpha((h')^k)^{\gamma(h)}\gamma(h)\gamma((h')^k)\delta(k)\delta(k') \\
&= \alpha(h)\gamma(h)\alpha(h')^{\delta(k)}\gamma(h')^{\delta(k)}\delta(k)\delta(k') \\
&= \alpha(h)\gamma(h)\delta(k)\alpha(h')\gamma(h')\delta(k') \\
&= \Theta(hk)\Theta(h'k').
\end{aligned}$$

Thus Θ is a group homomorphism. Using $(C4)$, it is clear that Θ is a bijection. Thus $\Theta \in Aut(G)$. Since $\Theta(K) = \delta(K)$ and $\delta \in Aut(K)$, $\Theta(K) = K$. Hence, $\Theta \in Aut_K(G)$. This shows that the map ψ is a bijection. Now, let $\Theta'(hk) = \alpha'(h)\gamma'(h)\delta'(k)$. Then we have

$$\begin{aligned}
\Theta\Theta'(hk) &= \Theta(\Theta'(hk)) \\
&= \Theta(\alpha'(h)\gamma'(h)\delta'(k)) \\
&= \alpha(\alpha'(h))\gamma(\alpha'(h))\delta(\gamma'(h)\delta'(k)) \\
&= \alpha\alpha'(h)\gamma(\alpha'(h))\delta(\gamma'(h))\delta(\delta'(k)) \\
&= \alpha\alpha'(h)(\gamma\alpha' + \delta\gamma')(h)\delta\delta'(k).
\end{aligned}$$

Write $\begin{pmatrix} h \\ k \end{pmatrix}$ for hk, then

$$\begin{pmatrix} \alpha' & 0 \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \alpha(h) \\ \gamma(h)\delta(k) \end{pmatrix}$$

and $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \alpha(h) \\ \gamma(h)\delta(k) \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha' + \delta\gamma' & \delta\delta' \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$

for all $h \in H$ and $k \in K$. Therefore, $\psi(\Theta\Theta') = \begin{pmatrix} \alpha\alpha' & 0 \\ \gamma\alpha' + \delta\gamma' & \delta\delta' \end{pmatrix} = \psi(\Theta)\psi(\Theta')$. Hence, ψ is an isomorphism of groups.

From now on we will identify automorphisms in $Aut_K(G)$ with the matrices in $\hat{\mathscr{M}}_K$. Now, we have the following remarks.

Remark. $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in Aut_K(G)$ if and only if $\alpha \in Aut(H)$ and $\alpha(h^k) = \alpha(h)^k$ for all $h \in H$ and $k \in K$.

Remark. $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \in Aut_K(G)$ if and only if $\gamma(H) \subseteq C_K(H)$ and $\gamma(h^k) = \gamma(h)^k$, for all $h \in H$ and $k \in K$, where $C_K(H) = \{k \in K \mid h^k = h, \forall\, h \in H\}$ is the centralizer of H in K.

Remark. $\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \in Aut_K(G)$ if and only if $k^{-1}\delta(k) \in C_K(H)$ for all $k \in K$.

Now, let us consider the following subsets of $Aut(H)$, $Aut(K)$ and $Aut(H) \times Aut(K)$,

$$\begin{aligned}
U &= \{\alpha \in Aut(H) \mid \alpha(h^k) = \alpha(h)^k, \forall h \in H, k \in K\}, \\
V &= \{\delta \in Aut(K) \mid k^{-1}\delta(k) \in C_K(H), \forall k \in K\}, \\
W &= \{(\alpha, \delta) \in Aut(H) \times Aut(K) \mid \alpha(h^k) = \alpha(h)^{\delta(k)}, \forall h \in H, k \in K\}.
\end{aligned}$$

Clearly, U, V and W are the subgroups of $Aut(H)$, $Aut(K)$, and $Aut(H)\times Aut(K)$, respectively. The corresponding subgroups of the group $Aut_K(G)$ are

$$A=\left\{\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\mid \alpha\in U\right\}, D=\left\{\begin{pmatrix}1 & 0\\ 0 & \delta\end{pmatrix}\mid \delta\in V\right\} \text{ and } E=\left\{\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\mid (\alpha,\delta)\in W\right\}.$$

Note that, if $\alpha\in U$ and $\delta\in V$, then $(\alpha,\delta)\in W$. Therefore, $U\times V\leq W$.

Clearly, E is a subgroup of $Aut_K(G)$. However, one can check that E need not be a normal subgroup of $Aut_K(G)$. Let

$$C=\left\{\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}\in Aut_K(G)\mid \gamma(H)\subseteq C_K(H) \text{ and } \gamma(h^k)=\gamma(h)^k, \forall h\in H \text{ and } k\in K\right\}.$$

Then, for all $\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}\in C$ and $\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\in E$, we have

$$\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}^{-1}=\begin{pmatrix}1 & 0\\ \delta\gamma\alpha^{-1} & 1\end{pmatrix} \tag{2}$$

Now, for all $h,h'\in H$ and $k\in K$, we have $h^{\delta\gamma\alpha^{-1}(h')}=h^{\delta(\gamma\alpha^{-1}(h'))}=h^{\gamma(\alpha^{-1}(h'))}=h$. This implies that $\delta\gamma\alpha^{-1}(h')\in C_K(H)$. Also, $\delta\gamma\alpha^{-1}(h^k)=\delta\gamma(\alpha^{-1}(h^k))=\delta(\gamma(\alpha^{-1}(h)^{\delta^{-1}(k)}))=\delta(\gamma(\alpha^{-1}(h))^{\delta^{-1}(k)})=\delta(\delta^{-1}(k)\gamma\alpha^{-1}(h)\delta^{-1}(k)^{-1})=k\delta\gamma\alpha^{-1}(h)k^{-1}=\delta\gamma\alpha^{-1}(h)^k$. Thus $\begin{pmatrix}1 & 0\\ \delta\gamma\alpha^{-1} & 1\end{pmatrix}\in C$ and so, C is a normal subgroup of the group $Aut_K(G)$. Clearly, $E\cap C=\{1\}$. Now, let $\begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}\in Aut_K(G)$. Then,

$$\begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}1 & 0\\ \gamma\alpha^{-1} & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\in CE.$$

Hence, $Aut_K(G)=CE$. Thus, we have proved the following theorem,

Theorem 1. *Let $G=H\rtimes K$ be the semidirect product. Then $Aut_K(G)\simeq C\rtimes E$.*

3 Computation of $Aut_K(G)$ for some groups

In this section, we will compute the automorphism group $Aut_K(G)$ for non-abelian metacyclic p-groups and non-abelian p-groups ($p\geq 5$) of order p^4, where p is a prime. The notation $\mathbb{Z}_m$ will denote the cyclic group of order m.

Metacyclic p-groups

First, assume that p is odd. A non-abelian split metacyclic p-group G is of the form $G=\langle a,b\mid a^{p^m}=1=b^{p^n}, a^b=a^{1+p^{m-r}}\rangle$, where $m\geq 2, n\geq 1$, and $1\leq r\leq \min\{m-1,n\}$. Let $H=\langle a\rangle$, $K=\langle b\rangle$ and $\phi: K\longrightarrow Aut(H)$ be defined by $\phi(b)(a)=a^{1+p^{m-r}}$. Then $G=H\rtimes_\phi K$.

Note that $[H,K]=\langle a^{p^{m-r}}\rangle\simeq\mathbb{Z}_{p^r}$. Since K is abelian, by [2, Corollary 2.2, p. 490], $\gamma(h^k)=\gamma(h)$ is equivalent to $\gamma\in Hom(H/[H,K],K)$. Define $\gamma_i: H\to K$ by $\gamma_i(a)=b^i, 1\leq i\leq p^n$ when $m-r\geq n$ and by $\gamma_i(a)=b^{ip^{n-m+r}}, 1\leq i\leq p^{m-r}$ when $m-r<n$. Since $[H,K]\subseteq Ker\gamma_i$, it will induce a homomorphism from $H/[H,K]$ to K. Let $\hat{\gamma}_1=\begin{pmatrix}1 & 0\\ \gamma_1 & 1\end{pmatrix}$. Then, one can easily observe that $\gamma_1(H)\subseteq C_K(H)$. Therefore, $Hom(H/[H,K],K)\simeq C=\langle\hat{\gamma}_1\rangle\simeq\mathbb{Z}_{p^{\min\{m-r,n\}}}$. Also, $C_K(H)=\langle b^{p^r}\rangle\simeq\mathbb{Z}_{p^{n-r}}$ and for $b\in K$, $b^{-1}\delta(b)\in C_K(H)$. Therefore, there are p^{n-r} choices for $\delta(b)$. If $\delta_1(b)=b^{1+p^r}$, then $V=\langle\delta_1\rangle\simeq\mathbb{Z}_{p^{n-r}}$ and so, $D\simeq\mathbb{Z}_{p^{n-r}}$. Now, for all $\alpha\in Aut(H)$, $\alpha(a^b)=\alpha(a^{1+p^{m-r}})=\alpha(a)^{1+p^{m-r}}=\alpha(a)^b$. Therefore, $U=Aut(H)\simeq\mathbb{Z}_{p^{m-1}(p-1)}$ and so, $A\simeq\mathbb{Z}_{p^{m-1}(p-1)}$. Then, by Theorem [5, Theorem 2, p. 207], $E=A\times D$. Now, by Theorem 1, $Aut_K(G)\simeq\mathbb{Z}_{p^{\min\{m-r,n\}}}\rtimes(\mathbb{Z}_{p^{m-1}(p-1)}\times\mathbb{Z}_{p^{n-r}})$. Hence, $Aut_K(G)$ is a subgroup of index $p^{\min\{m,n\}}$ in the group $Aut(G)$.

Now, assume $p=2$. Then, as given in [4], the non-abelian split metacyclic 2-group is one of the following three forms,

$(i) G=\langle a,b\mid a^{2^m}=1=b^{2^n}, a^b=a^{1+2^{m-r}}\rangle, 1\leq r\leq\min\{m-2,n\}, m\geq 3, n\geq 1.$

$(ii) G = \langle a,b \mid a^{2^m} = 1 = b^{2^n}, a^b = a^{-1+2^{m-r}} \rangle, 1 \le r \le \min\{m-2,n\}, m \ge 3, n \ge 1.$
$(iii) G = \langle a,b \mid a^{2^m} = 1 = b^{2^n}, a^b = a^{-1} \rangle, m \ge 2, n \ge 1.$

Let $H = \langle a \rangle \simeq \mathbb{Z}_{2^m}$ and $K = \langle b \rangle \simeq \mathbb{Z}_{2^n}$. We will compute the automorphism group, $Aut_K(G)$ in the above three cases $(i)-(iii)$. Using the similar argument as for odd prime p above, in the *case*(i), $[H,K] = \langle a^{2^{m-r}} \rangle \simeq \mathbb{Z}_{2^r}$ and $C_K(H) = \langle b^{2^r} \rangle \simeq \mathbb{Z}_{2^{n-r}}$. Then $Hom(H/[H,K],K) \simeq \mathbb{Z}_{2^{\min\{m-r,n\}}}$. Thus $C \simeq \mathbb{Z}_{2^{\min\{m-r,n\}}}$, $A \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ and $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}}$. Hence, $Aut_K(G) \simeq \mathbb{Z}_{2^{\min\{m-r,n\}}} \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}})$.

In the case (ii), $[H,K] = \langle a^2 \rangle \simeq \mathbb{Z}_{2^{m-1}}$ and $C_K(H) = \langle b^{2^r} \rangle \simeq \mathbb{Z}_{2^{n-r}}$. Thus, $C \simeq \mathbb{Z}_2, A \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ and $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}}$. Hence, $Aut_K(G) \simeq \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}})$. Similarly, in the *case*(iii), $[H,K] = \langle a^2 \rangle \simeq \mathbb{Z}_{2^{m-r}}$, and $C_K(H) = \langle b^2 \rangle \simeq \mathbb{Z}_{2^{n-1}}$. Thus, $C \simeq \mathbb{Z}_2$, $A \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}}$ and $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-r-1}}$. Hence, $Aut_K(G) \simeq \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})$.

Non-abelian p-groups of order p^4 $(p \ge 5)$

Burnside in [3] classified p-groups of order p^4, where p is a prime. Below, we list 10 non-abelian p-groups $(p \ge 5)$ of order p^4 up to isomorphism.

(i)$G_1 = \langle a,b \mid a^{p^3} = 1 = b^p, a^b = a^{1+p^2} \rangle$,
(ii)$G_2 = \langle a,b \mid a^{p^2} = 1 = b^{p^2}, a^b = a^{1+p} \rangle$,
(iii)$G_3 = \langle a,b,c \mid a^{p^2} = 1 = b^p = c^p, cb = a^p bc, ab = ba, ac = ca \rangle$,
(iv)$G_4 = \langle a,b,c \mid a^{p^2} = 1 = b^p = c^p, ca = a^{1+p} c, ab = ba, cb = bc \rangle$,
(v)$G_5 = \langle a,b,c \mid a^{p^2} = 1 = b^p = c^p, ca = abc, ab = ba, bc = cb \rangle$,
(vi)$G_6 = \langle a,b,c \mid a^{p^2} = 1 = b^p = c^p, ba = a^{1+p} b, ca = abc, bc = cb \rangle$,
(vii)$G_7 = \langle a,b,c \mid a^{p^2} = 1 = b^p = c^p, ba = a^{1+p} b, ca = a^{1+p} bc, cb = a^p bc \rangle$,
(viii)$G_8 = \langle a,b,c \mid a^{p^2} = 1 = b^p = c^p, ba = a^{1+p} b, ca = a^{1+dp} bc, cb = a^{dp} bc, d \not\equiv 0,1 \pmod p \rangle$,
(ix)$G_9 = \langle a,b,c,d \mid a^p = b^p = c^p = d^p = 1, dc = acd, bd = db, ad = da, bc = cb, ac = ca, ab = ba \rangle$,
(x)$G_{10} = \langle a,b,c,d \mid a^p = b^p = c^p = d^p = 1, dc = bcd, db = abd, ad = da, bc = cb, ac = ca, ab = ba \rangle$.

Observe that G_1 and G_2 are metacyclic p-groups. $Aut_K(G_1)$ and $Aut_K(G_2)$ (for the corresponding K) can be calculated as in the previous case.

The group G_3. Let $H = \langle a,b \mid a^{p^2} = b^p = 1, ab = ba \rangle$ and $K = \langle c \mid c^p = 1 \rangle$. Then $G_3 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(c)(a) = a$ and $\phi(c)(b) = a^p b$. Note that $[a^u b^v, c] = (a^u b^v) c (a^u b^v)^{-1} c^{-1} = a^u b^v (a^{u+pv} b^v)^{-1} = a^{-pv}$. Therefore, $[H,K] = \langle a^p \rangle \simeq \mathbb{Z}_p$. Also, if $c^s \in C_K(H)$, then $a^i b^j = c^s a^i b^j c^{-s} = a^{i+pjs} b^j$. Therefore, $js \equiv 0 \pmod p$ for all j and hence, $C_K(H) = \{1\}$. This implies that $Hom(H/[H,K],K)$ is the trivial group. Since K is abelian, by [2, Corollary 2.2, p. 490] C is the trivial group. Note that, each $\alpha \in Aut(H)$ defined by $\alpha(a) = a^i b^j$ and $\alpha(b) = a^{pm} b^l$ can be expressed as a matrix $\begin{pmatrix} i & j \\ m & l \end{pmatrix}$, where $0 \le i \le p^2 - 1, \gcd(p,i) = 1, 0 \le m, j \le p-1$ and $1 \le l \le p-1$. Also, let $\delta \in Aut(K) \simeq \mathbb{Z}_{p-1}$ be defined by $\delta(c) = c^r$, where $1 \le r \le p-1$. Now, if $(\alpha,\delta) \in W$, then (i) $\alpha(a^c) = \alpha(a)^{\delta(c)}$ and (ii) $\alpha(b^c) = \alpha(b)^{\delta(c)}$. By (i), $a^i b^j = \alpha(a) = \alpha(a^c) = \alpha(a)^{\delta(c)} = (a^i b^j)^{c^r} = a^i a^{prj} b^j = a^{i+prj} b^j$. Thus, $prj \equiv 0 \pmod{p^2}$ which implies that $j = 0$. Now, by (ii), $a^{pi+pm} b^l = \alpha(a^p b) = \alpha(b^c) = \alpha(b)^{\delta(c)} = (a^{pm} b^l)^{c^r} = a^{pm} (b^l)^{c^r} = a^{pm} a^{prl} b^l = a^{pm+prl} b^l$. Thus, $i \equiv rl \pmod p$. Let t be a primitive root of $1 \pmod p$ and $x = \left(\begin{pmatrix} t+p & 0 \\ 0 & t \end{pmatrix}, \delta_1 \right)$, $y = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \delta_1 \right)$ and $z = \left(\begin{pmatrix} t+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_t \right)$, where $\delta_\rho(c) = c^\rho$. Then $W \simeq \langle x,y,z \mid x^{p(p-1)} = 1 = y^p = z^{p(p-1)}, xz = zx, xy = yx, zyz^{-1} = y^{t^{-1}} \rangle$. Therefore, $W \simeq (\mathbb{Z}_p \times \mathbb{Z}_{p(p-1)}) \rtimes \mathbb{Z}_{p(p-1)}$ and so, $E \simeq (\mathbb{Z}_p \times \mathbb{Z}_{p(p-1)}) \rtimes \mathbb{Z}_{p(p-1)}$. Hence, by Theorem 1, $Aut_K(G_3) \simeq (\mathbb{Z}_p \times \mathbb{Z}_{p(p-1)}) \rtimes \mathbb{Z}_{p(p-1)}$.

The group G_4. Let $H = \langle a,b \mid a^{p^2} = b^p = 1, ab = ba \rangle$ and $K = \langle c \mid c^p = 1 \rangle$. Then $G_4 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(c)(a) = a^{1+p}$ and $\phi(c)(b) = b$. Note that $[H,K] = \langle a^p \rangle \simeq \mathbb{Z}_p$. By the similar argument as in the case G_3 above, $C_K(H) = \{1\}$. Since K is abelian, by [2, Corollary 2.2, p. 490] C is the trivial group. Note that, any $\alpha \in Aut(H)$ defined by, $\alpha(a) = a^i b^j$ and $\alpha(b) = a^{pm} b^l$ can be expressed as a matrix $\begin{pmatrix} i & j \\ m & l \end{pmatrix}$, where $0 \le i \le p^2 - 1, \gcd(p,i) = 1, 0 \le m, j \le p-1$ and $1 \le l \le p-1$. Also, let $\delta \in Aut(K) \simeq \mathbb{Z}_{p-1}$ be defined by $\delta(c) = c^r$, where $1 \le r \le p-1$. Now, if $(\alpha,\delta) \in W$, then (i) $\alpha(a^c) = \alpha(a)^{\delta(c)}$ and (ii) $\alpha(b^c) = \alpha(b)^{\delta(c)}$. Note that $\alpha(b^c) = \alpha(b) = a^{pm} b^l$ and $\alpha(b)^{\delta(c)} = (a^{pm} b^l)^{c^r} = (a^{pm})^{c^r} b^l = a^{pm(1+p)^r} b^l = a^{pm} b^l$. Therefore, each $\alpha \in Aut(H)$ satisfies (ii). Now, by (i), $(a^i b^j)^{1+p} = \alpha(a^{1+p}) = \alpha(a^c) = \alpha(a)^{\delta(c)} = (a^i b^j)^{c^r} = (a^i)^{c^r} b^j = a^{i(1+p)^r} b^j$. Thus, $i(p+1) \equiv i(1+p)^r \pmod{p^2}$ which implies that $r = 1$. Therefore, $W \simeq Aut(H) \simeq$

$\mathbb{Z}_{p-1} \times (((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$. Hence, $E \simeq \mathbb{Z}_{p-1} \times (((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$. Thus, $Aut_K(G_4) \simeq \mathbb{Z}_{p-1} \times (((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$.

The group G_5. Let $H = \langle b, c \mid b^p = c^p = 1, bc = cb \rangle$ and $K = \langle a \mid a^{p^2} = 1 \rangle$. Then $G_5 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(a)(b) = b$ and $\phi(a)(c) = b^{-1}c$. Note that $[H,K] = \langle b \rangle \simeq \mathbb{Z}_p$. Also, if $a^s \in C_K(H)$, then $b^i c^j = a^s b^i c^j a^{-s} = b^{i-js}c^j$. Therefore, $s \equiv 0 \pmod p$ and $C_K(H) = \langle a^p \rangle$. Since K is abelian, by [2, Corollary 2.2, p. 490], $\gamma(h^k) = \gamma(h)$ is equivalent to $\gamma \in Hom(H/[H,K],K)$. Define $\gamma_k \in Hom(H/[H,K],K)$ by $\gamma_k(b) = 1$ and $\gamma_k(c) = a^{pk}$ for all $0 \le k \le p-1$. Since $[H,K] \subseteq Ker\gamma_k$, it will induce a homomorphism from $H/[H,K]$ to K. Let $\hat{\gamma}_1 = \begin{pmatrix} 1 & 0 \\ \gamma_1 & 1 \end{pmatrix}$. Then, one can easily observe that $\gamma_1(H) \subseteq C_K(H)$. Therefore, $Hom(H/[H,K],K) \simeq C = \langle \hat{\gamma}_1 \rangle \simeq \mathbb{Z}_p$. Note that, any $\alpha \in Aut(H) \simeq GL(2,p)$ defined as, $\alpha(b) = b^i c^j$ and $\alpha(c) = b^l c^m$ can be represented as a matrix, $\begin{pmatrix} i & j \\ l & m \end{pmatrix}$, where $0 \le l, j \le p-1$ and $1 \le i, m \le p-1$. Also, let $\delta \in Aut(K) \simeq \mathbb{Z}_{p(p-1)}$ be defined by $\delta(a) = a^r$, where $r \in \mathbb{Z}_{p^2}, \gcd(p,r) = 1$. Now, if $(\alpha, \delta) \in W$, then (i) $\alpha(b^a) = \alpha(b)^{\delta(a)}$ and (ii) $\alpha(c^a) = \alpha(c)^{\delta(a)}$. By (i), $b^i c^j = \alpha(b) = \alpha(b^a) = \alpha(b)^{\delta(a)} = (b^i c^j)^{a^r} = b^i b^{-rj} c^j = b^{i-rj}c^j$. Thus, $rj \equiv 0 \pmod p$ which implies that $j = 0$. Now, by (ii), $b^{-i+l}c^m = \alpha(b^{-1}c) = \alpha(c^a) = \alpha(c)^{\delta(a)} = (b^l c^m)^{a^r} = b^l (c^m)^{a^r} = b^l b^{-rm} c^m = b^{l-rm}c^m$. Thus, $i \equiv rm \pmod p$. Let t be a primitive root of $1 \pmod p$ and $x = \left(\begin{pmatrix} t & 0 \\ 1 & t \end{pmatrix}, \delta_1 \right)$, and $y = \left(\begin{pmatrix} t+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_t \right)$, where $\delta_\rho(a) = a^\rho$. Then $W \simeq \langle x, y \mid x^{p(p-1)} = 1, y^{p(p-1)} = 1, yxy^{-1} = x^\lambda \rangle$, where $x^\lambda = \begin{pmatrix} t & 0 \\ (t+p)^{-1} & t \end{pmatrix}$. Then $W \simeq \mathbb{Z}_{p(p-1)} \rtimes \mathbb{Z}_{p(p-1)}$ and so, $E \simeq \mathbb{Z}_{p(p-1)} \rtimes \mathbb{Z}_{p(p-1)}$. Hence, $Aut_K(G_5) \simeq \mathbb{Z}_p \rtimes (\mathbb{Z}_{p(p-1)} \rtimes \mathbb{Z}_{p(p-1)})$.

The group G_6. Let $H = \langle a, b \mid a^{p^2} = b^p = 1, ba = a^{1+p}b \rangle$ and $K = \langle c \mid c^p = 1 \rangle$. Then $G_6 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(c)(a) = ab$ and $\phi(c)(b) = b$. Note that $[H,K] = \langle b^{-1}, a^p \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. By the similar argument as in the case G_3 above, $C_K(H) = \{1\}$ and hence C is the trivial group. Now, $\alpha \in Aut(H)$ as given in [2] can be expressed as a matrix $\begin{pmatrix} \eta & \beta \\ \xi & 1 \end{pmatrix}$, where $\eta(a) = a^i, 0 \le i \le p^2-1, \gcd(p,i) = 1, \beta(b) = a^{pj}, 0 \le j \le p-1, \xi(a) = b^k, 0 \le k \le p-1$, and $1(b) = b$. Also, $\delta \in Aut(K)$ is given by $\delta(c) = c^r, 1 \le r \le p-1$. Now, if $(\alpha, \delta) \in W$, then (i) $\alpha(a^c) = \alpha(a)^{\delta(c)}$ and (ii) $\alpha(b^c) = \alpha(b)^{\delta(c)}$. Note that $\alpha(b^c) = \alpha(b) = a^{pj}b$ and $\alpha(b)^{\delta(c)} = (a^{pj}b)^{c^r} = (c^r a c^{-r})^{pj} b = (ab^r)^{pj} b = a^{pj + rp\frac{pj(pj-1)}{2}} b^{pjr+1} = a^{pj}b$. Therefore, each $\alpha \in Aut(H)$ satisfies (ii). Now, by (i), $a^{i+pj}b^{k+1} = \alpha(ab) = \alpha(a^c) = \alpha(a)^{\delta(c)} = (a^i b^k)^{c^r} = (c^r a c^{-r})^i b^k = (ab^r)^i b^k = a^{i+rp\frac{i(i-1)}{2}} b^{ri+k}$. Thus, $ri \equiv 1 \pmod p$ which gives that $i \equiv 2j+1 \pmod p$. Therefore, $i \in \{(2j+1) + \lambda p \mid \lambda \in \mathbb{Z}_p\}$. Let t be a primitive root of $1 \pmod p$ and $x = \left(\begin{pmatrix} t+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_t \right)$, $y = \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \delta_1 \right)$, and $z = \left(\begin{pmatrix} 1+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_1 \right)$, where $\delta_\rho(c) = c^\rho$. Then $W \simeq \langle x, y, z \mid x^{p(p-1)} = 1 = y^p = z^p, xyx^{-1} = y^e, xz = zx, yz = zy \rangle$, where $y^e = \begin{pmatrix} 1 & 0 \\ (t+p)^{-1} & 1 \end{pmatrix}$. Hence, $W \simeq \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$ and so, $E \simeq \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$. Thus, $Aut_K(G_6) \simeq \mathbb{Z}_p \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$.

The group G_7. Let $H = \langle a, b \mid a^{p^2} = b^p = 1, ba = a^{1+p}b \rangle$ and $K = \langle c \mid c^p = 1 \rangle$. Then, $G_7 \simeq H \rtimes_\phi K$, where $\phi : K \longrightarrow Aut(H)$ is given by $\phi(c)(a) = a^{1+p}b$ and $\phi(c)(b) = a^p b$. Note that $[H,K] = \langle b, a^p \rangle \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. By the similar argument as in the case G_3 above, $C_K(H) = \{1\}$ and hence C is the trivial group. Each $\alpha \in Aut(H)$ can be expressed as a matrix $\begin{pmatrix} \eta & \beta \\ \xi & 1 \end{pmatrix}$, where $\eta(a) = a^i, 0 \le i \le p^2-1, \gcd(i,p) = 1$, $\beta(b) = a^{pj}, \xi(a) = b^k, 0 \le j, k \le p-1$, and $1(b) = b$. Also, $\delta \in Aut(K)$ is given by $\delta(c) = c^r, 1 \le r \le p-1$. Now, if $(\alpha, \delta) \in W$, then (i) $\alpha(a^c) = \alpha(a)^{\delta(c)}$ and $(ii) \alpha(b^c) = \alpha(b)^{\delta(c)}$.

By (ii), $a^{pi+pj}b = \alpha(a^p b) = \alpha(b^c) = \alpha(b)^{\delta(c)} = (a^{pj}b)^{c^r} = (c^r a c^{-r})^{pj}(c^r b c^{-r}) = (a^{1+p\frac{r(r+1)}{2}} b^r)^{pj}(a^{rp}b) = a^{p^2 j \frac{r(r+1)}{2}}(ab^r)^{pj} a^{rp} b = a^{pj+pr}b$. Thus $i \equiv r \pmod p$. Now, by (i), $a^{i(1+p)+pj}b^{k+1} = \alpha(a^{1+p}b) = \alpha(a^c) = \alpha(a)^{\delta(c)} = (a^i b^k)^{c^r} = (c^r a c^{-r})^i (c^r b c^{-r})^k = a^{i+pri\frac{r+i}{2}} b^{ri} (a^{rp}b)^k = a^{i+rpi(\frac{r+i}{2})+rpk} b^{ri+k}$. Thus, $ri \equiv 1 \pmod p$ and $ip + pj \equiv pri(\frac{r+i}{2}) + rpk \pmod{p^2}$ implies that $i + j \equiv r + rk \pmod p$. So, $j \equiv rk \pmod p$. Using $r \equiv i \pmod p$ and $ri \equiv 1 \pmod p$, we get $i^2 \equiv 1 \pmod{p^2}$. Let t be a primitive root of $1 \pmod p$ and $x = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \delta_1 \right)$, $y = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \delta_1 \right)$ and $z = \left(\begin{pmatrix} t+p & 0 \\ 0 & 1 \end{pmatrix}, \delta_t \right)$, where $\delta_\rho(c) = c^\rho$. Then $W \simeq \langle x, y, z \mid x^p = 1 = y^2 = z^{p(p-1)}, xy = yx^{-1}, zxz^{-1} = x^t, yz = zy \rangle \simeq \mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_p \rtimes \mathbb{Z}_2)$ and so, $E \simeq \mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_p \rtimes \mathbb{Z}_2)$. Hence, $Aut_K(G_7) \simeq \mathbb{Z}_{p(p-1)} \times (\mathbb{Z}_p \rtimes \mathbb{Z}_2) \simeq D_{2p} \times \mathbb{Z}_{p(p-1)}$, where D_{2p} is the dihedral group of order $2p$.

The group G_8. Let $H=\langle a,b \mid a^{p^2}=b^p=1, ba=a^{1+p}b\rangle$ and $K=\langle c \mid c^p=1\rangle$. Then $G_8\simeq H\rtimes_\phi K$, where $\phi: K\longrightarrow Aut(H)$ is given by $\phi(c)(a)=a^{1+dp}b$ and $\phi(c)(b)=a^{dp}b$, $d\not\equiv 0,1 \pmod p$. By the similar argument as for the group G_7, we get, C is the trivial group and $E\simeq \mathbb{Z}_{p(p-1)}\times(\mathbb{Z}_p\rtimes\mathbb{Z}_2)$. Hence, $Aut_K(G_8)\simeq D_{2p}\times\mathbb{Z}_{p(p-1)}$.

The group G_9. Let $H=\langle a,b,c \mid a^p=b^p=c^p=1, ab=ba, bc=cb, ac=ca\rangle$, and $K=\langle d \mid d^p=1\rangle$. Then $G_9\simeq H\rtimes_\phi K$, where $\phi: K\longrightarrow Aut(H)$ is given by $\phi(d)(a)=a, \phi(d)(b)=b$, and $\phi(d)(c)=ac$.

Note that $[H,K]=\langle a\rangle\simeq\mathbb{Z}_p$. By the similar argument as in the case G_3 above, $C_K(H)=\{1\}$ and hence C is the trivial group. Note that, $Aut(H)\simeq GL(3,p)$ and $Aut(K)\simeq\mathbb{Z}_{p-1}$. So, any automorphism $\alpha\in Aut(H)$ can be identified as an element $\begin{pmatrix} i & j & k\\ l & m & n\\ \lambda & \mu & \rho\end{pmatrix}$ in $GL(3,p)$. Let $\alpha\in Aut(H)$ and $\delta\in Aut(K)$ be defined as, $\alpha(a)=a^ib^jc^k, \alpha(b)=a^lb^mc^n, \alpha(c)=a^\lambda b^\mu c^\rho$, and $\delta(d)=d^r$, where $1\le i,m,\rho,r\le p-1$ and $0\le j,k,l,n,\lambda,\mu\le p-1$. Now, if $(\alpha,\delta)\in W$, then (i) $\alpha(a^d)=\alpha(a)^{\delta(d)}$, (ii) $\alpha(b^d)=\alpha(b)^{\delta(d)}$ and (iii) $\alpha(c^d)=\alpha(c)^{\delta(d)}$.

Note that, $d^rcd^{-r}=a^rc$. By (i), $a^ib^jc^k=\alpha(a)=\alpha(a^d)=\alpha(a)^{\delta(d)}=(a^ib^jc^k)^{d^r}=a^ib^j(d^rcd^{-r})^k=a^ib^j(a^rc)^k=a^{i+rk}b^jc^k$. Therefore, $rk\equiv 0(\mod p)$ which implies that $k=0$. Now, by (ii), $a^lb^mc^n=\alpha(b)=\alpha(b^d)=\alpha(b)^{\delta(d)}=(a^lb^mc^n)^{d^r}=a^{l+rn}b^mc^n$. Therefore, $rn\equiv 0(\mod p)$ which implies that $n=0$. By (iii), $a^{i+\lambda}b^{j+\mu}c^\rho=\alpha(ac)=\alpha(c^d)=\alpha(c)^{\delta(d)}=(a^\lambda b^\mu c^\rho)^{d^r}=a^{\lambda+r\rho}b^\mu c^\rho$. Thus, $i=r\rho$ and $j=0$. So, we have, $\alpha=\begin{pmatrix} r\rho & 0 & 0\\ l & m & 0\\ \lambda & \mu & \rho\end{pmatrix}$. Let t be a primitive root of 1 (mod p) and $u=\left(\begin{pmatrix} t&0&0\\0&1&0\\0&0&t\end{pmatrix},\delta_1\right)$, $v=\left(\begin{pmatrix} 1&0&0\\0&t&0\\0&0&1\end{pmatrix},\delta_1\right)$, $w=\left(\begin{pmatrix} t&0&0\\0&1&0\\0&0&1\end{pmatrix},\delta_t\right)$, $x=\left(\begin{pmatrix} 1&0&0\\1&1&0\\0&0&1\end{pmatrix},\delta_1\right)$, $y=\left(\begin{pmatrix} 1&0&0\\0&1&0\\1&0&1\end{pmatrix},\delta_1\right)$, $z=\left(\begin{pmatrix} 1&0&0\\0&1&0\\0&1&1\end{pmatrix},\delta_1\right)$, where $\delta_s(d)=d^s$. Then $W\simeq\langle u,v,w,x,y,z \mid u^{p-1}=1=v^{p-1}=w^{p-1}=x^p=y^p=z^p, uv=vu, uw=wu, uy=yu, vw=wv, vy=yv, wz=zw, xy=yx, yz=zy, uxu^{-1}=x^{t^{-1}}, uzu^{-1}=z^t, vxv^{-1}=x^t, vzv^{-1}=z^{t^{-1}}, wxw^{-1}=x^{t^{-1}}, wyw^{-1}=y^{t^{-1}}, zx=xyz\rangle\simeq(((\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_p)\times\mathbb{Z}_{p-1})\rtimes(\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1})$ and so, $E\simeq(((\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_p)\times\mathbb{Z}_{p-1})\rtimes(\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1})$. Hence, $Aut_K(G_9)\simeq(((\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_p)\times\mathbb{Z}_{p-1})\rtimes(\mathbb{Z}_{p-1}\times\mathbb{Z}_{p-1})$.

The group G_{10}. Let $H=\langle a,b,c \mid a^p=b^p=c^p=1, ab=ba, bc=cb, ac=ca\rangle$ and $K=\langle d \mid d^p=1\rangle$. Then, $G_{10}\simeq H\rtimes_\phi K$, where $\phi: K\longrightarrow Aut(H)$ is given by $\phi(d)(a)=a, \phi(d)(b)=ab$, and $\phi(d)(c)=bc$.

Note that $[H,K]=\langle a,b\rangle\simeq\mathbb{Z}_p\times\mathbb{Z}_p$. By the similar argument as above, C is the trivial group. Note that, $Aut(H)\simeq GL(3,p)$ and $Aut(K)\simeq\mathbb{Z}_{p-1}$. So, any automorphism $\alpha\in Aut(H)$ can be identified as an element $\begin{pmatrix} i & j & k\\ l & m & n\\ \lambda & \mu & \rho\end{pmatrix}$ in $GL(3,p)$. Let $\alpha\in Aut(H)$ and $\delta\in Aut(K)$ be defined as, $\alpha(a)=a^ib^jc^k, \alpha(b)=a^lb^mc^n, \alpha(c)=a^\lambda b^\mu c^\rho$, and $\delta(d)=d^r$, where $1\le i,m,\rho,r\le p-1$ and $0\le j,k,l,n,\lambda,\mu\le p-1$. Now, if $(\alpha,\delta)\in W$, then (i) $\alpha(a^d)=\alpha(a)^{\delta(d)}$, (ii) $\alpha(b^d)=\alpha(b)^{\delta(d)}$ and (iii) $\alpha(c^d)=\alpha(c)^{\delta(d)}$.

Note that, $d^rbd^{-r}=a^rb$ and $d^rcd^{-r}=a^{\frac{r(r-1)}{2}}b^rc$. By (i), $a^ib^jc^k=\alpha(a)=\alpha(a^d)=\alpha(a)^{\delta(d)}=(a^ib^jc^k)^{d^r}=a^i(d^rbd^{-r})^j(d^rcd^{-r})^k=a^i(a^rb)^j(a^{\frac{r(r-1)}{2}}b^rc)^k=a^{i+rj+k\frac{r(r-1)}{2}}b^{j+rk}c^k$. Thus $k=0$ and $j=0$. Now, by (ii), $a^{i+l}b^mc^n=\alpha(ab)=\alpha(b^d)=\alpha(b)^{\delta(d)}=(a^lb^mc^n)^{d^r}=a^{l+rm+n\frac{r(r-1)}{2}}b^{m+rn}c^n$. Thus, $n=0$ and $i=rm$. By (iii), $a^{l+\lambda}b^{m+\mu}c^\rho=\alpha(bc)=\alpha(c^d)=\alpha(c)^{\delta(d)}=(a^\lambda b^\mu c^\rho)^{d^r}=a^{\lambda+r\mu+\rho\frac{r(r-1)}{2}}b^{\mu+r\rho}c^\rho$. Thus, $m=r\rho$ and $l=r\mu+\rho\frac{r(r-1)}{2}$ (mod p). So, we have, $\alpha=\begin{pmatrix} r^2\rho & 0 & 0\\ l & r\rho & 0\\ \lambda & \mu & \rho\end{pmatrix}$, where $l=r\mu+\rho\frac{r(r-1)}{2}$ (mod p). Let t be a primitive root of 1 (mod p) and $x=\left(\begin{pmatrix} t&0&0\\0&t&0\\0&0&t\end{pmatrix},\delta_1\right)$, $y=\left(\begin{pmatrix} t^2&0&0\\0&t&0\\0&0&1\end{pmatrix},\delta_t\right)$ and $z=\left(\begin{pmatrix} 1&0&0\\1&1&0\\1&1&1\end{pmatrix},\delta_1\right)$, where $\delta_s(d)=d^s$. Note that,

$\langle z\rangle = \langle \begin{pmatrix} 1&0&0\\ 1&1&0\\ 0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\ 0&1&0\\ 1&0&1 \end{pmatrix} \rangle$ is an abelian group of order p^2. Therefore, $W \simeq \langle x,y,z \mid x^{p-1} = y^{p-1} = z^p, xy = yx, xz = zx, yzy^{-1} = z^u\rangle$, where $z^u = \begin{pmatrix} 1&0&0\\ t^{-1}&1&0\\ t^{-2}&t^{-1}&1 \end{pmatrix}$. Thus $W \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1})$ and so, $E \simeq (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1})$. Hence, $Aut_K(G_{10}) \simeq \mathbb{Z}_{p-1} \times ((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_{p-1})$.

Declarations

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On the Incomplete $(p,q)-$Fibonacci and $(p,q)-$ Lucas Numbers

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Abstract: In this present work, the incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers are defined. We examine their recurrence relations as well as some of their properties. We derive their generating functions.

Keywords: Fibonacci numbers; Lucas numbers; generating function.

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1 Introduction

The Fibonacci sequence and its generalizations offer a variety of intriguing features and uses in science and art (see, e.g., [9,10]). The Fibonacci and Lucas numbers $\{f_h\}$ and $\{l_h\}$ are expressed as the recurrence relations, respectively, for $h \geq 0$

$$f_{h+2} = f_{h+1} + f_h \text{ with initial conditions } \quad f_0 = 0 \quad \text{and} \quad f_1 = 1,$$
$$l_{h+2} = l_{h+1} + l_h \text{ with initial conditions } \quad l_0 = 2 \quad \text{and} \quad l_1 = 1$$

Filipponi [5] introduced the incomplete Fibonacci and Lucas numbers. The incomplete Fibonacci numbers $F_h(u)$ and Lucas numbers $L_h(v)$ are expressed, respectively, by

$$F_h(u) = \sum_{i=0}^{u} \binom{h-1-i}{i}, \quad \left(\lfloor \frac{h-1}{2} \rfloor \leq u \leq h-1\right)$$

and

$$L_h(v) = \sum_{i=0}^{v} \frac{h}{h-i} \binom{h-i}{i}, \quad \left(\lfloor \frac{h}{2} \rfloor \leq v \leq h-1\right)$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. It is obvious that

$$F_h(\lfloor \frac{h-1}{2} \rfloor) = f_h \quad \text{and} \quad L_h(\lfloor \frac{h}{2} \rfloor) = l_h$$

where the $h-$th Fibonacci and Lucas numbers are denoted by f_h and l_h, respectively.
The generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers were examined by Djordjevic [3]. The incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers were defined and studied by Djordjevic and Srivastava [4]. The generating functions of the incomplete Fibonacci and Lucas numbers were discovered by Pintr and Srivastava [18]. Ramrez [14] presented the bi-periodic incomplete Fibonacci sequences, the incomplete $k-$Fibonacci and $k-$Lucas numbers [15]. The incomplete Tribonacci numbers and polynomials were introduced by

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Ramirez and Sirvent [16]. The incomplete Fibonacci and Lucas $p-$numbers were defined by Tasci and Firengiz [26]. The incomplete bivariate Fibonacci and Lucas $p-$polynomials were defined by Tasci et al. [27]. We refer to other studies on incompletes of some impressive numbers and polynomials [2,11,12,13,17,20,25].
In [6,22,23,28], the $(p,q)-$Fibonacci and $(p,q)-$ Lucas sequences are defined, respectively, by

$$F_h^{(p,q)} = pF_{h-1}^{(p,q)} + qF_{h-2}^{(p,q)}, \quad F_0^{(p,q)} = 0, \quad F_1^{(p,q)} = 1, \tag{1}$$

and

$$L_h^{(p,q)} = pL_{h-1}^{(p,q)} + qL_{h-2}^{(p,q)}, \quad L_0^{(p,q)} = 2, \quad L_1^{(p,q)} = p.$$

where p and q are real coefficients.
In [8,19,24], the $(p,q)-$Fibonacci and $(p,q)-$ Lucas sequences are also given by the well-known formulas

$$F_h^{(p,q)} = \sum_{j=0}^{\lfloor \frac{h-1}{2} \rfloor} \binom{h-j-1}{j} p^{h-2j-1} q^j, \quad h \geq 1$$

and

$$L_h^{(p,q)} = \sum_{j=0}^{\lfloor \frac{h}{2} \rfloor} \frac{h}{h-j} \binom{h-j}{j} p^{h-2j} q^j, \quad h \geq 1.$$

Note that $F_h^{(p,q)}$ and $L_h^{(p,q)}$ reduce to the Fibonacci and Lucas sequences F_h and L_h, respectively, when $p = q = 1$; see, respectively, sequences A000045 and A000032 in [29].
According to Filipponi, the specific use of well-known combinatorial expressions for Fibonacci and Lucas numbers yields two interesting classes of integers (specifically, $F_n(k)$ and $L_n(k)$) ruled by the integral parameters n and k [5]. In this paper, we examine how the specific application of combinatorial phrases for $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers yields to two interesting classes of integers governed by the integral parameters n and k. Moreover, we derive some identities and the generating functions of the incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers.

2 The Incomplete $(p,q)-$Fibonacci Numbers

Definition 1. *The incomplete $(p,q)-$Fibonacci numbers $F_{(h,k)}^{(p,q)}$ are defined as*

$$F_{(h,k)}^{(p,q)} = \sum_{j=0}^{k} \binom{h-j-1}{j} p^{h-2j-1} q^j, \quad \left(1 \leq h; 0 \leq k \leq \lfloor \frac{h-1}{2} \rfloor = \hat{h}\right). \tag{2}$$

The numbers $F_{(h,k)}^{(p,q)}$ are displayed in Table 1. It shows the first few h values and the corresponding permissible k values:

Table 1: The first few values of the incomplete $(p,q)-$Fibonacci Numbers

$h \backslash k$	0	1	2	3
1	1			
2	p			
3	p^2	p^2+q		
4	p^3	p^3+2pq		
5	p^4	p^4+3p^2q	$p^4+3p^2q+q^2$	
6	p^5	p^5+4p^3q	$p^5+4p^3q+3pq^2$	
7	p^6	p^6+5p^4q	$p^6+5p^4q+6p^2q^2$	$p^6+5p^4q+6p^2q^2+q^3$
8	p^7	p^7+6p^5q	$p^7+6p^5q+10p^3q^2$	$p^7+6p^5q+10p^3q^2+4pq^3$
9	p^8	p^8+7p^6q	$p^8+7p^6q+15p^4q^2$	$p^8+7p^6q+15p^4q^2+10p^2q^3$
10	p^9	p^9+8p^7q	$p^9+8p^7q+21p^5q^2$	$p^9+8p^7q+21p^5q^2+20p^3q^3$

The relation (2) has some special cases as follows:

1. $F_{(h,0)}^{(p,q)} = p^{h-1}, \quad (h \geq 1)$
2. $F_{(h,1)}^{(p,q)} = p^{h-1} + (h-2)p^{h-3}q, \quad (h \geq 3)$
3. $F_{(h,2)}^{(p,q)} = p^{h-1} + (h-2)p^{h-3}q + \frac{(h-4)(h-3)}{2}p^{h-5}q^2, \quad (h \geq 5)$
4. $F_{(h,\hat{h})}^{(p,q)} = F_h^{(p,q)}, \quad (h \geq 1)$
5. $F_{(h,\hat{h}-1)}^{(p,q)} = \begin{cases} F_h^{(p,q)} - \frac{h}{2}pq^{(\frac{h}{2}-1)} & (h \quad even) \\ F_h^{(p,q)} - q^{(\frac{h-1}{2})} & (h \quad odd) \end{cases}, \quad (h \geq 3)$

2.1 Some identities of the numbers $F_{(n,k)}^{(p,q)}$

Proposition 1. *The incomplete $(p,q)-$Fibonacci numbers $F_{(h,k)}^{(p,q)}$ can be given by the recurrence relation*

$$F_{(h+2,k+1)}^{(p,q)} = pF_{(h+1,k+1)}^{(p,q)} + qF_{(h,k)}^{(p,q)}, \quad 0 \leq k \leq \hat{h}. \tag{3}$$

Proof. Using Definition (3), we obtain the desired equality as follows:

$$\begin{aligned} pF_{(h+1,k+1)}^{(p,q)} + qF_{(h,k)}^{(p,q)} &= \sum_{j=0}^{k+1}\binom{h-j}{j}p^{h-2j+1}q^j + \sum_{j=0}^{k}\binom{h-j-1}{j}p^{h-2j-1}q^{j+1} \\ &= \sum_{j=0}^{k+1}\binom{h-j}{j}p^{h-2j+1}q^j + \sum_{j=1}^{k+1}\binom{h-j}{j-1}p^{h-2j+1}q^j \\ &= \sum_{j=0}^{k+1}\left[\binom{h-j}{j} + \binom{h-j}{j-1}\right]p^{h-2j+1}q^j - \binom{h}{-1} \\ &= \sum_{j=0}^{k+1}\binom{h-j+1}{j}p^{h-2j+1}q^j - 0 \\ &= F_{(h+2,k+1)}^{(p,q)} \end{aligned}$$

Proposition 2. *The following identity holds:*

$$F_{(h+2,k)}^{(p,q)} = pF_{(h+1,k)}^{(p,q)} + qF_{(h,k)}^{(p,q)} - \binom{h-k-1}{k}p^{h-2k-1}q^{k+1} \tag{4}$$

Proof. it is clear that

$$\begin{aligned} F_{(h+2,k)}^{(p,q)} &= \sum_{j=0}^{k}\binom{h-j+1}{j}p^{h-2j+1}q^j \\ &= \sum_{j=0}^{k}\left[\binom{h-j}{j} + \binom{h-j}{j-1}\right]p^{h-2j+1}q^j \\ &= \sum_{j=0}^{k}\binom{h-j}{j}p^{h-2j+1}q^j + \sum_{j=0}^{k}\binom{h-j}{j-1}p^{h-2j+1}q^j \\ &= pF_{(h+1,k)}^{(p,q)} + \sum_{j=-1}^{k-1}\binom{h-j-1}{j}p^{h-2j-1}q^{j+1} \\ &= pF_{(h+1,k)}^{(p,q)} + \binom{h}{-1}p^{h+1} + q\sum_{j=0}^{k}\binom{h-j-1}{j}p^{h-2j-1}q^j - \binom{h-k-1}{k}p^{h-2k-1}q^{k+1} \\ &= pF_{(h+1,k)}^{(p,q)} + qF_{(h,k)}^{(p,q)} - \binom{h-k-1}{k}p^{h-2k-1}q^{k+1} \end{aligned}$$

Proposition 3. *The following identity holds:*

$$\sum_{j=0}^{r}\binom{r}{j}p^{j}q^{r-j}F_{(h+j,k+j)}^{(p,q)}=F_{(h+2r,k+r)}^{(p,q)},\quad 0\leq k\leq\frac{h-r-1}{2} \tag{5}$$

Proof. We use induction on r. The sum (5) is plainly valid for $r=1$; Assume it is true for a specific case $r>1$. In order to perform the inductive step $r\to r+1$, we get

$$\begin{aligned}
F_{(h+2(r+1),k+(r+1))}^{(p,q)} &= \sum_{j=0}^{r+1}\left[\binom{r}{j}+\binom{r}{j-1}\right]p^{j}q^{r-j+1}F_{(h+j,k+j)}^{(p,q)}\\
&= q\sum_{j=0}^{r}\binom{r}{j}p^{j}q^{r-j}F_{(h+j,k+j)}^{(p,q)}+\binom{r}{r+1}p^{r+1}q^{2}F_{(h+r+1,k+r+1)}^{(p,q)}\\
&+\sum_{j=0}^{r+1}\binom{r}{j-1}p^{j}q^{r-j+1}F_{(h+j,k+j)}^{(p,q)}\\
&= qF_{(h+2r,k+r)}^{(p,q)}+0+\sum_{j=-1}^{r}\binom{r}{j}p^{j+1}q^{r-j}F_{(h+j+1,k+j+1)}^{(p,q)}\\
&= qF_{(h+2r,k+r)}^{(p,q)}+p\sum_{j=0}^{r}\binom{r}{j}p^{j}q^{r-j}F_{(h+j+1,k+j+1)}^{(p,q)}+\binom{r}{-1}p^{-1}q^{r+1}F_{(h,k)}^{(p,q)}\\
&= qF_{(h+2r,k+r)}^{(p,q)}+pF_{(h+2r+1,k+r+1)}^{(p,q)}+0
\end{aligned}$$

Proposition 4. *For $n\geq 2k+2$, we have*

$$\sum_{j=0}^{r-1}p^{r-j-1}q^{j+2}F_{(h+j,k)}^{(p,q)}=q^{r+1}F_{(h+r+1,k+1)}^{(p,q)}-p^{r}F_{(h+1,k+1)}^{(p,q)}. \tag{6}$$

Proof. We use induction on r. The sum (5) is plainly valid for $r=1$; Assume it is true for a specific case $r>1$. In order to perform the inductive step $r\to r+1$, we obtain

$$\begin{aligned}
\sum_{j=0}^{r}p^{r-j}q^{j+2}F_{(h+j,k)}^{(p,q)} &= p\sum_{j=0}^{r-1}p^{r-j-1}q^{j+2}F_{(h+j,k)}^{(p,q)}+q^{r+2}F_{(h+r,k)}^{(p,q)}\\
&= p\left(q^{r+1}F_{(h+r+1,k+1)}^{(p,q)}-p^{r}F_{(h+1,k+1)}^{(p,q)}\right)+q^{r+2}F_{(h+r,k)}^{(p,q)}\\
&= q^{r+1}\left(pF_{(h+r+1,k+1)}^{(p,q)}+qF_{(h+r,k)}^{(p,q)}\right)-p^{r+1}F_{(h+1,k+1)}^{(p,q)}\\
&= q^{r+1}F_{(h+r+1,k+1)}^{(p,q)}-p^{r+1}F_{(h+1,k+1)}^{(p,q)}
\end{aligned}$$

In [24,1], note that if p and q in (1) are real variables, then $F_h^{(p,q)}=F_h(x,y)$ and hence they correspond to the bivariate Fibonacci polynomials expressed as

$$F_h(x,y)=xF_{h-1}(x,y)+yF_{h-2}(x,y),\quad F_0(x,y)=0,\quad F_1(x,y)=1,\quad h\geq 2.$$

Lemma 1. *In [1], the following relation holds:*

$$\frac{\partial F_h^{(p,q)}}{\partial p}=\frac{hF_h^{(p,q)}+q(h-2)F_{h-2}^{(p,q)}-2pF_{h-1}^{(p,q)}}{p^2+4q}.$$

Lemma 2. *For $h\in\mathbb{Z}^+$, the following equality is true:*

$$\sum_{j=0}^{\hat{h}}j\binom{h-j-1}{j}p^{h-2j-1}q^{j}=\frac{((h-1)(p^2+4q)-hp)F_h^{(p,q)}-pq(h-2)F_{h-2}^{(p,q)}+2p^2F_{h-1}^{(p,q)}}{2(p^2+4q)}$$

*Proof.*We are aware that

$$pF_h^{(p,q)} = \sum_{j=0}^{\hat{h}} \binom{h-j-1}{j} p^{h-2j} q^j$$

By derivating into the previous equation with respect to p, we get

$$\begin{aligned} F_h^{(p,q)} + p\frac{\partial F_h^{(p,q)}}{\partial p} &= \sum_{j=0}^{\hat{h}} (h-2j) \binom{h-j-1}{j} p^{h-2j-1} q^j \\ &= hF_h^{(p,q)} - 2\sum_{j=0}^{\hat{h}} j \binom{h-j-1}{j} p^{h-2j-1} q^j \end{aligned}$$

From Lemma 1, the proof is completed.

Proposition 5.*For $h \in \mathbb{Z}^+$, the following equality is true:*

$$\sum_{k=0}^{\hat{h}} F_{(h,k)}^{(p,q)} = \frac{((2\hat{h}-h+3)(p^2+4q)+hp)F_h^{(p,q)} + pq(h-2)F_{h-2}^{(p,q)} - 2p^2F_{h-1}^{(p,q)}}{2(p^2+4q)}$$

*Proof.*From Lemma 1, we obtain

$$\begin{aligned} \sum_{k=0}^{\hat{h}} F_{(h,k)}^{(p,q)} &= F_{(h,0)}^{(p,q)} + F_{(h,1)}^{(p,q)} + \cdots + F_{(h,\hat{h})}^{(p,q)} \\ &= \binom{h-1}{0} p^{h-1} + \left[\binom{h-1}{0} p^{h-1} + \binom{h-2}{1} p^{h-3} q \right] \\ &+ \cdots + \left[\binom{h-1}{0} p^{h-1} + \binom{h-2}{1} p^{h-3} q + \cdots + \binom{h-\hat{h}-1}{\hat{h}} p^{h-2\hat{h}-1} q^{\hat{h}} \right] \\ &= (\hat{h}+1) \binom{h-1}{0} p^{h-1} + \hat{h} \binom{h-3}{1} p^{h-3} q + \cdots + \binom{h-\hat{h}-1}{\hat{h}} p^{h-2\hat{h}-1} q^{\hat{h}} \\ &= \sum_{j=0}^{\hat{h}} (\hat{h}-j+1) \binom{h-j-1}{j} p^{h-2j-1} q^j \\ &= (\hat{h}+1) \sum_{j=0}^{\hat{h}} \binom{h-j-1}{j} p^{h-2j-1} q^j - \sum_{j=0}^{\hat{h}} j \binom{h-j-1}{j} p^{h-2j-1} q^j \\ &= (hath+1)F_h^{(p,q)} - \frac{((h-1)(p^2+4q)-hp)F_h^{(p,q)} - pq(h-2)F_{h-2}^{(p,q)} + 2p^2F_{h-1}^{(p,q)}}{2(p^2+4q)} \\ &= \frac{((2\hat{h}-h+3)(p^2+4q)+hp)F_h^{(p,q)} + pq(h-2)F_{h-2}^{(p,q)} - 2p^2F_{h-1}^{(p,q)}}{2(p^2+4q)} \end{aligned}$$

3 The Incomplete $(p,q)-$Lucas Numbers

Definition 2.*The incomplete $(p,q)-$Lucas numbers $L_{(h,k)}^{(p,q)}$ are defined by*

$$L_{(h,k)}^{(p,q)} = \sum_{j=0}^{k} \frac{h}{h-j} \binom{h-j}{j} p^{h-2j} q^j, \quad \left(1 \le h; 0 \le k \le \lfloor \frac{h}{2} \rfloor = \tilde{h}\right). \tag{7}$$

The numbers $L_{(h,k)}^{(p,q)}$ are displayed in Table 2. It shows the first few h values and the corresponding permissible k values:

Table 2: The first few values of the incomplete $(p,q)-$Lucas Numbers

$n \backslash k$	0	1	2	3
1	p			
2	p^2	p^2+2q		
3	p^3	p^3+3pq		
4	p^4	p^4+4p^2q	$p^4+4p^2q+2q^2$	
5	p^5	p^5+5p^3q	$p^5+5p^3q+5pq^2$	
6	p^6	p^6+6p^4q	$p^6+6p^4q+9p^2q^2$	$p^6+6p^4q+9p^2q^2+2q^3$
7	p^7	p^7+7p^5q	$p^7+7p^5q+14p^3q^2$	$p^7+7p^5q+14p^3q^2+7pq^3$
8	p^8	p^8+8p^6q	$p^8+8p^6q+20p^4q^2$	$p^8+8p^6q+20p^4q^2+16p^2q^3$
9	p^9	p^9+9p^7q	$p^9+9p^7q+27p^5q^2$	$p^9+9p^7q+27p^5q^2+30p^3q^3$
10	p^10	$p^{10}+10p^8q$	$p^{10}+10p^8q+35p^6q^2$	$p^{10}+10p^8q+35p^6q^2+50p^4q^3$

The relation (7) has some special cases as follows:

–$L_{(h,0)}^{(p,q)} = p^h, \quad (h \geq 1)$

–$L_{(h,1)}^{(p,q)} = p^h + hp^{h-2}q, \quad (h \geq 2)$

–$L_{(h,2)}^{(p,q)} = p^h + hp^{h-2}q + \dfrac{h(h-3)}{2}p^{h-4}q^2, \quad (h \geq 5)$

–$L_{(h,\tilde{h})}^{(p,q)} = L_h^{(p,q)}, \quad (h \geq 1)$

–$L_{(h,\tilde{h}-1)}^{(p,q)} = \begin{cases} L_h^{(p,q)} - 2q^{(\frac{h}{2})} & (h \quad even) \\ L_h^{(p,q)} - hpq^{(\frac{h-1}{2})} & (h \quad odd) \end{cases}, \quad (h \geq 2)$

3.1 *Some identities of the numbers* $L_{(h,k)}^{(p,q)}$

Proposition 6. *The following identity holds:*

$$L_{(h,k)}^{(p,q)} = qF_{(h-1,k-1)}^{(p,q)} + F_{(h+1,k)}^{(p,q)}, \quad 0 \leq k \leq \tilde{h}. \tag{8}$$

Proof. Using Definition (2), we obtain the desired equality as follows:

$$\begin{aligned} qF_{(h-1,k-1)}^{(p,q)} + F_{(h+1,k)}^{(p,q)} &= q\sum_{j=0}^{k-1}\binom{h-j-2}{j}p^{h-2j-2}q^j + \sum_{j=0}^{k}\binom{h-j}{j}p^{h-2j}q^j \\ &= q\sum_{j=1}^{k}\binom{h-j-1}{j-1}p^{h-2j}q^{j-1} + \sum_{j=0}^{k}\binom{h-j}{j}p^{h-2j}q^j \\ &= \sum_{j=0}^{k}\left[\binom{h-j-1}{j-1} + \binom{h-j}{j}\right]p^{h-2j}q^j - \binom{h-1}{-1}p^h \\ &= \sum_{j=0}^{k}\frac{h}{h-j}\binom{h-j}{j}p^{h-2j}q^j - 0 = L_{(h,k)}^{(p,q)} \end{aligned}$$

Proposition 7. *The incomplete $(p,q)-$Lucas numbers $L_{(h,k)}^{(p,q)}$ can be given by the recurrence relation*

$$L_{(h+2,k+1)}^{(p,q)} = pL_{(h+1,k+1)}^{(p,q)} + qL_{(h,k)}^{(p,q)}, \quad 0 \leq k \leq \tilde{h}. \tag{9}$$

Proof. Relation (9) can be proved by using (8).

Proposition 8. *The following identity holds:*

$$L_{(h+2,k)}^{(p,q)} = pL_{(h+1,k)}^{(p,q)} + qL_{(h,k)}^{(p,q)} - \frac{h}{h-k}\binom{h-k}{k}p^{h-2k}q^{k+1} \tag{10}$$

Proof. Relation (10) can be proved by using (4) and (8).

Proposition 9. *The following identity holds:*

$$\sum_{j=0}^{r}\binom{r}{j}q^{r-j}p^{j}L_{(h+j,k+j)}^{(p,q)}=L_{(h+2r,k+r)}^{(p,q)},\quad 0\le k\le\frac{h-r}{2} \tag{11}$$

Proof. Relation (11) can be proved by using (5) and (8).

4 Generating Functions of the Incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas Numbers

The generating functions of the incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers are given in this section.

Lemma 3. *Assume $\{T_h\}_{h=0}^{\infty}$ is a complex sequence that obeys the non-homogeneous second-order recurrence relation:*

$$T_h=\alpha T_{h-1}+\beta T_{h-2}+R_h,\quad h>1,$$

where $\alpha,\beta\in\mathbb{C}$ (the field of complex numbers) and $R_h:\mathbb{N}\to\mathbb{C}$ is a sequence. Then the generating function $U(t)$ of T_h is

$$U(t)=\frac{G(t)+T_0-R_0+(T_1-\alpha S_0-R_1)t}{1-\alpha t-\beta t^2}$$

where the generating function of $\{R_h\}$ is denoted by $G(t)$ (See [18]).

Theorem 1. *The generating function of the incomplete $(p,q)-$Fibonacci numbers $F_{(h,k)}^{(p,q)}$ is*

$$G_{p,q,k}^{F}(x)=\frac{\frac{x^2q^{k+1}}{(1-px)^{k+1}}+F_{2k+1}^{(p,q)}+qF_{2k}^{(p,q)}x}{1-px-qx^2}$$

Proof. Assume k is a fixed positive integer. Using (2) and (4), $F_{(h,k)}^{(p,q)}=0$ for $0\le h<2k+1$, $F_{(2k+1,k)}^{(p,q)}=F_{p,q,2k+1}$, and $F_{(2k+2,k)}^{(p,q)}=F_{2k+2}^{(p,q)}$,

$$F_{(h,k)}^{(p,q)}=pF_{(h-1,k)}^{(p,q)}+qF_{(h-2,k)}^{(p,q)}-\binom{h-k-3}{k}p^{h-2k-3}q^{k+1}$$

Now consider $T_0=F_{(2k+1,k)}^{(p,q)}$, $T_1=F_{(2k+2,k)}^{(p,q)}$ and $T_h=F_{(h+2k+1,k)}^{(p,q)}$.
Also, consider $R_0=R_1=0$,

$$R_h=\binom{h+k-2}{h-2}p^{h-2}q^{k+1}.$$

Here,

$$G(x)=\frac{x^2q^{k+1}}{(1-px)^{k+1}}$$

is the generating function of the sequence $\{R_h\}$ (see [21]). As a result of Lemma 3, we obtain the generating function $G_{p,q,k}^{F}(x)$ of the sequence $\{T_h\}$.

Theorem 2. *The generating function of the incomplete $(p,q)-$Lucas numbers $F_{(h,k)}^{(p,q)}$ is*

$$G_{p,q,k}^{L}(x)=\frac{\frac{x^2(2-px)q^{k+1}}{(1-px)^{k+1}}+L_{2k}^{(p,q)}+qL_{2k-1}^{(p,q)}x}{1-px-qx^2}$$

*Proof.*Assume k is a fixed positive integer. Using (2) and (4), $L_{(h,k)}^{(p,q)}=0$ for $0\le h<2k$, $L_{(2k,k)}^{(p,q)}=L_{2k}^{(p,q)}$, and $L_{(2k+1,k)}^{(p,q)}=L_{2k+1}^{(p,q)}$,

$$L_{(h,k)}^{(p,q)}=pL_{(h-1,k)}^{(p,q)}+qL_{(h-2,k)}^{(p,q)}-\frac{h-2}{h-k-2}\binom{h-k-2}{h-2k-2}p^{h-2k-2}q^{k+1}$$

Now consider $T_0=L_{(2k,k)}^{(p,q)}$, $T_1=L_{(2k+1,k)}^{(p,q)}$ and $T_n=L_{(n+2k,k)}^{(p,q)}$.
Also, consider $R_0=R_1=0$,

$$R_h=\frac{n+2k-2}{h+k-2}\binom{h+k-2}{h-2}p^{h-2}q^{k+1}$$

Here,

$$G(x)=\frac{x^2(2-px)q^{k+1}}{(1-px)^{k+1}}$$

is the generating function of the sequence $\{R_h\}$ (see [21]). As a result of Lemma 3, we get the generating function $G_{p,q,k}^{L}(x)$ of the sequence $\{T_h\}$.

5 Conclusion

In this paper, the incomplete $(p,q)-$Fibonacci and $(p,q)-$Lucas numbers are defined. Some properties and identities for them are given. The generating functions are derived. From these results, we can reach familiar results for some special numbers, such as Fibonacci, Lucas, Pell, and Jacobsall, as special cases

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On Recursive Computation of Moments of Generalized Order Statistics for a Transmuted Pareto Distribution and Characterization

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Abstract: In this paper, some relations are obtained for recursive computation of moments of generalized order statistics for a modified Pareto distribution. The recursive methods are obtained for raw and product moments. We have also obtained the recursive methods to compute the moments of specific situations which include order statistics and record values. We have also given some characterization results for the modified Pareto distribution.

Keywords: Transmuted Pareto Distribution, Generalized Order Statistics, Recursive Computation, Characterization.

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1 Introduction

A single combined method for random variables arranged in increasing order is known as the generalized order statistics (*gos*). This method is proposed by [1] and provides certain other methods for ordered data as special case. The density function of all of the variables in a *gos* is given by [1] as

$$g_{1,\dots,n:n,t,\kappa}(y_1,\dots,y_n)=\kappa\left(\prod_{j=1}^{n-1}\gamma_j\right)[1-G(y_n)]^{\kappa-1}g(y_n)$$
$$\times\prod_{j=1}^{n-1}\left[\{1-G(y_j)\}^t g(y_j)\right], \tag{1}$$

where the quantities n, t and κ are the parameters of the density function such that $\gamma_h=\kappa+(n-h)(t+1)$. The *gos* produces different other methods for ordered data for different values of the parameters. The most popular of these are ordinary order statistics, kth record values; by [2]; and simple record values by [3].

The probability density function of a single *gos* is

$$g_{p:n,t,\kappa}(y)=\frac{C_{p-1}}{(p-1)!}g(y)[1-G(y)]^{\gamma_p-1}f_t^{p-1}[G(y)], \tag{2}$$

where $C_{p-1}=\prod_{j=1}^{p}\gamma_j$ and

$$f_t(u)=h_t(u)-h_t(0)=\begin{cases}\left[1-(1-u)^{t+1}\right]\Big/(t+1)\ ;\ t\neq-1\\ -\ln(1-u)\ ;\ t=-1.\end{cases}$$

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The probability density of any two *gos* is given as

$$g_{p,q:n,t,\kappa}(y_1,y_2) = \frac{C_{q-1}}{(p-1)!\,(q-p-1)!} g(y_1)\, g(y_2)\, [1-G(y_1)]^t f_t^{p-1}[G(y_1)] \times [1-G(y_2)]^{\gamma_q-1} [h_t\{G(y_2)\} - h_t\{G(y_1)\}]^{q-p-1}, \quad (3)$$

for $-\infty < y_1 < y_2 < \infty$ and

$$h_t(u) = \begin{cases} -(1-u)^{t+1}\Big/(t+1)\ ;\ t \neq -1 \\ -\ln(1-u)\ ;\ t = -1. \end{cases}$$

The *gos* is a single general method to model the data arranged in increasing order. Different simple methods are the special cases of this method for specific values of the parameters. The simple order statistics is obtained from *gos* if $t = 0$ and $\kappa = 1$. The model reduces to kth record values of [2] for $t = -1$. The simple upper record values; [3]; are obtained from *gos* for $t = -1$ and $\kappa = 1$. Some more details can be found in [4] and [5].

Several authors have used different distributions to study their properties in context of the *gos*. A lot of work has been done in obtaining expressions to compute moments of *gos* recursively for specific distributions. A general expression for relations between moments of *gos* for any parent distribution has been obtained by [6] and [7]. Pareto and related distributions have attracted several authors to obtained the recurrence relations for moments of gos and its special cases. The relations for moments of generalized Pareto distribution were obtained by [8]. The expressions for recursive computation of moments of record values for Pareto and generalized Pareto distribution were obtained by [9] and [10]. The recursive expression for moments of *gos* for Pareto distribution are developed by [10]. The relations for recursive computation of moments of *gos* for Kumaraswamy Pareto distribution are developed by [11]. Some characterizations for the distributions using *gos* have been given by [12].

The area of recursive computation of moments for transmuted distributions is yet to be explored. This paper deals with developing some recursive methods to compute moments of *gos* for a transmuted Pareto distribution. A brief about the distribution is first given in the following section.

2 The Transmuted Pareto Distribution

The Pareto distribution; [13]; has tremendous applications in economics and finance. The distribution has been proposed as a suitable distribution for modeling income. The distribution has following density and distribution function

$$f(y;k,\alpha) = \frac{\alpha c^\alpha}{y^{\alpha+1}}\ ;\ y \geq c\ ,\ (\alpha,c) > 0$$

and

$$F(y;k;\alpha) = 1 - \left(\frac{c}{y}\right)^\alpha\ ;\ y \geq c\ ,\ (\alpha,c) > 0.$$

The distribution is studied extensively by several authors. Various authors has given different modifications of the distribution. A modification of the Pareto distribution has been given by [14] by using the technique of [15] and is referred to as the transmuted Pareto distribution. The density and distribution functions of this transmuted Pareto distribution are

$$g(y) = \frac{\alpha c^\alpha}{y^{\alpha+1}} \left[1 + \lambda - 2\lambda \left\{1 - \left(\frac{c}{y}\right)^\alpha\right\}\right]\ ;\ y \geq c\ ,\ (\alpha,c) > 0, \quad (4)$$

and

$$G(y) = \left[1 - \left(\frac{c}{y}\right)^\alpha\right] + \lambda \left(\frac{c}{y}\right)^\alpha \left[1 - \left(\frac{c}{y}\right)^\alpha\right]\ ;\ y \geq c\ ,\ (\alpha,c) > 0, \quad (5)$$

where λ is the transmutation parameter such that $-1 \leq \lambda \leq 1$. The transmuted Pareto distribution has wide spread applications in modeling of financial and geological data. It is easy to see that (4) and (5) are related as

$$1 - G(y) = \frac{y}{\alpha} g(y) - \lambda \left(\frac{c}{y}\right)^{2\alpha}. \quad (6)$$

This paper deals with obtaining recursive expressions to compute moments of *gos* for the transmuted Pareto distribution. The distribution has also been characterized on the basis of these recursive expressions of single and joint moments. The recursive expressions are obtained in Sections 3 and 4 below.

3 Relations for Simple and Reciprocal Moments

This section deals with obtaining recursive expressions to compute simple and reciprocal moments of *gos* for a transmuted Pareto distribution are given. These recursive expressions are obtained in the Theorem and the resulting corollaries, below.

Theorem 1. *The simple moments of gos for transmuted Pareto distribution can be recursively computed as*

$$\mu^{r}_{p:n,t,\kappa}=\frac{\alpha\gamma_p}{\alpha\gamma_p-r}\left[\mu^{r-1}_{p:n,t,\kappa}-\frac{\lambda c^{2\alpha}r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\left\{\mu^{r-2\alpha}_{p:n,t,\kappa-1}-\mu^{r-2\alpha}_{p-1:n,t,\kappa-1}\right\}\right], \tag{7}$$

where $\gamma_{h(\kappa-1)}=(\kappa-1)+(n-h)(t+1)$ *and* $C_{p-1(\kappa-1)}=\prod_{h=1}^{p}\gamma_{h(\kappa-1)}$.

Proof. It is shown by [7] that the recursive expression for moments of *gos* for any distribution can be obtained by using

$$\mu^{r}_{p:n,t,\kappa}-\mu^{r}_{p-1:n,t,\kappa}=\frac{rC_{p-1}}{\gamma_p(p-1)!}\int_{-\infty}^{\infty}y^{r-1}[1-G(y)]^{\gamma_p}f_t^{p-1}[G(y)]\,dy, \tag{8}$$

where $\mu^{r}_{p:n,t,\kappa}=E\left(Y^{r}_{p:n,t,\kappa}\right)$ and $Y^{r}_{p:n,t,\kappa}$ is the pth *gos*. The relation (8) can be written as

$$\mu^{r}_{p:n,t,\kappa}-\mu^{r}_{p-1:n,t,\kappa}=\frac{rC_{p-1}}{\gamma_p(p-1)!}\int_{-\infty}^{\infty}y^{r-1}[1-G(y)][1-G(y)]^{\gamma_p-1}\times f_t^{p-1}[G(y)]\,dy.$$

Now, using (6) in above equation, we have

$$\begin{aligned}\mu^{r}_{p:n,t,\kappa}-\mu^{r}_{p-1:n,t,\kappa}&=\frac{rC_{p-1}}{\gamma_p(p-1)!}\int_{c}^{\infty}y^{r-1}\left[\frac{y}{\alpha}g(y)-\lambda\left(\frac{c}{y}\right)^{2\alpha}\right]\\&\quad\times[1-G(y)]^{\gamma_p-1}f_t^{p-1}[G(y)]\,dy.\\&=\frac{rC_{p-1}}{\gamma_p(p-1)!}\int_{c}^{\infty}y^{r}g(y)[1-G(y)]^{\gamma_p-1}f_t^{p-1}[G(y)]\,dy\\&\quad-\frac{\lambda c^{2\alpha}rC_{p-1}}{\gamma_p(p-1)!}\int_{c}^{\infty}y^{r-2\alpha-1}[1-G(y)]^{\gamma_p-1}f_t^{p-1}[G(y)]\,dy,\end{aligned}$$

or

$$\begin{aligned}\mu^{r}_{p:n,t,\kappa}-\mu^{r}_{p-1:n,t,\kappa}&=\frac{r}{\alpha\gamma_p}\mu^{r}_{p:n,t,\kappa}-\frac{\lambda c^{2\alpha}r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\frac{(r-2\alpha)C_{p-1(\kappa-1)}}{\gamma_{p(\kappa-1)}(p-1)!}\\&\quad\times\int_{c}^{\infty}y^{r-2\alpha-1}[1-G(y)]^{\gamma_{p(k-1)}}f_t^{p-1}[G(y)]\,dy,\end{aligned}$$

where $\gamma_{h(\kappa-1)}=(\kappa-1)+(n-h)(t+1)$ and $C_{p-1(\kappa-1)}=\prod_{h=1}^{p}\gamma_{h(\kappa-1)}$. Now, again using (6), we have

$$\mu^{r}_{p:n,t,\kappa}-\mu^{r}_{p-1:n,t,\kappa}=\frac{r}{\alpha\gamma_p}\mu^{r}_{p:n,t,\kappa}-\frac{\lambda c^{2\alpha}r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\left[\mu^{r-2\alpha}_{p:n,t,\kappa-1}-\mu^{r-2\alpha}_{p-1:n,t,\kappa-1}\right]$$

or

$$\mu^{r}_{p:n,t,\kappa}=\frac{\alpha\gamma_p}{\alpha\gamma_p-r}\left[\mu^{r}_{p-1:n,t,\kappa}-\frac{\lambda c^{2\alpha}r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\left\{\mu^{r-2\alpha}_{p:n,t,\kappa-1}-\mu^{r-2\alpha}_{p-1:n,t,\kappa-1}\right\}\right],$$

which is (7) and the proof is complete.

The recursive expression for simple moments of *gos* for Pareto distribution, given by [10], is readily obtained from (7) by using $\lambda=0$.

Following corollaries are immediately obtained from Theorem 1.

Corollary 1. *Using $-r$ instead of r in (7), we have following recursive expression for the reciprocal moments of gos for the transmuted Pareto distribution*

$$\mu^{-r}_{p:n,t,\kappa}=\frac{\alpha\gamma_p}{\alpha\gamma_p+r}\left[\mu^{-r}_{p-1:n,t,\kappa}-\frac{\lambda c^{2\alpha}r}{(r+2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}}\left\{\mu^{-(r+2\alpha)}_{p:n,t,\kappa-1}-\mu^{-(r+2\alpha)}_{p-1:n,t,\kappa-1}\right\}\right]. \tag{9}$$

Corollary 2. *Using $t=-1$ in (7), following relation for moments of kth upper record value for transmuted Pareto distribution is obtained*

$$\mu_{K(p)}^{r}=\frac{\alpha\kappa}{\alpha\kappa-r}\left[\mu_{K(p-1)}^{r}-\frac{\kappa^{p-1}\lambda c^{2\alpha}r}{(\kappa-1)^{p-1}(r-2\alpha)}\left\{\mu_{K-1(p)}^{r-2\alpha}-\mu_{K-1(p-1)}^{r-2\alpha}\right\}\right]. \tag{10}$$

The recursive expression for moments of kth record value for Pareto distribution; given by [9]; is obtained as a special case of (10) by using $\lambda=0$.

Corollary 3. *The recursive expression for simple moments of order statistics is derived by using $t=0$ and $\kappa=1$ in (7) and is*

$$\mu_{p:n}^{r}=\frac{\alpha(n-p+1)}{\alpha(n-p+1)-p}\left[\mu_{p-1:n}^{r}-\frac{n\lambda c^{2\alpha}r}{(r-2\alpha)(n-p+1)}\left\{\mu_{p:n}^{r-2\alpha}-\mu_{p-1:n}^{r-2\alpha}\right\}\right]. \tag{11}$$

The recursive expression for moments of order statistics for Pareto distribution is obtained by setting $\lambda=0$ in (11).

Corollary 4. *The recursive expression for reciprocal moments of transmuted Pareto distribution are obtained by using $t=-1$ in (9) and is*

$$\mu_{K(p)}^{-r}=\frac{\alpha\kappa}{\alpha\kappa-r}\left[\mu_{K(p-1)}^{-r}-\frac{\kappa^{p-1}\lambda c^{2\alpha}r}{(\kappa-1)^{p-1}(r+2\alpha)}\left\{\mu_{K-1(p)}^{-(r+2\alpha)}-\mu_{K-1(p-1)}^{-(r+2\alpha)}\right\}\right]. \tag{12}$$

Corollary 5. *The recursive expression for reciprocal moments of order statistics is obtained by using $t=0$ and $\kappa=1$ in (9) and is*

$$\mu_{p:n}^{-r}=\frac{\alpha(n-p+1)}{\alpha(n-p+1)+p}\left[\mu_{p-1:n}^{-r}-\frac{n\lambda c^{2\alpha}r}{(r+2\alpha)(n-p+1)}\left\{\mu_{p:n}^{-(r+2\alpha)}-\mu_{p-1:n}^{-(r+2\alpha)}\right\}\right]. \tag{13}$$

We will now obtain recursive expression for joint and ratio moments of *gos* for transmuted Pareto distribution.

4 Recursive Computation of Joint and Ratio Moments

The recursive relations for joint moments of *gos* for a transmuted Pareto distribution is obtained in the following theorem.

Theorem 2. *The joint moments of gos for transmuted Pareto distribution can be recursively computed by using*

$$\mu_{r,s:n,t,\kappa}^{p,q}=\frac{\alpha\gamma_s}{\alpha\gamma_s-q}\left[\mu_{r,s-1:n,t,\kappa}^{p,q}-\frac{\lambda c^{2\alpha}q}{(q-2\alpha)}\frac{\gamma_{s(\kappa-1)}C_{s-1}}{\gamma_s C_{s-1(\kappa-1)}}\right.$$
$$\left.\times\left\{\mu_{r,s:n,t,\kappa-1}^{p,q-2\alpha}-\mu_{r,s-1:n,t,\kappa-1}^{p,q-2\alpha}\right\}\right], \tag{14}$$

where $\mu_{p,q:n,t,\kappa}^{r,s}=E\left(Y_{p:n,t,\kappa}^{r}Y_{q:n,t,\kappa}^{s}\right)$ and $p<q$.

Proof. The joint moments of *gos* for any distribution are related as; see [7];

$$\mu_{p,q:n,t,\kappa}^{r,s}-\mu_{p,q-1:n,t,\kappa}^{r,s}=\frac{sC_{q-1}}{\gamma_q(p-1)!(q-p-1)!}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}y_1^r y_2^{s-1}g(y_1)$$
$$\times[1-G(y_1)]^t f_t^{p-1}[G(y_1)][1-G(y_2)]^{\gamma_q}$$
$$\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1, \tag{15}$$

where $\mu_{p,q:n,t,\kappa}^{r,s}=E\left(Y_{p:n,t,\kappa}^{r}Y_{q:n,t,\kappa}^{s}\right)$. The relation (15) can also be written as

$$\mu_{p,q:n,t,\kappa}^{r,s}-\mu_{p,q-1:n,t,\kappa}^{r,s}=\frac{sC_{q-1}}{\gamma_q(p-1)!(q-p-1)!}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}y_1^r y_2^{s-1}g(y_1)$$
$$\times f_t^{p-1}[G(y_1)][1-G(y_1)]^t$$
$$\times[1-G(y_2)][1-G(y_2)]^{\gamma_q-1}$$
$$\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1.$$

Using (6), we have

$$\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=\frac{sC_{q-1}}{\gamma_q(p-1)!(q-p-1)!}\int_c^\infty\int_{y_1}^\infty y_1^r y_2^{s-1}g(y_1)$$
$$\times f_t^{p-1}[G(y_1)][1-G(y_1)]^t[1-G(y_2)]^{\gamma_q-1}$$
$$\times\left[\frac{y_2}{\alpha}g(y_2)-\lambda\left(\frac{c}{y_2}\right)\right]$$
$$\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1$$

or

$$\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=\frac{sC_{q-1}}{\gamma_q(p-1)!(q-p-1)!}\int_c^\infty\int_{y_1}^\infty y_1^r y_2^{s-1}g(y_1)g(y_2)$$
$$\times[1-G(y_1)]^t f_t^{p-1}[G(y_1)][1-G(y_2)]^{\gamma_q-1}$$
$$\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1$$
$$-\frac{\lambda c^{2\alpha}sC_{q-1}}{\gamma_q(p-1)!(q-p-1)!}\int_c^\infty\int_{y_1}^\infty y_1^r y_2^{s-1}g(y_1)$$
$$\times[1-G(y_1)]^t f_t^{p-1}[G(y_1)][1-G(y_2)]^{\gamma_q-1}$$
$$\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1$$

or

$$\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=\frac{s}{\alpha\gamma_q}\mu^{r,s}_{p,q:n,t,\kappa}-\frac{\lambda c^{2\alpha}sC_{q-1}}{\gamma_q(p-1)!(q-p-1)!}$$
$$\times\int_c^\infty\int_{y_1}^\infty y_1^r y_2^{s-1}g(y_1)[1-G(y_1)]^t$$
$$\times f_t^{p-1}[G(y_1)][1-G(y_2)]^{\gamma_q-1}$$
$$\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1$$

or

$$\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=\frac{s}{\alpha\gamma_q}\mu^{r,s}_{p,q:n,t,\kappa}-\frac{\lambda c^{2\alpha}s\gamma_{q(\kappa-1)}C_{q-1}}{(s-2\alpha)\gamma_q C_{q-1(\kappa-1)}}$$
$$\times\frac{(s-2\alpha)C_{q-1(\kappa-1)}}{\gamma_{q(\kappa-1)}(p-1)!(q-p-1)!}\int_c^\infty\int_{y_1}^\infty y_1^r y_2^{s-1}$$
$$\times g(y_1)[1-G(y_1)]^t f_t^{p-1}[G(y_1)][1-G(y_2)]^{\gamma_q-1}$$
$$\times[h_t\{G(y_2)\}-h_t\{G(y_1)\}]^{q-p-1}dy_2dy_1,$$

where $\gamma_{h(\kappa-1)}=(\kappa-1)+(n-h)(t+1)$ and $C_{q-1(\kappa-1)}=\prod_{h=1}^q\gamma_{h(\kappa-1)}$. Again using (15), we have

$$\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=\frac{s}{\alpha\gamma_q}\mu^{r,s}_{p,q:n,t,\kappa}-\frac{\lambda c^{2\alpha}s\gamma_{q(\kappa-1)}C_{q-1}}{(s-2\alpha)\gamma_q C_{q-1(\kappa-1)}}$$
$$\left[\mu^{r,s-2\alpha}_{p,q:n,t,\kappa-1}-\mu^{r,s-2\alpha}_{p,q-1:n,t,\kappa-1}\right]$$

or

$$\mu^{r,s}_{p,q:n,t,\kappa}=\frac{\alpha\gamma_q}{\alpha\gamma_q-s}\left[\mu^{r,s}_{p,q-1:n,t,\kappa}-\frac{\lambda c^{2\alpha}s}{(s-2\alpha)}\frac{\gamma_{q(\kappa-1)}C_{q-1}}{\gamma_q C_{q-1(\kappa-1)}}\right.$$
$$\left.\times\left\{\mu^{r,s-2\alpha}_{p,q:n,t,\kappa-1}-\mu^{r,s-2\alpha}_{p,q-1:n,t,\kappa-1}\right\}\right].$$

This is (14) and the proof is complete.

The relation (14) transforms to the relation for joint moments of *gos* from Pareto distribution, obtained by [10], for $\lambda=0$ as it should be.

Theorem 2 provides following corollaries.

Corollary 6.*Substituting $-s$ in (14), the recursive expression for the ratio moments of gos for transmuted Pareto distribution is*

$$\mu^{r,-s}_{p,q:n,t,\kappa} = \frac{\alpha\gamma_q}{\alpha\gamma_q + s}\left[\mu^{r,-s}_{p,q-1:n,t,\kappa} - \frac{\lambda c^{2\alpha}s}{(s+2\alpha)}\frac{\gamma_{q(\kappa-1)}C_{q-1}}{\gamma_q C_{q-1(\kappa-1)}} \times\left\{\mu^{r,-(s+2\alpha)}_{p,q:n,t,\kappa-1} - \mu^{r,-(s+2\alpha)}_{p,q-1:n,t,\kappa-1}\right\}\right]. \tag{16}$$

Corollary 7.*Substituting $t = -1$ in (14), following recursive expression for (r,s)th moments of (p,q)th upper record values for transmuted Pareto distribution is obtained*

$$\mu^{r,s}_{K(p,q)} = \frac{\alpha\kappa}{\alpha\kappa - s}\left[\mu^{r,s}_{K(p,q-1)} - \frac{\kappa^{q-1}\lambda c^{2\alpha}s}{(\kappa-1)^{q-1}(s-2\alpha)}\left\{\mu^{r,s-2\alpha}_{K-1(p,q)} - \mu^{r,s-2\alpha}_{K-1(p,q-1)}\right\}\right]. \tag{17}$$

Corollary 8.*Substituting $t = 0$ and $\kappa = 1$ in (14), following recursive expression for joint moments of order statistics for transmuted Pareto distribution is obtained*

$$\mu^{r,s}_{p,q:n} = \frac{\alpha(n-q+1)}{\alpha(n-q+1)-s}\left[\mu^{r,s}_{p,q-1:n} - \frac{n\lambda c^{2\alpha}s}{(n-q+1)(s-2\alpha)}\left\{\mu^{r,s-2\alpha}_{p,q:n} - \mu^{r,s-2\alpha}_{p,q-1:n}\right\}\right]. \tag{18}$$

Corollary 9.*Substituting $t = -1$ in (16), following recursive expression for the ratio moments of kth record value for transmuted Pareto distribution is obtained*

$$\mu^{r,-s}_{K(p,q)} = \frac{\alpha\kappa}{\alpha\kappa + s}\left[\mu^{r,-s}_{K(p,q-1)} - \frac{\kappa^{q-1}\lambda c^{2\alpha}s}{(\kappa-1)^{q-1}(s+2\alpha)}\left\{\mu^{r,-(s+2\alpha)}_{K-1(p,q)} - \mu^{r,-(s+2\alpha)}_{K-1(p,q-1)}\right\}\right]. \tag{19}$$

Corollary 10.*Substituting $t = 0$ and $\kappa = 1$ in (16), the following expression for recursive computation of ratio moments of order statistics for transmuted Pareto distribution is obtained*

$$\mu^{r,-s}_{p,q:n} = \frac{\alpha(n-q+1)}{\alpha(n-q+1)+s}\left[\mu^{r,-s}_{p,q-1:n} - \frac{n\lambda c^{2\alpha}s}{(n-q+1)(s+2\alpha)}\left\{\mu^{r,-(s+2\alpha)}_{p,q:n} - \mu^{r,-(s+2\alpha)}_{p,q-1:n}\right\}\right]. \tag{20}$$

The above relations are useful for recursive computation of moments.

5 Some Characterizations

Some characterizations of the transmuted Pareto distribution in terms of simple and joint moments of *gos* are given in the following theorems.

Theorem 3.*For a random variable X to have the density and distribution functions given in (4) and (5) respectively, the simple moments of its gos should be related as*

$$\mu^{r}_{p:n,t,\kappa} - \mu^{r}_{p-1:n,t,\kappa} = \frac{r}{\alpha\gamma_p}\mu^{r}_{p:n,t,\kappa} - \frac{\lambda c^{2\alpha}r}{(r-2\alpha)}\frac{\gamma_{p(\kappa-1)}C_{p-1}}{\gamma_p C_{p-1(\kappa-1)}} \times\left[\mu^{r-2\alpha}_{p:n,t,\kappa-1} - \mu^{r-2\alpha}_{p-1:n,t,\kappa-1}\right].$$

*Proof.*The necessary part of the Theorem is easily proved from Theorem 1. The sufficient condition is proved by considering (7); with $\bar{G}(x) = 1 - G(x)$; as

$$\frac{rC_{p-1}}{\gamma_p(p-1)!}\int_c^\infty y^{r-1}\left\{\bar{G}(y)\right\}^{\gamma_p} f_t^{p-1}[G(y)]\,dy$$
$$= \frac{rC_{p-1}}{\gamma_p(p-1)!}\int_c^\infty y^{y-1}\left\{\bar{G}(y)\right\}^{\gamma_p-1} f_t^{p-1}[G(y)] \times\left[\frac{y}{\alpha}g(y) - \lambda\left(\frac{c}{y}\right)^{2\alpha}\right]dy$$

or

$$\frac{rC_{p-1}}{\gamma_p\,(p-1)!}\int_c^{\infty} y^{y-1}\left\{\bar{G}(y)\right\}^{\gamma_p-1} f_t^{p-1}\left[G(y)\right]\left[\bar{G}(y)-\left\{\frac{y}{\alpha}g(y)-\lambda\left(\frac{c}{y}\right)^{2\alpha}\right\}\right]dy=0.$$

Applying *Müntz–Száz* theorem; see [16]. We have; from above equation;

$$\bar{G}(y)=\left\{\frac{y}{\alpha}g(y)-\lambda\left(\frac{c}{y}\right)^{2\alpha}\right\}.$$

The above is (6) and hence the proof is complete.

Theorem 4.*For a random variable X to have the density and distribution functions given in (4) and (5) respectively, the joint moments of its gos should be related as*

$$\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=\frac{s}{\alpha\gamma_q}\mu^{r,s}_{p,q:n,t,\kappa}-\frac{\lambda c^{2\alpha}s\gamma_{q(k-1)}C_{q-1}}{(s-2\alpha)\,\gamma_q C_{q-1(k-1)}}\left[\mu^{r,s-2\alpha}_{p,q:n,t,\kappa-1}-\mu^{r,s-2\alpha}_{p,q-1:n,t,\kappa-1}\right].$$

*Proof.*The necessity is readily proved from Theorem 2. For sufficiency consider (15) as

$$\begin{aligned}\mu^{r,s}_{p,q:n,t,\kappa}-\mu^{r,s}_{p,q-1:n,t,\kappa}=&\frac{sC_{q-1}}{\gamma_q\,(p-1)!\,(q-p-1)!}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}y_1^r y_2^{s-1}g(y_1)\\&\times\left[\bar{G}(y_1)\right]^t f_t^{p-1}\left[G(y_1)\right]\left[\bar{G}(y_2)\right]^{\gamma_q}\\&\times\left[h_t\left\{G(y_2)\right\}-h_t\left\{G(y_1)\right\}\right]^{q-p-1}dy_2dy_1,\end{aligned}$$

Now, using above relation with (6) we have

$$\begin{aligned}&\frac{sC_{q-1}}{\gamma_q\,(p-1)!\,(q-p-1)!}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}y_1^r y_2^{s-1}g(y_1)\left[\bar{G}(y_1)\right]^t f_t^{p-1}\left[G(y_1)\right]\\&\times\left[h_t\left\{G(y_2)\right\}-h_t\left\{G(y_1)\right\}\right]^{q-p-1}\left[\bar{G}(y_2)\right]^{\gamma_q}dy_2dy_1\\&=\frac{sC_{q-1}}{\gamma_q\,(p-1)!\,(q-p-1)!}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}y_1^r y_2^{s-1}g(y_1)\left[\bar{G}(y_1)\right]^t f_t^{p-1}\left[G(y_1)\right]\\&\times\left[h_t\left\{G(y_2)\right\}-h_t\left\{G(y_1)\right\}\right]^{q-p-1}\left[\bar{G}(y_2)\right]^{\gamma_q-1}\\&\times\left\{\frac{y_2}{\alpha}g(y_2)-\lambda\left(\frac{c}{y_2}\right)^{2\alpha}\right\}dy_2dy_1\end{aligned}$$

or

$$\begin{aligned}&\frac{sC_{q-1}}{\gamma_q\,(p-1)!\,(q-p-1)!}\int_{-\infty}^{\infty}\int_{y_1}^{\infty}y_1^r y_2^{s-1}g(y_1)\left[\bar{G}(y_1)\right]^t f_t^{p-1}\left[G(y_1)\right]\\&\times\left[h_t\left\{G(y_2)\right\}-h_t\left\{G(y_1)\right\}\right]^{q-p-1}\left[\bar{G}(y_2)\right]^{\gamma_q-1}\\&\times\left[\bar{G}(y_2)-\left\{\frac{y_2}{\alpha}g(y_2)-\lambda\left(\frac{c}{y_2}\right)^{2\alpha}\right\}\right]dy_2dy_1\\&=0.\end{aligned}$$

Applying *Müntz–Száz* theorem; see [16]. We have; from above equation;

$$\bar{G}(y_2)=\frac{y_2}{\alpha}g(y_2)-\lambda\left(\frac{c}{y_2}\right)^{2\alpha}.$$

The above is (6) and hence the proof is complete.

6 Conclusion

This paper deals with obtaining some recursive expressions to compute the simple and joint moments of *gos* for a transmuted Pareto distribution. These expressions can be used to recursively compute the higher moments from the lower moments. We have also given the recursive expressions for the simple and joint moments of the specific cases of *gos*. The simple and joint moments are also used to obtain some characterization results. We have found that the recursive expressions for simple and joint moments of *gos* for the Pareto distribution appear as a special case. These relations are also useful in studying certain properties of the transmuted Pareto distribution.

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Some Common Fixed-Point Theorems of Kannan-type Contractions with CLR and EA Property on Fuzzy Metric Spaces

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Abstract: This manuscript explores generalized Kannan-type contractions in the framework of fuzzy metric spaces. We prove several common fixed-point results of these new contractive mappings via common limit in the range and El Moutawakil-Aamri properties. We provide examples to clarify our results. Our results improvise and generalize a few common fixed-point theorems established by earlier studies.

Keywords: Fuzzy metric spaces; Common fixed points; Kannan-type contraction; CLR property; EA property.

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1 Introduction

Fuzzy set is an idea proposed by Zadeh [35] in 1965. Since then, this concept has been widely acknowledged by researchers and utilized in diverse branches of mathematics as well as real life applications. At a later point, Kramosil and Michálek [18] presented fuzzy metric spaces as an extension for the probabilistic metric spaces with the perspective of fuzzy sets. Their notion was later modified by George and Veeramani [8] so that Hausdorff topology can be studied on this space. Grabiec [13] pioneered the investigation of fixed-point theory on fuzzy metric spaces. Consequently, researchers studied fixed-point theory intensively on this abstract spaces and its generalized spaces. A few fixed-point theories on these spaces may be seen in [10], [9], [22], [23], [25], [26], and [33].

In fixed-point theory, one of the well-known contraction mappings is Kannan-type contractive mapping introduced by Kannan [16, 17]. There are several thoughts about the important of Kannan-type contractions, especially under the scope of metric fixed-point theory. One of the reasons is the famous Banach contraction by Banach [3] requires continuous mapping, but Kannan-type contractive mapping needs not to be continuous. Another reason is the relationship between Kannan-type contractive mapping and the completeness of the metric spaces. Connell [7] gave an illustration of metric space that is not complete and yet any Banach contractive mapping assigned on it have fixed point. However, this is not the case for Kannan-type contraction mappings in metric spaces. Subrahmanyam [30] demonstrated that metric space is complete implies and is implied by all Kannan-type contractive mappings in this space contain fixed points. Recent works related to Kannan-type contractive mappings can found in [6], [11], [12], [20], and [36].

Aamri and El Moutawakil [1] proposed El Moutawakil-Aamri (E.A. for short) property for noncompatible self-mapping on metric space in 2002. This (E.A.) property allows one to acquire fixed point results without the completeness of the space. However, it requires a condition of closeness of range for fixed point to exist. Later, Sintunavarat and Kumam [28] proposed a novel property, dubbed "common limit in the range" (CLR for short) that is

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more versatile compared to (E.A.) property, as it no longer needs the condition of closeness of range. These two properties are studied extensively in different spaces (see [2], [4], [14], [19], [21], [29], [31] and [32]).

The objective of this research is to validate several common fixed-point theorems for generalized Kannan-type contractive mappings equipped with a common limits in the range or (E.A.) properties on fuzzy metric spaces. This manuscript is arranged into four main sections as follows: Section 1 presents introduction. Section 2 provides preliminary definitions and notions. Section 3 contains primary findings and their proofs. Section 4 is the conclusion and open problems.

2 Preliminaries

We recollect some terminologies from the fuzzy fixed point theory that will be employed in this manuscript.

Definition 1([16]). *Let (X,δ) denoted as metric space and $\mathscr{T}:E\to E$ be a self-mapping. Then, $\mathscr{T}$ is called a Kannan-type contractive mapping if there exist $k\in[0,\frac{1}{2})$ satisfy*

$$\delta(\mathscr{T}\xi,\mathscr{T}\chi)\leq k[\delta(\xi,\mathscr{T}\xi)+\delta(\chi,\mathscr{T}\chi)]$$

for all $\xi,\chi\in X$.

Definition 2([27]). *A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is referred to as continuous t-norm if the conditions below hold:*

1. *$a*1=a$ for every a in $[0,1]$;*
2. *$*$ is associative and commutative;*
3. *$a*b\leq i*j$ provided $a\leq i$ and $b\leq j$, where $a,i,b,j\in[0,1]$;*
4. *$*$ is continuous.*

Definition 3([8]). *Let E be a nonempty set, $*$ be a continuous t-norm and Γ be a fuzzy set defined on $E\times E\times(0,\infty)$ such that the following conditions hold:*

1. *$0<\Gamma(\varpi,\omega,\varkappa)$;*
2. *$\Gamma(\varpi,\omega,\varkappa)=1 \iff \varpi=\omega$;*
3. *$\Gamma(\varpi,\omega,\varkappa)=\Gamma(\omega,\varpi,\varkappa)$;*
4. *$\Gamma(\varpi,\vartheta,\varkappa+\varsigma)\geq\Gamma(\varpi,\omega,\varkappa)*\Gamma(\omega,\vartheta,\varsigma)$;*
5. *$\Gamma(\varpi,\omega,\cdot):(0,\infty)\to(0,1]$ is continuous,*

for every $\varpi,\omega,\vartheta\in E$ and any $\varkappa,\varsigma>0$. Then, an ordered triple $(E,\Gamma,*)$ is called a fuzzy metric space.

Lemma 1([13]). *If $(E,\Gamma,*)$ is a fuzzy metric space, then $\Gamma(\varpi,\omega,\varkappa)$ is increasing for any pair of ϖ,ω in E.*

Definition 4([8]). *Let $(E,\Gamma,*)$ be a fuzzy metric space and $\{\varpi_n\}$ be a sequence in E. Then,*

1. *$\{\varpi_n\}$ is convergent provided there exists $x\in E$ satisfies $\lim_{n\to\infty}\Gamma(\varpi_n,x,\varkappa)=1$ for any $\varkappa>0$;*
2. *$\{\varpi_n\}$ is called Cauchy sequence provided that for any $0<\varepsilon<1$ and $\varkappa>0$, there is $n_0\in\mathbb{N}$ satisfies $\Gamma(\varpi_n,\varpi_m,\varkappa)>1-\varepsilon$ for every $n,m\geq n_0$;*
3. *$(E,\Gamma,*)$ is complete whenever each Cauchy sequence in E is convergent.*

Consider $\mathscr{F},\mathscr{G}:E\to E$ where E is a nonempty set and consider an element $\omega\in E$. We say that ω is a fixed point of $\mathscr{F}$ if it satisfies $\mathscr{F}\omega=\omega$. For the case where $\mathscr{F}\omega=\mathscr{G}\omega$, ω is called a coincidence point of $\mathscr{F}$ and $\mathscr{G}$. Moreover, if $\mathscr{F}\omega=\omega=\mathscr{G}\omega$, then ω is known as the common fixed point of $\mathscr{F}$ and $\mathscr{G}$.

Definition 5([15]). *Let E be a nonempty set. Two self-mappings $\mathscr{F},\mathscr{G}:E\to E$ are weakly compatible if both $\mathscr{F}$ and $\mathscr{G}$ commute at the coincidence point of $\mathscr{F}$ and $\mathscr{G}$, for instance, $\mathscr{F}\omega=\mathscr{G}\omega$ for some ω in E implies that $\mathscr{F}\mathscr{G}\omega=\mathscr{G}\mathscr{F}\omega$.*

The following definitions are (E.A.) and CLR property defined on two and four self-mappings. It is notable that definitions below are written under the framework of fuzzy metric space instead of the space where they originally defined.

Definition 6([1]). *For a fuzzy metric space $(E,\Gamma,*)$, a pair $(\mathscr{F},\mathscr{T})$ of self-mappings satisfy the (E.A.) property if there is a sequence $\{\varpi_n\}\subset E$ such that*

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa)=1$$

for some $z\in E$ and for all $\varkappa>0$.

Definition 7([21]). *For a fuzzy metric space $(E,\Gamma,*)$, two pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ of self-mappings satisfy the common (E.A.) property if there are two sequences $\{\varpi_n\},\{\omega_n\} \subset E$ such that*

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{G}\omega_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{S}\omega_n,z,\varkappa)=1$$

for some $z\in E$ and for all $\varkappa>0$.

Definition 8([28]). *For a fuzzy metric space $(E,\Gamma,*)$, a pair $(\mathscr{F},\mathscr{T})$ of self-mappings satisfy the common limit in the range of $\mathscr{T}$ property, denoted by (CLR$_{\mathscr{T}}$) if there is a sequence $\{\varpi_n\}\subset E$ such that*

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa)=1$$

for some $z\in \mathscr{T}E$ and for all $\varkappa>0$.

Definition 9([34]). *For a fuzzy metric space $(E,\Gamma,*)$, two pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ of self-mappings satisfy the common limit in the range of $\mathscr{T}$ and $\mathscr{S}$ property, denoted by (CLR$_{\mathscr{T}\mathscr{S}}$) if there are two sequences $\{\varpi_n\},\{\omega_n\}\subset E$ such that*

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{G}\omega_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{S}\omega_n,z,\varkappa)=1$$

for some $z\in \mathscr{T}E\cap\mathscr{S}E$ and for all $\varkappa>0$.

Definition 10([24]). *For a fuzzy metric space $(E,\Gamma,*)$, assume $\mathscr{F},\mathscr{T}$, and $\mathscr{S}$ are three self-mappings of E. The pair $(\mathscr{F},\mathscr{T})$ satisfy the common limit in the range of $\mathscr{S}$ property, denoted by (CLR$_{(\mathscr{F},\mathscr{T}),\mathscr{S}}$), if there exists sequence $\{\varpi_n\}\subset E$ such that*

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa)=1$$

for some $z\in \mathscr{T}E\cap\mathscr{S}E$ and for all $\varkappa>0$.

*Remark.*Using condition (2) in Definition 3, Definition 6 can be expressed in a way similar to its metric counterpart, that is, the pair $(\mathscr{F},\mathscr{T})$ satisfies the (E.A.) property if there is a sequence $\{\varpi_n\}\subset E$ such that for some $z\in E$,

$$\lim_{n\to\infty}\mathscr{F}\varpi_n=\lim_{n\to\infty}\mathscr{T}\varpi_n=z.$$

This is applicable to Definitions 7, 8, 9, and 10 as well.

By setting $\mathscr{F}=\mathscr{G}$ and $\mathscr{T}=\mathscr{S}$ in Definition 7 and Definition 9, one can obtain Definition 6 and Definition 8, respectively. Moreover, we can see that Definition 9 implies Definition 10, but this is not the case for converse. This is shown in the examples below.

*Example 1.*Suppose $(E,\Gamma,*)$ is a fuzzy metric space where $E=[0,\infty)$, Γ is a fuzzy set on $E\times E\times(0,\infty)$ and $*$ is a continuous t-norm. In addition, consider $\mathscr{F},\mathscr{G},\mathscr{T},\mathscr{S}:E\to E$ expressed as:

$$\mathscr{F}(\varpi)=\frac{7\varpi}{8},$$
$$\mathscr{G}(\varpi)=\varpi^2,$$
$$\mathscr{T}(\varpi)=\frac{\varpi}{8},$$
$$\mathscr{S}(\varpi)=5\varpi^2.$$

We have $\mathscr{T}E\cap\mathscr{S}E=[0,\infty)$. Define sequences $\{\varpi_n\}=\{\frac{1}{n}\}$ and $\{\omega_n\}=\{\frac{1}{n^2}\}$ for every $n\in\mathbb{N}$. Considering that

$$\lim_{n\to\infty}\mathscr{F}\varpi_n=\lim_{n\to\infty}\mathscr{T}\varpi_n=\lim_{n\to\infty}\mathscr{G}\omega_n=\lim_{n\to\infty}\mathscr{S}\omega_n=0$$

and $0\in\mathscr{T}E\cap\mathscr{S}E$, both $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ satisfy the (CLR$_{\mathscr{T}\mathscr{S}}$) property. Moreover, $(\mathscr{F},\mathscr{T})$ satisfy (CLR$_{(\mathscr{F},\mathscr{T}),\mathscr{S}}$) property.

Example 2. Suppose $(E,\Gamma,*)$ is a fuzzy metric space where $E=[0,\infty)$, Γ is a fuzzy set on $E\times E\times(0,\infty)$ and $*$ is a continuous t-norm. Furthermore, consider $\mathscr{F},\mathscr{G},\mathscr{T},\mathscr{S}:E\to E$ expressed as:

$$\begin{aligned}\mathscr{F}(\varpi)&=\varpi+2,\\ \mathscr{G}(\varpi)&=\frac{\varpi+1}{2},\\ \mathscr{T}(\varpi)&=3\varpi,\\ \mathscr{S}(\varpi)&=\varpi+3.\end{aligned}$$

We have $\mathscr{T}E=[0,\infty)$ and $\mathscr{S}E=[3,\infty)$ which implies $\mathscr{T}E\cap\mathscr{S}E=[3,\infty)$. Consider a sequence $\{\varpi_n\}=\{\frac{n+1}{n}\}$. It is clear that

$$\lim_{n\to\infty}\mathscr{F}\varpi_n=\lim_{n\to\infty}\mathscr{T}\varpi_n=3$$

and $3\in\mathscr{T}E\cap\mathscr{S}E$. Thus, the pair $(\mathscr{F},\mathscr{T})$ satisfy the (CLR$_{\mathscr{F},\mathscr{T}),\mathscr{S}}$) property.

If we let sequence $\{\omega_n\}=\{\frac{1}{n}\}$, we get

$$\lim_{n\to\infty}\mathscr{G}\omega_n=\frac{1}{2}\text{ and }\lim_{n\to\infty}\mathscr{S}\omega_n=3$$

which means that $\lim_{n\to\infty}\mathscr{G}\omega_n\neq\lim_{n\to\infty}\mathscr{S}\omega_n$. This concludes both $(\mathscr{F},\mathscr{T})$, $(\mathscr{G},\mathscr{S})$ do not satisfy (CLR$_{\mathscr{T}\mathscr{S}}$) property.

The function below will be utilized in our later results.

Definition 11. *A mapping $\psi:[0,1]\times[0,1]\to[0,1]$ is called as Ψ-function if:*

1. *$\psi(u,v)$ is monotonically nondecreasing in both u and v variables;*
2. *$\psi(u,v)$ is lower semicontinuous in both u and v variables;*
3. *$\psi(v,v)>v$ for every $v\in(0,1)$;*
4. *$\psi(1,1)=1$ and $\psi(0,0)=0$.*

Ψ_f is denoted as the collection of all Ψ-functions. Examples of Ψ-functions are $\psi(u,v)=\frac{k\sqrt{u}+l\sqrt{v}}{k+l}$ where $k,l\in\mathbb{R}^+$, $\psi(u,v)=\sqrt{uv}$, and $\psi(u,v)=\min\{u,v\}$ for all $u,v\in[0,1]$.

3 Main Results

Theorem 1. *Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{S},\mathscr{T}$ are self-mappings of E satisfying the following condition:*

$$\begin{aligned}&\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa)+h(1-\max\{\Gamma(\mathscr{T}\varpi,\mathscr{G}\omega,\varkappa),\Gamma(\mathscr{S}\omega,\mathscr{F}\varpi,\varkappa),\Gamma(\mathscr{T}\varpi,\mathscr{S}\omega,\varkappa)\})\\ &\geq\psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)\end{aligned}\tag{1}$$

for any $\varpi,\omega\in E$ and $\varkappa>0$ where $h\geq0,\varkappa,\varkappa_1,\varkappa_2>0$ with $\varkappa=\varkappa_1+\varkappa_2$, $p,q>0$ with $p+q\in(0,1)$ and $\psi\in\Psi_f$. Assume that both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ satisfy the (CLR$_{\mathscr{T}\mathscr{S}}$) property, then the pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ have a coincidence point in E.

Proof. Given that both pairs $(\mathscr{F},\mathscr{T})$, $(\mathscr{G},\mathscr{S})$ satisfy the (CLR$_{\mathscr{T}\mathscr{S}}$) property, there exist sequences $\{\varpi_n\}$ and $\{\omega_n\}$ in E such that for all $\varkappa>0$,

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{G}\omega_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{S}\omega_n,z,\varkappa)=1$$

for some $z\in\mathscr{T}E\cap\mathscr{S}E$. This means that

$$\lim_{n\to\infty}\mathscr{F}\varpi_n=\lim_{n\to\infty}\mathscr{T}\varpi_n=\lim_{n\to\infty}\mathscr{G}\omega_n=\lim_{n\to\infty}\mathscr{S}\omega_n=z.$$

As $z\in\mathscr{T}E$, one can find an element $u\in E$ satisfy $z=\mathscr{T}u$. We will show that $\mathscr{F}u=\mathscr{T}u$. Assume $\mathscr{F}u\neq\mathscr{T}u$, which means, $0<\Gamma(\mathscr{F}u,\mathscr{T}u,\varkappa)<1$ for some $\varkappa>0$. Using inequality (1), for all $\varkappa>0$, we yield

$$\begin{aligned}&\Gamma(\mathscr{F}u,\mathscr{G}\omega_n,\varkappa)+h(1-\max\{\Gamma(\mathscr{T}u,\mathscr{G}\omega_n,\varkappa),\Gamma(\mathscr{S}\omega_n,\mathscr{F}u,\varkappa),\Gamma(\mathscr{T}u,\mathscr{S}\omega_n,\varkappa)\})\\ &\geq\psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega_n,\mathscr{G}\omega_n,\frac{\varkappa_2}{q}\right)\right).\end{aligned}\tag{2}$$

Let $\varkappa_1 = \frac{p\varkappa}{p+q}, \varkappa_2 = \frac{q\varkappa}{p+q}$ and $r = p+q$. Clearly, we have $\frac{\varkappa_1}{p} = \frac{\varkappa_2}{q} = \frac{\varkappa}{r}$ and $0 < r < 1$. Then, from (2) we can obtain the following:

$$\begin{aligned}&\Gamma(\mathscr{F}u,\mathscr{G}\omega_n,\varkappa)+h(1-\max\{\Gamma(\mathscr{T}u,\mathscr{G}\omega_n,\varkappa),\Gamma(\mathscr{S}\omega_n,\mathscr{F}u,\varkappa),\Gamma(\mathscr{T}u,\mathscr{S}\omega_n,\varkappa)\})\\ &\geq \psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{S}\omega_n,\mathscr{G}\omega_n,\frac{\varkappa}{r}\right)\right).\end{aligned}$$

By taking the limit as $n\to\infty$, we yield

$$\begin{aligned}&\Gamma(\mathscr{F}u,z,\varkappa)+h(1-\max\{\Gamma(\mathscr{T}u,z,\varkappa),\Gamma(z,\mathscr{F}u,\varkappa),\Gamma(\mathscr{T}u,z,\varkappa)\})\\ &\geq \psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right),\Gamma\left(z,z,\frac{\varkappa}{r}\right)\right)\\ &= \psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right),1\right).\end{aligned}$$

Since $z = \mathscr{T}u$, the inequality above can be rewritten as

$$\begin{aligned}\Gamma(\mathscr{F}u,\mathscr{T}u,\varkappa)+h(1-\max\{1,\Gamma(z,\mathscr{F}u,\varkappa),1\}) &\geq \psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right),1\right)\\ \Gamma(\mathscr{F}u,\mathscr{T}u,\varkappa)+h(1-1) &\geq \psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right),1\right)\\ \Gamma(\mathscr{F}u,\mathscr{T}u,\varkappa) &\geq \psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right),1\right).\end{aligned}$$

By Ψ-function's properties and Lemma 1, we yield

$$\begin{aligned}\Gamma(\mathscr{F}u,\mathscr{T}u,\varkappa) &\geq \psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right),1\right)\\ &\geq \psi\left(\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right)\right)\\ &> \Gamma\left(\mathscr{T}u,\mathscr{F}u,\frac{\varkappa}{r}\right)\\ &> \Gamma(\mathscr{T}u,\mathscr{F}u,\varkappa)\\ &= \Gamma(\mathscr{F}u,\mathscr{T}u,\varkappa)\end{aligned}$$

which leads to a contradiction. As a result, $\Gamma(\mathscr{F}u,\mathscr{T}u,\varkappa) = 1$ for each $\varkappa > 0$. By the condition (2) from Definition 3, we yield $\mathscr{F}u = \mathscr{T}u = z$. This implies that point u is a coincidence point of the pair $(\mathscr{F},\mathscr{T})$.

Additionally, since $z \in \mathscr{S}E$, one can find an element $v \in E$ satisfy $z = \mathscr{S}v$. We will show that $\mathscr{G}v = \mathscr{S}v$. Assume $\mathscr{G}v \neq \mathscr{S}v$, which means, $0 < \Gamma(\mathscr{G}v,\mathscr{S}v,\varkappa) < 1$ for some $\varkappa > 0$. Using inequality (1), for each $\varkappa > 0$, it follows that

$$\begin{aligned}&\Gamma(\mathscr{F}\varpi_n,\mathscr{G}v,\varkappa)+h(1-\max\{\Gamma(\mathscr{T}\varpi_n,\mathscr{G}v,\varkappa),\Gamma(\mathscr{S}v,\mathscr{F}\varpi_n,\varkappa),\Gamma(\mathscr{T}\varpi_n,\mathscr{S}v,\varkappa)\})\\ &\geq \psi\left(\Gamma\left(\mathscr{T}\varpi_n,\mathscr{F}\varpi_n,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa_2}{q}\right)\right).\end{aligned} \tag{3}$$

Again let $\varkappa_1 = \frac{p\varkappa}{p+q}, \varkappa_2 = \frac{q\varkappa}{p+q}$ and $r = p+q$. Then, from (3) we can obtain the following:

$$\begin{aligned}&\Gamma(\mathscr{F}\varpi_n,\mathscr{G}v,\varkappa)+h(1-\max\{M(\mathscr{T}\varpi_n,\mathscr{G}v,\varkappa),\Gamma(\mathscr{S}v,\mathscr{F}\varpi_n,\varkappa),\Gamma(\mathscr{T}\varpi_n,\mathscr{S}v,\varkappa)\})\\ &\geq \psi\left(\Gamma\left(\mathscr{T}\varpi_n,\mathscr{F}\varpi_n,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\right).\end{aligned}$$

By taking the limit as $n\to\infty$, we yield

$$\begin{aligned}&\Gamma(z,\mathscr{G}v,\varkappa)+h(1-\max\{\Gamma(z,\mathscr{G}v,\varkappa),\Gamma(\mathscr{S}v,z,\varkappa),\Gamma(z,\mathscr{S}v,\varkappa)\})\\ &\geq \psi\left(\Gamma\left(z,z,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\right)\\ &= \psi\left(1,\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\right).\end{aligned}$$

Since $z = \mathscr{S}v$, the inequality above can be rewritten as

$$\begin{aligned}\Gamma(\mathscr{S}v,\mathscr{G}v,\varkappa)+h(1-\max\{\Gamma(\mathscr{S}v,\mathscr{G}v,\varkappa),1,1\}) &\geq \psi\left(1,\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\right)\\ \Gamma(\mathscr{S}v,\mathscr{G}v,\varkappa)+h(1-1) &\geq \psi\left(1,\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\right)\\ \Gamma(\mathscr{S}v,\mathscr{G}v,\varkappa) &\geq \psi\left(1,\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\right).\end{aligned}$$

Due to Ψ-function's properties and Lemma 1, we yield

$$\begin{aligned}\Gamma(\mathscr{S}v,\mathscr{G}v,\varkappa) &\geq \psi\left(1,\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\right)\\ &\geq \psi\left(\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\right)\\ &> \Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa}{r}\right)\\ &> \Gamma(\mathscr{S}v,\mathscr{G}v,\varkappa)\end{aligned}$$

which leads to a contradiction. As a result, $\Gamma(\mathscr{G}v,\mathscr{S}v,\varkappa)=1$ for each $\varkappa>0$. By using the condition (2) from Definition 3, we yield $\mathscr{G}v=\mathscr{S}v=z$. So v is a coincidence point of the pair $(\mathscr{G},\mathscr{S})$.

*Remark.*It is possible to obtain Theorem 2.2 in Choudhury et al. [5] if we let $\mathscr{F}=\mathscr{G}$, $\mathscr{T}=\mathscr{S}$ and $\max\{\Gamma(\mathscr{T}\varpi,\mathscr{G}\omega,\varkappa),\Gamma(\mathscr{S}\omega,\mathscr{F}\varpi,\varkappa),\Gamma(\mathscr{T}\varpi,\mathscr{S}\omega,\varkappa)\}=\max\{\Gamma(\mathscr{T}\varpi,\mathscr{G}\omega,\varkappa),\Gamma(\mathscr{S}\omega,\mathscr{F}\varpi,\varkappa)\}$ in our Theorem 1 above. In addition to that, they require the fuzzy metric space to be equipped with Hadzic type t-norm, whereas in our result the t-norm for fuzzy metric space picked is arbitrary. Hence, our results improvises their results without t-norm restriction and completeness on fuzzy metric space.

We deduce the subsequent corollary from Theorem 1.

Corollary 1.*Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{S},\mathscr{T}$ are self-mappings of E satisfying the following condition:*

$$\begin{aligned}&\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa)+h(1-\max\{\Gamma(\mathscr{T}\varpi,\mathscr{G}\omega,\varkappa),\Gamma(\mathscr{S}\omega,\mathscr{F}\varpi,\varkappa),\Gamma(\mathscr{T}\varpi,\mathscr{S}\omega,\varkappa)\})\\ &\geq \psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)\end{aligned}\tag{4}$$

for any $\varpi,\omega\in E$ and $\varkappa>0$ where $h\geq 0,\varkappa,\varkappa_1,\varkappa_2>0$ with $\varkappa=\varkappa_1+\varkappa_2$, $p,q>0$ with $p+q\in(0,1)$ and $\psi\in\Psi_f$. Assume that $\mathscr{T}E$, $\mathscr{S}E$ are closed subsets of E and the pairs $(\mathscr{F},\mathscr{T})$, $(\mathscr{G},\mathscr{S})$ satisfy common (E.A.) property, then both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ have a coincidence point.

*Proof.*As both pairs $(\mathscr{F},\mathscr{T}),(\mathscr{G},\mathscr{S})$ fulfill common (E.A.) property, we have some sequences $\{\varpi_n\},\{\omega_n\}\subset E$ such that for all $\varkappa>0$,

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{G}\omega_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{S}\omega_n,z,\varkappa)=1$$

for some z in E. This means that

$$\lim_{n\to\infty}\mathscr{F}\varpi_n=\lim_{n\to\infty}\mathscr{S}\omega_n=\lim_{n\to\infty}\mathscr{G}\omega_n=\lim_{n\to\infty}\mathscr{T}\varpi_n=z.$$

Given that $\mathscr{T}E$ is closed set, there is an element $u\in E$ satisfy $z=\mathscr{T}u$. Moreover, since $\mathscr{S}E$ is closed, we can identify an element $v\in E$ satisfy $z=\mathscr{S}v$. Hence, $z\in\mathscr{T}E\cap\mathscr{S}E$. This concludes that both pairs $(\mathscr{F},\mathscr{T}),(\mathscr{G},\mathscr{S})$ satisfy the (CLR$_{\mathscr{T}\mathscr{S}}$) property. The remaining of this proof follows from Theorem 1.

Theorem 2.*Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{S},\mathscr{T}$ are self-mappings of E satisfying the following condition:*

$$\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa)\geq\psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)\tag{5}$$

for all $\varpi,\omega\in E$ and $\varkappa>0$ where $t_1,t_2>0$ with $\varkappa=\varkappa_1+\varkappa_2$, $p,q>0$ with $p+q\in(0,1)$ and $\psi\in\Psi_f$. Assume that both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ satisfy the (CLR$_{\mathscr{T}\mathscr{S}}$) property, then both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ have a coincidence point. Furthermore, if both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ are weakly compatible, this implies that mappings $\mathscr{F},\mathscr{T},\mathscr{G},\mathscr{S}$ have a unique common fixed point in E.

*Proof.*To show both pairs $(\mathscr{F},\mathscr{T}),(\mathscr{G},\mathscr{S})$ possess a coincidence point, consider $h=0$ in (1) and the proof follows as in Theorem 1.

For the rest of the Theorem, as $(\mathscr{F},\mathscr{T})$ is weakly compatible and $\mathscr{F}u=\mathscr{T}u=z$, it follows that $\mathscr{T}z=\mathscr{T}\mathscr{F}u=\mathscr{F}\mathscr{T}u=\mathscr{F}z$. We say that point z is the common fixed point of $(\mathscr{F},\mathscr{T})$. Using (5) and Ψ-function's property, for each

$\varkappa > 0$, we yield

$$\begin{aligned}\Gamma(\mathscr{F}z,z,\varkappa)=\Gamma(\mathscr{F}z,\mathscr{G}v,\varkappa)&\geq\psi\left(\Gamma\left(\mathscr{T}z,\mathscr{F}z,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}v,\mathscr{G}v,\frac{\varkappa_2}{q}\right)\right)\\&=\psi\left(\Gamma\left(\mathscr{F}z,\mathscr{F}z,\frac{\varkappa_1}{p}\right),\Gamma\left(z,z,\frac{\varkappa_2}{q}\right)\right)\\&=\psi(1,1)\\&=1.\end{aligned}$$

Thus, $\Gamma(\mathscr{F}z,z,\varkappa)=1$ for each $\varkappa>0$, which means, $\mathscr{F}z=\mathscr{T}z=z$. So, z is a common fixed point of $\mathscr{F}$ and $\mathscr{T}$.

Also, since $(\mathscr{G},\mathscr{S})$ is weakly compatible and $\mathscr{G}v=\mathscr{S}v=z$, this implies that $\mathscr{S}z=\mathscr{S}\mathscr{G}v=\mathscr{G}\mathscr{S}v=\mathscr{G}z$. We say that point z is a common fixed point of pair $(\mathscr{G},\mathscr{S})$. Using (5) and Ψ-function's property, for each $\varkappa>0$, it follows that

$$\begin{aligned}\Gamma(z,\mathscr{G}z,\varkappa)=\Gamma(\mathscr{F}z,\mathscr{G}z,\varkappa)&\geq\psi\left(\Gamma\left(\mathscr{T}z,\mathscr{F}z,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}z,\mathscr{G}z,\frac{\varkappa_2}{q}\right)\right)\\&=\psi\left(\Gamma\left(z,z,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{G}z,\mathscr{G}z,\frac{\varkappa_2}{q}\right)\right)\\&=\psi(1,1)\\&=1.\end{aligned}$$

As a result, $\Gamma(z,\mathscr{G}z,\varkappa)=1$ for every $\varkappa>0$, which means, $\mathscr{G}z=z=\mathscr{S}z$. Thus, z is a common fixed point of pair $(\mathscr{G},\mathscr{S})$. This shows that z is a common fixed point of mappings $\mathscr{F},\mathscr{G},\mathscr{T},\mathscr{S}$.

For the uniqueness, assume two common fixed points $z_1,\ z_2\in E$ are distinct, for instance, $0<\Gamma(z_1,z_2,\varkappa)<1$ for some $\varkappa>0$. Using (5), for any $\varkappa>0$, we get

$$\begin{aligned}\Gamma(z_1,z_2,\varkappa)&=\Gamma(\mathscr{F}z_1,\mathscr{G}z_2,\varkappa)\\&\geq\psi\left(\Gamma\left(\mathscr{T}z_1,\mathscr{F}z_1,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}z_2,\mathscr{G}z_2,\frac{\varkappa_2}{q}\right)\right)\\&=\psi\left(\Gamma\left(z_1,z_1,\frac{\varkappa_1}{p}\right),\Gamma\left(z_2,z_2,\frac{\varkappa_2}{q}\right)\right)\\&=\psi(1,1)\\&=1\end{aligned}$$

which is contradict to our assumption. Thus, $z_1=z_2$ which proves the common fixed point is unique.

By substituting $\mathscr{G}$ with $\mathscr{F}$ and $\mathscr{S}$ with $\mathscr{T}$ in the theorem above, we deduce the subsequent corollary.

Corollary 2.*Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{T}$ are self-mappings of E satisfying the following condition:*

$$\Gamma(\mathscr{F}\varpi,\mathscr{F}\omega,\varkappa)\geq\psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{T}\omega,\mathscr{F}\omega,\frac{\varkappa_2}{q}\right)\right)$$

for all $\varpi,\omega\in E$ and $\varkappa>0$ where $\varkappa_1,\varkappa_2>0$ with $\varkappa=\varkappa_1+\varkappa_2$, $p,q>0$ with $p+q\in(0,1)$ and $\psi\in\Psi_f$. Consider the pair $(\mathscr{F},\mathscr{T})$ satisfies ($CLR_{\mathscr{T}}$) property, then the pair $(\mathscr{F},\mathscr{T})$ has a coincidence point. Furthermore, if the pair $(\mathscr{F},\mathscr{T})$ is weakly compatible, this implies that both mappings $\mathscr{F}$ and $\mathscr{T}$ have a unique common fixed point.

Theorem 3.*Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{S},\mathscr{T}$ are self-mappings of E satisfying the following condition:*

$$\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa)\geq\psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)$$

for all $\varpi,\omega\in E$ and $\varkappa>0$ where $\varkappa_1,\varkappa_2>0$ with $\varkappa=\varkappa_1+\varkappa_2$, $p,q>0$ with $p+q\in(0,1)$ and $\psi\in\Psi_f$. Assume that $\mathscr{T}E$ and $\mathscr{S}E$ are closed subsets of E and the pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ satisfy common (E.A.) property, then both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ have a coincidence point. Furthermore, if both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ are weakly compatible, this implies that mappings $\mathscr{F},\mathscr{T},\mathscr{G},\mathscr{S}$ have a unique common fixed point.

Proof. As both $(\mathscr{F},\mathscr{T}),(\mathscr{G},\mathscr{S})$ satisfy common (E.A.) property, there exist $\{\varpi_n\},\{\omega_n\}\subset E$ such that for all $\varkappa>0$,

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{G}\omega_n,z,\varkappa)=\lim_{n\to\infty}\Gamma(\mathscr{S}\omega_n,z,\varkappa)=1$$

for some $z\in X$. This means that

$$\lim_{n\to\infty}\mathscr{F}\varpi_n=\lim_{n\to\infty}\mathscr{S}\omega_n=\lim_{n\to\infty}\mathscr{G}\omega_n=\lim_{n\to\infty}\mathscr{T}\varpi_n=z.$$

As $\mathscr{T}E$ is closed, there is an element $u\in E$ satisfy $z=\mathscr{T}u$. Moreover, since $\mathscr{S}E$ is closed, there is an element $v\in E$ satisfy $z=\mathscr{S}v$. Hence, $z\in\mathscr{T}E\cap\mathscr{S}E$ which means that both $(\mathscr{F},\mathscr{T}),(\mathscr{G},\mathscr{S})$ satisfy (CLR$_{\mathscr{T}\mathscr{S}}$) property. The rest of the proof follows from Theorem 2.

By substituting $\mathscr{G}$ with $\mathscr{F}$ and $\mathscr{S}$ with $\mathscr{T}$ in Theorem above, we obtain corollary below.

Corollary 3. *Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{T}$ are self-mappings of E satisfying the following condition:*

$$\Gamma(\mathscr{F}\varpi,\mathscr{F}\omega,\varkappa)\geq\psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{T}\omega,\mathscr{F}\omega,\frac{\varkappa_2}{q}\right)\right)$$

for all $\varpi,\omega\in E$ and $\varkappa>0$ where $\varkappa_1,\varkappa_2>0$ with $\varkappa=\varkappa_1+\varkappa_2$, $p,q>0$ with $p+q\in(0,1)$ and $\psi\in\Psi_f$. Assume that the pair $(\mathscr{F},\mathscr{T})$ satisfies the (E.A.) property, then the pair $(\mathscr{F},\mathscr{T})$ has a coincidence point. Furthermore, if the pair $(\mathscr{F},\mathscr{T})$ is weakly compatible, then mappings $\mathscr{F}$ and $\mathscr{T}$ have a unique common fixed point.

We will present an example below to demonstrate our Theorem 2.

Example 3. Suppose that $(E,\Gamma,*)$ is a fuzzy metric space with $E=[2,8)$, $*$ is a product continuous t-norm, that is, $a*b=ab$ for any $a,b\in[0,1]$ and $\Gamma(\varpi,\omega,\varkappa)=\frac{\varkappa}{\varkappa+|\varpi-\omega|}$ for every $\varpi,\omega\in E$, $\varkappa>0$. Let $\mathscr{F},\mathscr{G},\mathscr{T},\mathscr{S}:E\to E$ define as follows:

$$\mathscr{F}(\varpi)=\begin{cases}2 & \text{if } \varpi\in\{2\}\cup(7,8),\\ 2.3 & \text{if } \varpi\in(2,7],\end{cases}$$

$$\mathscr{G}(\varpi)=\begin{cases}2 & \text{if } \varpi\in\{2\}\cup(7,8),\\ 2.5 & \text{if } \varpi\in(2,7],\end{cases}$$

$$\mathscr{T}(\varpi)=\begin{cases}2 & \text{if } \varpi\in\{2\},\\ 4 & \text{if } \varpi\in(2,7),\\ 5 & \text{if } \varpi\in\{7\},\\ \frac{\varpi+3}{5} & \text{if } \varpi\in(7,8),\end{cases}$$

$$\mathscr{S}(\varpi)=\begin{cases}2 & \text{if } \varpi\in\{2\},\\ 6 & \text{if } \varpi\in(2,7),\\ 7 & \text{if } \varpi\in\{7\},\\ \frac{\varpi+3}{5} & \text{if } \varpi\in(7,8),\end{cases}$$

and $\psi(u,v)=\sqrt{uv}$ where $u,v\in E$. One can easily validate that inequality (5) is satisfied for every ϖ,ω in E and for all $\varkappa>0$. Now, we pick sequences $\{\varpi_n\}=\left\{7+\frac{1}{n}\right\}$ and $\{\omega_n\}=\{2\}$. It is clear that we have

$$\lim_{n\to\infty}\mathscr{F}\varpi_n=\lim_{n\to\infty}\mathscr{T}\varpi_n=\lim_{n\to\infty}\mathscr{G}\omega_n=\lim_{n\to\infty}\mathscr{S}\omega_n=2.$$

Since $2\in\mathscr{T}E\cap\mathscr{S}E$, it implies that both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ satisfy (CLR$_{\mathscr{T}\mathscr{S}}$) property. Furthermore, it is straightforward to verify that both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ are weakly compatible. Hence, each conditions of Theorem 2 hold. Furthermore, 2 is the unique common fixed point of $\mathscr{F},\mathscr{G},\mathscr{T}$ and $\mathscr{S}$. Figures 1, 2 and 3 provide a visual representation of the inequality with specific assigned values.

Remark. It is obvious that Theorem 3 cannot be applied on example above because both $\mathscr{T}E$, $\mathscr{S}E\subset E$ are not closed.

Before we proceed further, we present two lemmas that are needed the next results related to four mappings but only two mappings satisfying (CLR) or (E.A.) property, respectively.

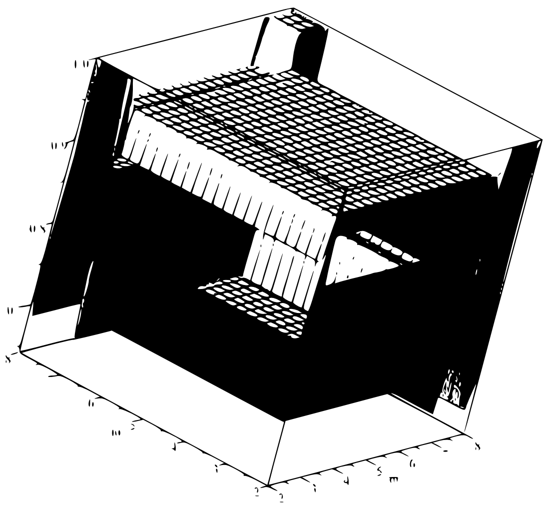

Fig. 1: Graphical view of inequality $\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)$, where the orange plane represents the left-hand side and the blue plane represents the right-hand side, with specific values assigned as follows: $\varkappa = 5, \varkappa_1 = 3, \varkappa_2 = 2, p = 0.5$, and $q = 0.3$.

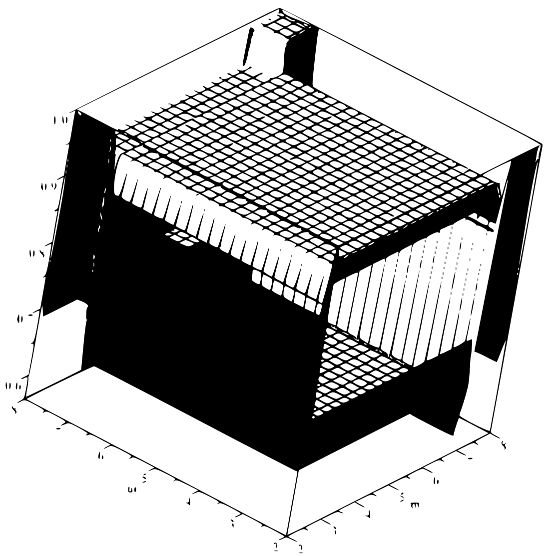

Fig. 2: Graphical view of inequality $\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)$, where the orange plane represents the left-hand side and the blue plane represents the right-hand side, with specific values assigned as follows: $\varkappa = 5, \varkappa_1 = 1, \varkappa_2 = 4, p = 0.5$, and $q = 0.3$.

Lemma 2. *Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{T},\mathscr{S}$ are four self-mappings of E such that the following conditions hold:*

1. the pair $(\mathscr{F},\mathscr{T})$ (or $(\mathscr{G},\mathscr{S})$) satisfies the $(CLR_{\mathscr{T}})$ (or $(CLR_{\mathscr{S}})$) property;

2. $\mathscr{F}E \subset \mathscr{S}E$ (or $\mathscr{G}E \subset \mathscr{T}E$);

3. $\mathscr{S}E \subset E$ closed;

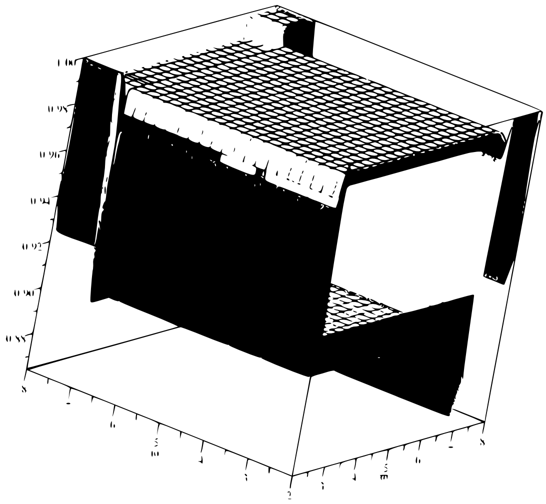

Fig. 3: Graphical view of inequality $\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)$, where the orange plane represents the left-hand side and the blue plane represents the right-hand side, with specific values assigned as follows: $\varkappa = 30, \varkappa_1 = 5, \varkappa_2 = 25, p = 0.5$, and $q = 0.3$.

4. *$\{\mathscr{G}\omega_n\}$ converges for all sequences $\{\omega_n\}$ in E provided $\{\mathscr{S}\omega_n\}$ converges (or $\{\mathscr{F}\omega_n\}$ converges for all sequences $\{\omega_n\}$ in E provided $\{\mathscr{T}\omega_n\}$ converges);*

5. *$\mathscr{F},\mathscr{G},\mathscr{T}$ and $\mathscr{S}$ satisfy inequality (5) for every $\varpi,\omega \in E$ and any $\varkappa > 0$.*

Then, both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ satisfy the ($CLR_{\mathscr{T}\mathscr{S}}$) property.

Proof. As $(\mathscr{F},\mathscr{T})$ satisfy ($CLR_{\mathscr{T}}$) property, there is $\{\varpi_n\} \subset E$ satisfy

$$\lim_{n\to\infty} \Gamma(\mathscr{F}\varpi_n, z, \varkappa) = \lim_{n\to\infty} \Gamma(\mathscr{T}\varpi_n, z, \varkappa) = 1$$

for some $z \in \mathscr{T}E$. This means that

$$\lim_{n\to\infty} \mathscr{F}\varpi_n = \lim_{n\to\infty} \mathscr{T}\varpi_n = z.$$

Since $\mathscr{F}E \subset \mathscr{S}E$, for each ϖ_n, there is an element $\omega_n \in E$ satisfy $\mathscr{F}\varpi_n = \mathscr{S}\omega_n$ for every $n \in \mathbb{N}$. Thus, we yield

$$\lim_{n\to\infty} \mathscr{F}\varpi_n = \lim_{n\to\infty} \mathscr{S}\omega_n = z.$$

Since $\mathscr{S}E$ is closed, the convergent point z is in $\mathscr{S}E$. Therefore we have $z \in \mathscr{T}E \cap \mathscr{S}E$ and

$$\mathscr{F}\varpi_n \to z, \mathscr{T}\varpi_n \to z, \text{ and } \mathscr{S}\omega_n \to z$$

as we let $n \to \infty$. Due to condition (4), sequence $\{\mathscr{G}\omega_n\}$ converges, which means, there is a point $\theta \in E$ satisfy

$$\lim_{n\to\infty} \mathscr{G}\omega_n = \theta.$$

We claim that $\theta = z$. Otherwise, let $\theta \neq z$. This implies that $0 < \Gamma(\theta, z, \varkappa) < 1$ for every $\varkappa > 0$. Using inequality (5), we have

$$\Gamma(\mathscr{F}\varpi_n, \mathscr{G}\omega_n, \varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi_n, \mathscr{F}\varpi_n, \frac{\varkappa_1}{p}\right), \Gamma\left(\mathscr{S}\omega_n, \mathscr{G}\omega_n, \frac{\varkappa_2}{q}\right)\right)$$

where $\varkappa_1, \varkappa_2 > 0$ with $\varkappa = \varkappa_1 + \varkappa_2$, $p, q > 0$ with $p + q \in (0,1)$ and $\psi \in \Psi_f$. Let $\varkappa_1 = \frac{p\varkappa}{p+q}, \varkappa_2 = \frac{q\varkappa}{p+q}$ and $r = p + q$, the inequality above can be rewrite as

$$\Gamma(\mathscr{F}\varpi_n, \mathscr{G}\omega_n, \varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi_n, \mathscr{F}\varpi_n, \frac{\varkappa}{r}\right), \Gamma\left(\mathscr{S}\omega_n, \mathscr{G}\omega_n, \frac{\varkappa}{r}\right)\right).$$

Let $n \to \infty$,

$$\begin{aligned}\Gamma(z,\theta,\varkappa) &\geq \psi\left(\Gamma\left(z,z,\frac{\varkappa}{r}\right),\Gamma\left(z,\theta,\frac{\varkappa}{r}\right)\right)\\ &= \psi\left(1,\Gamma\left(z,\theta,\frac{\varkappa}{r}\right)\right).\end{aligned}$$

Due to the properties of Ψ-function and Lemma 1, it follows that

$$\begin{aligned}\Gamma(z,\theta,\varkappa) &\geq \psi\left(1,\Gamma\left(z,\theta,\frac{\varkappa}{r}\right)\right)\\ &\geq \psi\left(\Gamma\left(z,\theta,\frac{\varkappa}{r}\right),\Gamma\left(z,\theta,\frac{\varkappa}{r}\right)\right)\\ &> \Gamma\left(z,\theta,\frac{\varkappa}{r}\right)\\ &> \Gamma(z,\theta,\varkappa)\end{aligned}$$

which leads to a contradiction. As a result, $\Gamma(w,z,\varkappa) = 1$ for any $\varkappa > 0$ which means $\theta = z$. Hence, we conclude

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{G}\omega_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{S}\omega_n,z,\varkappa) = 1$$

which means that both $(\mathscr{F},\mathscr{T}),(\mathscr{G},\mathscr{S})$ satisfy (CLR$_{\mathscr{T}\mathscr{S}}$) property.

Lemma 3. *Supose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{T},\mathscr{S}$ are four self-mappings of E such that the following conditions hold:*

1. the pair $(\mathscr{F},\mathscr{T})$ (or $(\mathscr{G},\mathscr{S})$) satisfies (E.A.) property;

2. $\mathscr{F}E \subset \mathscr{S}E$ (or $\mathscr{G}E \subset \mathscr{T}E$);

3. $\{\mathscr{G}\omega_n\}$ converges for all sequences $\{\omega_n\}$ in E provided $\{\mathscr{S}\omega_n\}$ converges (or $\{\mathscr{F}\omega_n\}$ converges for all sequences $\{\omega_n\}$ in E provided $\{\mathscr{T}\omega_n\}$ converges);

4. $\mathscr{F},\mathscr{G},\mathscr{T}$ and $\mathscr{S}$ satisfy inequality (5) for every $\varpi,\omega \in E$ and any $\varkappa > 0$.

Then, both pairs $(\mathscr{F},\mathscr{T})$, $(\mathscr{G},\mathscr{S})$ satisfy common (E.A.) property.

Proof. The proof is similar to Lemma 2 so we omit here to avoid repetition.

Theorem 4. *Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{S},\mathscr{T}$ are self-mappings of E satisfy the following conditions:*

1. the pair $(\mathscr{F},\mathscr{T})$ (or $(\mathscr{G},\mathscr{S})$) satisfies the (CLR$_{\mathscr{T}}$) (or (CLR$_{\mathscr{S}}$)) property;

2. $\mathscr{F}E \subset \mathscr{S}E$ (or $\mathscr{G}E \subset \mathscr{T}E$);

3. $\{\mathscr{G}\omega_n\}$ converges for all sequences $\{\omega_n\}$ in E provided $\{\mathscr{S}\omega_n\}$ converges (or $\{\mathscr{F}\omega_n\}$ converges for all sequences $\{\omega_n\}$ in E provided $\{\mathscr{T}\omega_n\}$ converges);

4. mappings $\mathscr{F},\mathscr{G},\mathscr{T}$ and $\mathscr{S}$ satisfy

$$\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)$$

for every $\varpi,\omega \in E$ and any $\varkappa > 0$ where $\varkappa_1,\varkappa_2 > 0$ with $\varkappa = \varkappa_1 + \varkappa_2$, $p,q > 0$ with $p+q \in (0,1)$ and $\psi \in \Psi_f$.

Then, both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ have a coincidence point. Furthermore, if both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ are weakly compatible, this implies that mappings $\mathscr{F},\mathscr{T},\mathscr{G},\mathscr{S}$ have a unique common fixed point.

Proof. By Lemma 2, both pairs $(\mathscr{F},\mathscr{T}),(\mathscr{G},\mathscr{S})$ satisfy (CLR$_{\mathscr{T}\mathscr{S}}$) property. Hence, there are $\{\varpi_n\},\{\omega_n\} \subset E$ such that for all $\varkappa > 0$,

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{G}\omega_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{S}\omega_n,z,\varkappa) = 1$$

for some $z \in \mathscr{T}E \cap \mathscr{S}E$. The remaining of this proof follows from Theorem 2.

Theorem 5. *Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{S},\mathscr{T}$ are self-mappings of E satisfy the following conditions:*

1. *the pair $(\mathscr{F},\mathscr{T})$ (or $(\mathscr{G},\mathscr{S})$) satisfies the (E.A.) property;*
2. *$\mathscr{F}E \subset \mathscr{S}E$ (or $\mathscr{G}E \subset \mathscr{T}E$);*
3. *$\{\mathscr{G}\omega_n\}$ converges for all sequences $\{\omega_n\}$ in E provided $\{\mathscr{S}\omega_n\}$ converges (or $\{\mathscr{F}\omega_n\}$ converges for all sequences $\{\omega_n\}$ in E provided $\{\mathscr{T}\omega_n\}$ converges);*
4. *mappings $\mathscr{F},\mathscr{G},\mathscr{T}$ and $\mathscr{S}$ satisfy*

$$\Gamma(\mathscr{F}\varpi,\mathscr{G}\omega,\varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi,\mathscr{F}\varpi,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}\omega,\mathscr{G}\omega,\frac{\varkappa_2}{q}\right)\right)$$

for every $\varpi,\omega \in E$ and any $\varkappa > 0$ where $\varkappa_1,\varkappa_2 > 0$ with $\varkappa = \varkappa_1+\varkappa_2$, $p,q>0$ with $p+q \in (0,1)$ and $\psi \in \Psi_f$.

Then, both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ have a coincidence point. Furthermore, if both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ are weakly compatible, this implies that mappings $\mathscr{F},\mathscr{T},\mathscr{G},\mathscr{S}$ have a unique common fixed point.

Proof. In view of Lemma 3, both $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ satisfy common (E.A.) property. Hence, there are $\{\varpi_n\}$, $\{\omega_n\} \subset E$ such that for all $\varkappa > 0$,

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{G}\omega_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{S}\omega_n,z,\varkappa) = 1$$

for some $z \in E$. The remaining of this proof follows from Theorem 3.

Theorem 6. *Suppose that $(E,\Gamma,*)$ is a fuzzy metric space and $\mathscr{F},\mathscr{G},\mathscr{S},\mathscr{T}$ are self-mappings of E satisfy inequality (5) for every for every $\varpi,\omega \in E$ and any $\varkappa > 0$. Assume the pair $(\mathscr{F},\mathscr{T})$ satisfy ($CLR_{(\mathscr{F},\mathscr{T}),\mathscr{S}}$) property, then both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ have a coincidence point. Furthermore, if both pairs $(\mathscr{F},\mathscr{T})$ and $(\mathscr{G},\mathscr{S})$ are weakly compatible, this implies that mappings $\mathscr{F},\mathscr{T},\mathscr{G},\mathscr{S}$ have a unique common fixed point.*

Proof. Consider the pair $(\mathscr{F},\mathscr{T})$ satisfy ($\text{CLR}_{(\mathscr{F},\mathscr{T}),\mathscr{S}}$) property, we have a sequence $\{\varpi_n\} \in E$ such that for each $\varkappa > 0$,

$$\lim_{n\to\infty}\Gamma(\mathscr{F}\varpi_n,z,\varkappa) = \lim_{n\to\infty}\Gamma(\mathscr{T}\varpi_n,z,\varkappa) = 1$$

for some $z \in \mathscr{T}E \cap \mathscr{S}E$. This means that

$$\lim_{n\to\infty}\mathscr{F}\varpi_n = \lim_{n\to\infty}\mathscr{T}\varpi_n = z.$$

As $z \in \mathscr{S}E$, there is an element u in E satisfy $z = \mathscr{S}u$. We will show that $\mathscr{G}u = \mathscr{S}u$. Assume $\mathscr{G}u \neq \mathscr{S}u$, which means, $0 < \Gamma(\mathscr{G}u,\mathscr{S}u,\varkappa) < 1$ for some $\varkappa > 0$. Using inequality (5), for any $\varkappa > 0$, it leads to

$$\Gamma(\mathscr{F}\varpi_n,\mathscr{G}u,\varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi_n,\mathscr{F}\varpi_n,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa_2}{q}\right)\right). \tag{6}$$

Let $\varkappa_1 = \frac{p\varkappa}{p+q}, \varkappa_2 = \frac{q\varkappa}{p+q}$ and $r = p+q$. Then, we obtain

$$\Gamma(\mathscr{F}\varpi_n,\mathscr{G}u,\varkappa) \geq \psi\left(\Gamma\left(\mathscr{T}\varpi_n,\mathscr{F}\varpi_n,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa}{r}\right)\right).$$

As we let $n \to \infty$, it follows that

$$\begin{aligned}\Gamma(\mathscr{S}u,\mathscr{G}u,\varkappa) &\geq \psi\left(\Gamma\left(z,z,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa}{r}\right)\right)\\ &= \psi\left(1,\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa}{r}\right)\right).\end{aligned}$$

Due to properties of Ψ-function and Lemma 1, we yield

$$\begin{aligned}\Gamma(\mathscr{S}u,\mathscr{G}u,\varkappa) &\geq \psi\left(1,\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa}{r}\right)\right)\\ &\geq \psi\left(\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa}{r}\right)\right)\\ &> \Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa}{r}\right)\\ &> \Gamma(\mathscr{S}u,\mathscr{G}u,\varkappa)\end{aligned}$$

which leads to a contradiction. Therefore, $\Gamma(\mathscr{S}u,\mathscr{G}u,\varkappa)=1$ for any $\varkappa>0$, which means, $\mathscr{G}u=\mathscr{S}u=z$. So u is a coincidence point of pair $(\mathscr{G},\mathscr{S})$.

Moreover, since $z\in\mathscr{T}E$, we can find an element v in E satisfy $z=\mathscr{T}v$. We will validate that $\mathscr{F}v=\mathscr{T}v$. Assume $\mathscr{F}v\neq\mathscr{T}v$, which means, $0<\Gamma(\mathscr{F}v,\mathscr{T}v,\varkappa)<1$ for some $\varkappa>0$. Using inequality (5), for all $\varkappa>0$, it leads to

$$\Gamma(\mathscr{F}v,\mathscr{G}u,\varkappa)\geq\psi\left(\Gamma\left(\mathscr{T}v,\mathscr{F}v,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa_2}{q}\right)\right). \tag{7}$$

Again let $\varkappa_1=\frac{p\varkappa}{p+q},\varkappa_2=\frac{q\varkappa}{p+q}$ and $r=p+q$. Then, we obtain

$$\Gamma(\mathscr{F}v,\mathscr{G}u,\varkappa)\geq\psi\left(\Gamma\left(\mathscr{T}v,\mathscr{F}v,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa}{r}\right)\right)\quad=\psi\left(\Gamma\left(\mathscr{T}v,\mathscr{F}v,\frac{\varkappa}{r}\right),1\right).$$

Since $\mathscr{G}u=z=\mathscr{T}v$, by Ψ-function's properties and Lemma 1, it follows that

$$\begin{aligned}\Gamma(\mathscr{F}v,\mathscr{T}v,\varkappa)&\geq\psi\left(\Gamma\left(\mathscr{T}v,\mathscr{F}v,\frac{\varkappa}{r}\right),1\right)\\&\geq\psi\left(\Gamma\left(\mathscr{T}v,\mathscr{F}v,\frac{\varkappa}{r}\right),\Gamma\left(\mathscr{T}v,\mathscr{F}v,\frac{\varkappa}{r}\right)\right)\\&>\Gamma\left(\mathscr{T}v,\mathscr{F}v,\frac{\varkappa}{r}\right)\\&>\Gamma(\mathscr{T}v,\mathscr{F}v,\varkappa)\\&=\Gamma(\mathscr{F}v,\mathscr{T}v,\varkappa)\end{aligned}$$

which leads to a contradiction. As a result, $\Gamma(\mathscr{F}v,\mathscr{T}v,\varkappa)=1$ for any $\varkappa>0$, for instance, $\mathscr{F}v=\mathscr{T}v=z$. So v is a coincidence point of the pair $(\mathscr{F},\mathscr{T})$.

As $(\mathscr{F},\mathscr{T})$ is weakly compatible and $\mathscr{F}v=\mathscr{T}v$, these lead to $\mathscr{T}z=\mathscr{T}\mathscr{F}v=\mathscr{F}\mathscr{T}v=\mathscr{F}z$. We say that z is a common fixed point of $(\mathscr{F},\mathscr{T})$. Using inequality (5) and the property of Ψ-function, for any $\varkappa>0$, we obtain

$$\begin{aligned}\Gamma(\mathscr{F}z,z,\varkappa)=\Gamma(\mathscr{F}z,\mathscr{G}u,\varkappa)&\geq\psi\left(\Gamma\left(\mathscr{T}z,\mathscr{F}z,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}u,\mathscr{G}u,\frac{\varkappa_2}{q}\right)\right)\\&=\psi\left(\Gamma\left(\mathscr{F}z,\mathscr{F}z,\frac{\varkappa_1}{p}\right),\Gamma\left(z,z,\frac{\varkappa_2}{q}\right)\right)\\&=\psi(1,1)\\&=1.\end{aligned}$$

Thus, $\Gamma(\mathscr{F}z,z,\varkappa)=1$ for all $\varkappa>0$,that is, $\mathscr{F}z=z=\mathscr{T}z$. Therefore, z is a common fixed point of the pair $(\mathscr{F},\mathscr{T})$.

Also, as the pair $(\mathscr{G},\mathscr{S})$ is weakly compatible and $\mathscr{G}u=\mathscr{S}u$, this implies that $\mathscr{S}z=\mathscr{S}\mathscr{G}u=\mathscr{G}\mathscr{S}u=\mathscr{G}z$. We say that z is a common fixed point of the pair $(\mathscr{G},\mathscr{S})$. Using inequality (5) and the property of Ψ-function, for any $\varkappa>0$, we obtain

$$\begin{aligned}\Gamma(z,\mathscr{G}z,\varkappa)=\Gamma(\mathscr{F}z,\mathscr{G}z,\varkappa)&\geq\psi\left(\Gamma\left(\mathscr{T}z,\mathscr{F}z,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}z,\mathscr{G}z,\frac{\varkappa_2}{q}\right)\right)\\&=\psi\left(\Gamma\left(z,z,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{G}z,\mathscr{G}z,\frac{\varkappa_2}{q}\right)\right)\\&=\psi(1,1)\\&=1.\end{aligned}$$

Thus, $\Gamma(z,\mathscr{G}z,\varkappa)=1$ for any $\varkappa>0$, which means, $\mathscr{G}z=z=Sz$. So z is a common fixed point of the pair $(\mathscr{G},\mathscr{S})$. This shows that z is a common fixed point of mappings $\mathscr{F},\mathscr{G},\mathscr{T},\mathscr{S}$.

For the uniqueness, consider two common fixed points z_1, z_2 are distinct, which means that $0<\Gamma(z_1,z_2,\varkappa)<1$ for some $\varkappa>0$. By inequality (5), for every $\varkappa>0$, we have

$$\begin{aligned}\Gamma(z_1,z_2,\varkappa)&=\Gamma(\mathscr{F}z_1,\mathscr{G}z_2,\varkappa)\\&\geq\psi\left(\Gamma\left(\mathscr{T}z_1,\mathscr{F}z_1,\frac{\varkappa_1}{p}\right),\Gamma\left(\mathscr{S}z_2,\mathscr{G}z_2,\frac{\varkappa_2}{q}\right)\right)\\&=\psi\left(\Gamma\left(z_1,z_1,\frac{\varkappa_1}{p}\right),\Gamma\left(z_2,z_2,\frac{\varkappa_2}{q}\right)\right)\\&=\psi(1,1)\\&=1\end{aligned}$$

which is contradict with our assumption. Thus, $z_1 = z_2$ which proves the common fixed point is unique.

4 Conclusion and Open Problem

Our paper generalized Kannan-type contractive mappings equipped with (CLR) or (E.A.) properties on fuzzy metric spaces and established several common fixed-point results of these mappings. Researchers can investigate the existence of fixed points for Kannan-type contractive mappings on more general setting, for example, fuzzy b-metric spaces, controlled fuzzy b-metric spaces, fuzzy bipolar metric spaces and etc. Additionally, Choudhury and Das [6] used h-coupled Kannan type mapping and obtained a common coupled fixed points for two mappings on partially ordered fuzzy metric space. This raise a question whether our results for four mappings able to expand to partially order fuzzy metric space. As mentioned in Section 1, Subrahmanyam [30] proved that the fixed point of Kannan-type contractive mappings implies the completeness for metric space. Therefore, we will end this paper with an open problem: Does the existence of fixed point for Kannan-type contractions imply the completeness on fuzzy metric space?

Declarations

Competing interests: The authors declare that there is no conflict of interest regarding the publication of this manuscript.
Authors' contributions: Conceptualization: Koon Sang Wong, Zabidin Salleh; Methodology: Koon Sang Wong; Formal analysis and investigation: Koon Sang Wong, Zabidin Salleh, Habibulla Akhadkulov; Writing - original draft preparation: Koon Sang Wong; Writing - review and editing: Zabidin Salleh, Habibulla Akhadkulov; Funding acquisition: Zabidin Salleh; Resources: Zabidin Salleh; Validation and Visualization: Habibulla Akhadkulov; Supervision: Zabidin Salleh. All authors reviewed the results and approved the final version of the manuscript.
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Transmuted Fuzzy Entropy: Another Look at Generalized Fuzzy Entropy

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Abstract: Generalizing a fuzzy entropy presents a better measure in theory and application. In this article we propose a new concept of generalizing an entropy on the basis of transmutation distribution function. Three transmuted fuzzy entropies are proposed and compared with well established generalized fuzzy entropies in the literature.

Keywords: Fuzzy Entropy; Transmuted Distribution; Generalized Fuzzy Entropy; Fuzzy Logic.

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1 Introduction

Fuzzy logic was established by [17], defining what is a fuzzy set and establishing most of the operations and properties. Later on, [4] extended the concept of a classical entropy to present what is known as the fuzzy entropy (FE).
In the following, we will define a fuzzy set and the fuzzy entropy:

Definition 1 *A fuzzy set A defined on a universe of discourse X is given by [17] as:*

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\}, \tag{1}$$

where, $\mu_A(x) \in [0,1]$ *is the membership function of A, it describes the degree of belongingness of an element x to the set A.*

Its quite important to note that, considering the elements $x_1, x_2, ..., x_n \in X$, the $\sum_{i=1}^{n} \mu_A(x_i)$ is not necessary equals to 1. Hence, $\mu_A(x)$ is not a probability.
The measure which quantify fuzzy information gained from a fuzzy set or is referred to as fuzzy entropy. In other words, a fuzzy entropy measures the average amount of knowledge or information from fuzzy data.Its different than the classical entropy which depend of a probabilistic concept, where FE is define on the basis of membership function.
[4] defined a fuzzy entropy (denoted by $H^{DT}(x)$) on the basis of shannon's entropy ([14]),

$$H^{DT}(x) = -n\sum_{i=1}^{n} \left[\mu_A(x_i)\log(\mu_A(x_i)) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right]. \tag{2}$$

They presented a set of axioms needed to be satisfied by any measure to be considered as an entropy; say $H(x)$ of a fuzzy set A. The axioms are

1.$H(x) = 0\ iff\ A$ is a non-fuzzy set (crisp set), i.e. $\mu_A(x_i) = 0$ or 1 $\forall x_i \in A$.
2.$H(x)$ is maximum iff $\mu_A(x_i) = 0.5, \forall x_i \in A$.

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3.$H(x) \geq H^*(x)$, where $H^*(x)$ is the entropy of A^* a sharpened version of A.
4.$H(x) = H^c(x)$, where $H^c(x)$ is the entropy of A^c; the complement set of A. $A^c = \{\langle x, 1-\mu_A(x)\rangle | x \in X\}$.

Many researchers and many articles studied fuzzy entropies and proposing modified and generalized versions of such entropies. [11], defined what is known now as a Hybrid entropy, where the fuzzy entropy is considered to be a generalization of the classical entropy (see, [2],[7]). This kind of entropy dealt with efficiencies of the total entropy which was proposed by [4].
In the same article, [11] introduced a higher order fuzzy entropy which measures the uncertainty associated with any subset with r combination. The entropy of order r of a fuzzy set A is

$$H^r(x) = -\frac{1}{t}\sum_{i=1}^{n} \mu(S_i^r)\log(\mu(S_i^r)) + (1-\mu(S_i^r))\log(1-\mu(S_i^r)),$$

where, S denotes the Shannon function, and $\mu(S_i^r)$ is the degree of membership through the function S.
[3] introduced a generalization of fuzzy entropy of order α based on Renyi's entropy of a fuzzy set A as

$$H^{BP}(x) = \frac{1}{n(\alpha-1)}\log\left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\right]; \quad \alpha \neq 1; \alpha > 0. \tag{3}$$

[9], [10] Introduced a fuzzy entropy based on the exponential behavior of information gain, of a fuzzy set A as

$$H^{PP}(x) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}\left[\mu_A(x_i)e^{(1-\mu_A(x_i))} + (1-\mu_A(x_i))e^{\mu_A(x_i)-1}\right], \tag{4}$$

Later on, [16] generalized the exponential fuzzy entropy of order–α, of a fuzzy set A is given as

$$H^{VS}(x) = \frac{1}{n(e^{(1-0.5^\alpha)}-1)}\sum_{i=1}^{n}\left[\mu_A(x_i)e^{(1-\mu_A^\alpha(x_i))} + (1-\mu_A(x_i))e^{(1-(1-\mu_A(x_i))^\alpha)-1}\right]; \quad \alpha > 0. \tag{5}$$

[1] proposed a fuzzy entropy of order α with a promising application in decision making. The measure of a fuzzy set A is given by

$$H^{NT}(x) = \sum_{i=1}^{n}\left[\frac{\mu_A^{\alpha/2}(x_i)(1-\mu_A(x_i))^{\alpha/2}}{\mu_A(x_i)e^{-\alpha(1-\mu_A(x_i))} + (1-\mu_A(x_i))e^{-\alpha\mu_A(x_i)}}\right]^{1/\alpha}; \quad \alpha > 0. \tag{6}$$

The reader may refer to [5], [6], [8] for more details.

2 Transmuted Fuzzy entropy

2.1 Quadratic Transmuted Fuzzy entropy

As noted earlier, generalizing fuzzy entropies is a common custom in fuzzy theory. Adding extra parameter(s) to an existing fuzzy entropy make it more flexible and hence secure all information from losing due fuzziness.
In a similar fashion, [15] introduced the quadratic transmuted family of distributions, where it enhances an existing distribution by adding additional variable, for solving drawbacks in financial mathematics field. The cdf for a distribution in the quadratic transmuted family is

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), \qquad x \in \mathbb{R},$$

where $\lambda \in [-1,1]$, and $G(x)$ is the cdf of baseline distribution.
Motivated by this family of distributions, we propose the Quadratic Transmuted Fuzzy Entropy (QTFE) defined below, also we study and prove that QTFE satisfies the axiomatic properties of [4].

Definition 2 *For a fuzzy entropy $H(x)$ of the Fuzzy set A, the transmuted fuzzy entropy of A is given by*

$$H_Q^T(x) = (1+\lambda)H(x) - \lambda H^2(x), \qquad x \in A, \lambda \in [-1,1]. \tag{7}$$

Theorem 1. *The Quadratic Transmuted Fuzzy Entropy $H_Q^T(x)$ is a fuzzy measure and satisfies The axiomatic properties [4].*

Proof. The set of four axiomatic properties are checked as follows

• $H_Q^T(x) = 0 \; iff$ A is a non-fuzzy set (crisp set), i.e. $\mu_A(x_i) = 0$ or $1 \; \forall x_i \in A$.
When, $\mu_A(x_i) = 0$ or 1 ,$\forall x_i \in A$ then it straight forward that $H(x) = 0$. Hence, $H_Q^T(x) = 0$. on the other hand, when $H_Q^T(x) = 0$, then $(1+\lambda)H(x) - \lambda H^2(x) = 0$, i.e.,

$$H(x) \times \big[(1+\lambda) - \lambda H(x)\big] = 0,$$

hence at least one of the factors is zero

$$H(x) = 0 \textbf{ or } (1+\lambda) - \lambda H(x) = 0, \forall x \in A.$$

If $H(x) = 0$, then $\mu_A(x_i) = 0$ or $1 \; \forall x_i \in A$, as $H(x)$ is a fuzzy entropy and satisfies this axiomatic property.
Now, when $(1+\lambda) - \lambda H(x) = 0$, so, $H(x) = \frac{1+\lambda}{\lambda}$. This conclusion is false, as it means that the fuzzy entropy is always constant and ranges between $[0,2]$.

• $H(x)$ is maximum iff $\mu_A(x_i) = 0.5, \forall x_i \in A$.
by differentiating $H_Q^T(x)$ with respect to $\mu_A(x_i)$, we get

$$\begin{aligned}\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} &= (1+\lambda)\frac{\partial H(x)}{\partial \mu_A(x_i)} - 2\lambda H(x) \cdot \frac{\partial H(x)}{\partial \mu_A(x_i)} \\ &= \frac{\partial H(x)}{\partial \mu_A(x_i)}\Big[(1+\lambda) - 2\lambda H(x)\Big] \\ &= \frac{\partial H(x)}{\partial \mu_A(x_i)}\Big[1+\lambda \cdot \big(1-2H(x)\big)\Big]\end{aligned}$$

Let, $0 \le \mu_A(x_i) < 0.5$.
Notice that $\frac{\partial H(x)}{\partial \mu_A(x_i)}$ is always greater than 0 as $H(x)$ satisfies this particular axiomatic property, since its a fuzzy measure.
i.e., the statement $1+\lambda \cdot \big(1-2H(x)\big)$ should be positive in order to have $\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} > 0$.
Case 1: $\lambda \ge 0, H(x) \le 0.5$. The statement is valid.
Case 2: $\lambda \le 0, H(x) \ge 0.5$. The statement is valid.
Case 3: $\lambda \ge 0, H(x) \ge 0.5$.
Now, we have

$$\begin{aligned} 0 \ge& (1-2H(x)) \ge -1 \\ 0 \ge& \lambda \cdot (1-2H(x)) \ge -\lambda > -1, \quad \text{(multipling by } \lambda) \\ 1 \ge& 1+\lambda \cdot (1-2H(x)) \ge 1-\lambda > 0, \quad \text{(adding 1)}\end{aligned}$$

hence, $1+\lambda \cdot (1-2H(x)) \ge 0$, The statement is valid.
Case 4: $\lambda \le 0, H(x) \le 0.5$.

$$\begin{aligned} 1 \ge& (1-2H(x)) \ge 0 \\ -1 <& \lambda \le \lambda \cdot (1-2H(x)) \le 0, \quad \text{(multipling by } \lambda) \\ 0 <& 1+\lambda \le 1+\lambda \cdot (1-2H(x)) \le 1, \quad \text{(adding 1)}\end{aligned}$$

hence, $1+\lambda \cdot (1-2H(x)) \ge 0$, The statement is valid.
Hence, $\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} > 0$ when $0 < \mu_A(x_i) \le 0.5$.
Now, let, $0.5 < \mu_A(x_i) \le 1.0$,
$\frac{\partial H(x)}{\partial \mu_A(x_i)}$ is always less than 0 as $H(x)$ satisfies this particular axiomatic property, since its a fuzzy measure. On the other hand and as explained earlier, the statement $1+\lambda \cdot \big(1-2H(x)\big)$ is positive. i.e.,

$$\frac{\partial H(x)}{\partial \mu_A(x_i)}\Big[1+\lambda \cdot \big(1-2H(x)\big)\Big]$$

Hence, $\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} < 0$.
Thus, $H_Q^T(x)$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$.

• $H_Q^T(x) \geq H^*(x)$, where $H^*(x)$ is the entropy of A^* a sharpened version of A.

As shown in the last point, $H_Q^T(x)$ is increasing on the interval $[0,0.5)$ and decreasing on the interval $(0.5,1]$. It follows that $\mu_A^*(x_i)$; the membership function entropy of A^*, is less the $\mu_A(x_i)$ in the interval $[0,0.5)$ and its greater than $\mu_A(x_i)$ in the interval $(0.5,1]$. Hence, $H_Q^T(x) \geq H^*(x)$

• $H_Q^T(x) = H_Q^{T^c}(x)$, where $H_Q^{T^c}(x)$ is the entropy of A^c; the complement set of A.

$$H_Q^T(x) = (1+\lambda)H(x) - \lambda H^2(x) = (1+\lambda)H^c(x) - \lambda H^{c2}(x),$$

as $H(x)$ is a fuzzy entropy. So,

$$(1+\lambda)H^c(x) - \lambda H^{c2}(x) = H_Q^{T^c}(x).$$

Hence the theorem is proved, the QTFE satisfies the axiomatic properties, i.e., $H_Q^T(x)$ is indeed a fuzzy entropy.

2.2 Cubic Transmuted Fuzzy Entropy

[12] and [13] introduced the idea of the cubic transmuted distribution function, The cdf for a distribution in the cubic transmuted family is

$$F(x) = (1+\lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x), \qquad x \in \mathbb{R},$$

where $\lambda_1 \in [-1,1]$, $\lambda_2 \in [-1,1]$ and $-2 < \sum_i^2 \lambda_i < 1$. $G(x)$ is the cdf of baseline distribution.
On the basis of the defined family of distribution, we define the Cubic Transmuted Fuzzy Entropy (CTFE).

Definition 3 *For a fuzzy entropy $H(x)$ of the Fuzzy set A, the Cubic Transmuted Fuzzy Entropy of A is given by*

$$H_C^T(x) = (1+\lambda_1)H(x) + (\lambda_2 - \lambda_1)H^2(x) - \lambda_2 H^3(x), \tag{8}$$

where, $\lambda_1, \lambda_2 \in [-1,1]$, and $-2 < \sum_i^2 \lambda_i < 1$, $x \in A$.

Theorem 2. *The Cubic Transmuted Fuzzy Entropy $H_C^T(x)$ is a fuzzy measure and satisfies The axiomatic properties [4].*

Proof. The axiomatic properties are checked as follows,

• $H_C^T(x) = 0$ iff A is a non-fuzzy set (crisp set), i.e. $\mu_A(x_i) = 0$ or 1 $\forall x_i \in A$.

When, $\mu_A(x_i) = 0$ or 1 $\forall x_i \in A$ then $H(x) = 0$. and hence $H_C^T(x) = 0$. on the other hand, when $H_C^T(x) = 0$, then $(1+\lambda_1)H(x) + (\lambda_2 - \lambda_1)H^2(x) - \lambda_2 H^3(x) = 0$, i.e.,

$$H(x) \times \left[(1+\lambda_1) + (\lambda_2 - \lambda_1)H(x) - \lambda_2 H^2(x)\right] = 0,$$

hence at least one of the factors is zero

$$H(x) = 0 \textbf{ or } (1+\lambda_1) + (\lambda_2 - \lambda_1)H(x) - \lambda_2 H^2(x) = 0, \forall x \in A.$$

If $H(x) = 0$, then $\mu_A(x_i) = 0$ or 1 $\forall x_i \in A$, as $H(x)$ satisfies this axiomatic property.
Now when $(1+\lambda_1) + (\lambda_2 - \lambda_1)H(x) - \lambda_2 H^2(x) = 0$, this will give a specific value of the entropy dependent on the choice of λ_1 and λ_2. And hence this factor is not zero.

• $H_C^T(x)$ is maximum iff $\mu_A(x_i) = 0.5, \forall x_i \in A$.
by differentiating $H_C^T(x)$ with respect to $\mu_A(x_i)$, we get

$$\begin{aligned}
\frac{\partial H_C^T(x)}{\partial \mu_A(x_i)} &= (1+\lambda_1)\frac{\partial H(x)}{\partial \mu_A(x_i)} + 2(\lambda_2 - \lambda_1)H(x) \cdot \frac{\partial H(x)}{\partial \mu_A(x_i)} - 3\lambda_2 H^2(x) \cdot \frac{\partial H(x)}{\partial \mu_A(x_i)} \\
&= \frac{\partial H(x)}{\partial \mu_A(x_i)}\left[(1+\lambda_1) + 2(\lambda_2 - \lambda_1)H(x) \cdot -3\lambda_2 H^2(x) \cdot \right] \\
&= \frac{\partial H(x)}{\partial \mu_A(x_i)}\left[1 + \lambda_1 \cdot (1 - 2H(x)) + \lambda_2 H(x) \cdot (2 - 3H(x))\right]
\end{aligned}$$

Let, $0 \le \mu_A(x_i) < 0.5$,
Notice that $\frac{\partial H(x)}{\partial \mu_A(x_i)}$ is always greater than 0 as $H(x)$ a fuzzy measure and hence it satisfies this axiomatic property, i.e., the statement $1+\lambda_1 \cdot (1-2H(x)) + \lambda_2 H(x) \cdot (2-3H(x))$ should be positive in order to have $\frac{\partial H_Q^T(x)}{\partial \mu_A(x_i)} > 0$.
Case 1: $\lambda_1, \lambda_2 \ge 0, H(x) \le 0.5$. The statement is valid.
Case 2: $\lambda_1, \lambda_2 \ge 0, 0.5 \le H(x) \le 2/3$. The statement is valid.
Case 3: $\lambda_1, \lambda_2 \le 0, H(x) \le 2/3$. The statement is valid.
Case 4: $\lambda_1, \lambda_2 \ge 0, H(x) \ge 2/3$.
we have

$$0 \ge (1-2H(x)) \ge -1 \quad , \quad 0 \ge H(x)(2-3H(x)) \ge -1$$
$$0 \ge \lambda_1 \cdot (1-2H(x)) \ge -\lambda_1 > -1 \quad , \quad 0 \ge \lambda_2 \cdot H(x)(2-3H(x)) \ge -\lambda_2 > 0$$

hence,

$$0 \ge \lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x)) > -1,$$

then,

$$1 \ge 1+\lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x)) > 0$$

hence, The statement is valid.
Case 5: $\lambda_1, \lambda_2 \le 0, H(x) \le 0.5$.
we have

$$1 \ge (1-2H(x)) \ge 0 \quad , \quad 1 \ge H(x)(2-3H(x)) \ge 0$$
$$-1 < \lambda_1 \ge \lambda_1 \cdot (1-2H(x)) \ge 0 \quad , \quad -1 \ge \lambda_2 \ge \lambda_2 \cdot H(x)(2-3H(x)) < 0$$

hence,

$$-1 \ge \lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x)) < 0,$$

then,

$$0 < 1+\lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x))$$

hence, The statement is valid.
Case 6: $\lambda_1, \lambda_2 \le 0, 0.5 \le H(x) \le 2/3$.
we have

$$1 \le (1-2H(x)) \le 0 \quad , \quad H(x)(2-3H(x)) \ge 0$$
$$1 > \lambda_1 \ge \lambda_1 \cdot (1-2H(x)) \ge 0 \quad , \quad -1 \le \lambda_2 \le \lambda_2 \cdot H(x)(2-3H(x)) < 0$$
$$1 > \lambda_1 \ge \lambda_1 \cdot (1-2H(x)) \ge 0 \quad , \quad 0 \le 1+\lambda_2 \le 1+\lambda_2 \cdot H(x)(2-3H(x)) < 1$$

then,

$$0 < 1+\lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x))$$

hence, The statement is valid, and hence $\frac{\partial H_C^T(x)}{\partial \mu_A(x_i)} > 0$.

Now, let, $0.5 < \mu_A(x_i) \le 1.0$,
$\frac{\partial H(x)}{\partial \mu_A(x_i)}$ is always less than 0 as $H(x)$ satisfies this particular axiomatic property, since its a fuzzy measure. On the other hand and as shown above, the statement $1+\lambda_1 \cdot (1-2H(x)) + \lambda_2 \cdot H(x)(2-3H(x))$ is positive. Hence, $\frac{\partial H_C^T(x)}{\partial \mu_A(x_i)} < 0$.
Thus, $H_C^T(x)$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$.

• $H_C^T(x) \ge H^*(x)$, where $H^*(x)$ is the entropy of A^* a sharpened version of A.

As shown in the last point, $H_C^T(x)$ is increasing on the interval $[0, 0.5)$ and decreasing on the interval $(0.5, 1]$. It follows that $\mu_A^*(x_i)$; the membership function of A^*, is less the $\mu_A(x_i)$ in the interval $[0, 0.5)$ and its greater than $\mu_A(x_i)$ in the interval $(0.5, 1]$. Hence, $H_C^T(x) \ge H^*(x)$.

• $H_C^T(x) = H_C^{T^c}(x)$, where $H_C^{T^c}(x)$ is the entropy of A^c; the complement set of A.

$$H_C^T(x) = (1+\lambda_1)H(x) + (\lambda_2 - \lambda_1)H^2(x) - \lambda_2 H^3(x),$$

as $H(x)$ is a fuzzy entropy; i.e., $H(x) = H^c(x)$. So,

$$(1+\lambda_1)H^c(x) + (\lambda_2 - \lambda_1)H^{2^c}(x) - \lambda_2 H^{3^c}(x) = H_C^{T^c}(x).$$

Hence the theorem is proved, the Cubic Transmuted Fuzzy entropy satisfies the axiomatic properties, i.e., $H_C^T(x)$ is a fuzzy measure.

2.3 k-Transmuted Fuzzy Entropy

[12] have introduced the k-Transmuted families of distributions, a generalization of transmuted families, defined as

$$F(x) = G(x) + [1 - G(x)] \sum_{i=1}^{k} \lambda_i G^i(x),$$

where, $\lambda_i \in [-1,1]$ and $-k < \sum_i^k \lambda_i < 1$. $G(x)$ is the cdf of baseline distribution.
Based on this family of distribution we define the kTransmuted Fuzzy Entropy.

Definition 4 *For a fuzzy entropy $H(x)$ of the Fuzzy set A, the k-Transmuted Fuzzy Entropy of A is given by*

$$H_k^T(x) = H(x) + [1 - H(x)] \sum_{i=1}^{k} \lambda_i H^i(x), \qquad x \in A. \tag{9}$$

Where, $\lambda_i \in [-1,1]$ and $-k < \sum_i^k \lambda_i < 1$, k=2,3,....

It follows that we have another version of cubic Transmuted Fuzzy Entropy, given as

$$H_3^T(x) = H(x) + [1 - H(x)] \sum_{i=1}^{3} \lambda_i H^i(x),$$

which after simplifying turned out to be equivalent to CTFE.
Lets define a Quartic Transmuted Fuzzy Entropy; a k-TFE at $k = 4$, denoted by $H_4^T(x)$ given by

$$H_4^T(x) = H(x) + [1 - H(x)] \sum_{i=1}^{4} \lambda_i H^i(x). \tag{10}$$

Theorem 3. *The k–Transmuted Fuzzy Entropy $H_k^T(x)$ is a fuzzy measure and satisfies The axiomatic properties [4].*

The proof is straight forward as done in the previous theorems.

3 Transmuted entropies of well known fuzzy measures

Fuzzy entropy literature is rich in many versions and generalizations of FE, among many of these measures, we presented the most known and applied measures in equations (2) - (6). The Quadratic, Cubic and K–Transmuted Fuzzy entropies of fourth order are generalizations of the fuzzy measures mentioned above are found by applying equation (7), (8) and (10), respectively.
Table 1 presents the values of Deluca and Terminin 1972 performance at different values of $\mu_A(x_i)$ (refereed to as Normalized Values) alongside its Transmuted Fuzzy entropies.

$\mu_A(x_i)$	H^{DT}	H_Q^{DT}	H_C^{DT}	H_4^{DT}
0	1.71×10^{-6}	3.08×10^{-6}	8.56×10^{-7}	8.56×10^{-7}
0.1	0.325	0.501	0.279	0.284
0.2	0.501	0.700	0.488	0.501
0.3	0.611	0.801	0.622	0.640
0.4	0.673	0.849	0.696	0.716
0.5	0.693	0.863	0.719	0.740
0.6	0.673	0.849	0.696	0.716
0.7	0.611	0.801	0.622	0.640
0.8	0.501	0.700	0.488	0.501
0.9	0.325	0.501	0.279	0.284
1.0	1.71×10^{-6}	3.08×10^{-6}	8.56×10^{-7}	8.56×10^{-7}

Table 1: Normalized Values of $H^{DT}(A)$ and its respected transmuted generalizations

All transmuted fuzzy entropies are performing better than the original entropy, as we expected. There is a significant increase in the values of the generalized entropies in comparison with De-Luca and Termini measure. QTFE enhances the performance of the entropy measure more than the others.
The difference in performance between QTFE and CTFE, *k*–TFE is noticeable. But does this remark going against what we believe in?, introducing more parameters will end up in a better performance of a measure?
In fact, parameter λ in QTFE is acting as an additional parameter to the measure, but in the cases of CTFE and *k*–TFE, more λ's is not considered more parameters, it is just a division of one parameter.
As shown below in Table 2, TFE is performing better than the original measure, with best performance for QTFE.

$\mu_A(x_i)$	H^{PP}	H_Q^{PP}	H_C^{PP}	H_4^{PP}
0	4.19×10^{-7}	$7.54\ \times10^{-7}$	2.09×10^{-7}	2.09×10^{-7}
0.1	0.370	0.558	0.332	0.338
0.2	0.651	0.833	0.670	0.689
0.3	0.846	0.950	0.880	0.899
0.4	0.962	0.991	0.975	0.982
0.5	1.000	1.000	1.000	1.00
0.6	0.962	0.991	0.975	0.982
0.7	0.846	0.950	0.880	0.899
0.8	0.651	0.833	0.670	0.689
0.9	0.370	0.558	0.332	0.338
1.0	4.19×10^{-7}	$7.54\ \times10^{-7}$	2.09×10^{-7}	2.09×10^{-7}

Table 2: Normalized Values of $H^{PP}(A)$ and its respected transmuted generalizations

$\mu_A(x_i)$	H^{BP}	H_Q^{BP}	H_C^{BP}	H_4^{BP}
0	2.00×10^{-7}	3.60×10^{-7}	1.00×10^{-7}	1.00×10^{-7}
0.1	0.198	0.325	0.147	0.149
0.2	0.385	0.575	0.349	0.356
0.3	0.545	0.743	0.542	0.557
0.4	0.654	0.835	0.674	0.693
0.5	0.693	0.863	0.719	0.740
0.6	0.654	0.835	0.674	0.693
0.7	0.545	0.743	0.542	0.557
0.8	0.385	0.575	0.349	0.356
0.9	0.198	0.325	0.147	0.149
1.0	2.00×10^{-7}	3.60×10^{-7}	1.00×10^{-7}	1.00×10^{-7}

Table 3: Normalized Values of $H^{BP}(A)$ and its respected transmuted generalizations

$\mu_A(x_i)$	H^{VS}	H_Q^{VS}	H_C^{VS}	H_4^{VS}
0	1.26×10^{-7}	5.99×10^{-7}	1.66×10^{-7}	1.66×10^{-7}
0.1	0.837	0.494	0.274	0.278
0.2	0.985	0.791	0.608	0.626
0.3	0.999	0.936	0.851	0.871
0.4	0.999	0.989	0.969	0.977
0.5	1.000	1.000	1.000	1.000
0.6	0.999	0.989	0.969	0.977
0.7	0.999	0.936	0.851	0.871
0.8	0.985	0.791	0.608	0.626
0.9	0.837	0.494	0.274	0.278
1.0	1.26×10^{-7}	5.99×10^{-7}	1.66×10^{-7}	1.66×10^{-7}

Table 4: Normalized Values of $H^{VS}(A)$ and its respected transmuted generalizations

$\mu_A(x_i)$	H^{NT}	H_Q^{NT}	H_C^{NT}	H_4^{NT}
0	3.16×10^{-4}	5.69×10^{-4}	1.58×10^{-4}	1.58×10^{-4}
0.1	0.346	0.527	0.303	0.308
0.2	0.527	0.726	0.520	0.534
0.3	0.677	0.852	0.701	0.721
0.4	0.784	0.919	0.819	0.840
0.5	0.824	0.940	0.859	0.879
0.6	0.784	0.919	0.819	0.840
0.7	0.677	0.852	0.701	0.721
0.8	0.527	0.726	0.520	0.534
0.9	0.346	0.527	0.303	0.308
1.0	3.16×10^{-4}	5.69×10^{-4}	1.58×10^{-4}	1.58×10^{-4}

Table 5: Normalized Values of $H^{NT}(A)$ and its respected transmuted generalizations

We state the following based on Table 1 and Table2;

$$H^{DT} < H_C^{DT} < H_4^{DT} < H_Q^{DT},$$

$$H^{PP} < H_C^{PP} < H_4^{PP} < H_Q^{PP}.$$

In the later tables we studied three generalized fuzzy entropies of order α, and we a reached similar conclusion, In Table 3

$$H^{BP} < H_C^{BP} < H_4^{BP} < H_Q^{BP}.$$

Also, in Table 4

$$H^{VS} < H_C^{VS} < H_4^{VS} < H_Q^{VS}.$$

And in Table 5

$$H^{NT} < H_C^{NT} < H_4^{NT} < H_Q^{NT}.$$

4 Conclusion

The proposed generalized entropy named Transmuted Fuzzy entropy is another form of generalized entropies. we presented 3 different measures; QTFE, CTFE and k–TFE with performance much better than the original FE, where QTFE presented the best enhancement.
As TPD was first introduced in financial mathematics and later applied in modeling lifetime and survival data, we are intrigued to apply TFE, and specially QTFE in these fields in future research.

Declarations

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Computing Nash Optimal Strategies for a Two-Player Positive Game

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Abstract: We consider a linear quadratic differential game on an infinite time horizon with two types of an information structure. The game models are considered in both information structures: the open loop design and feedback design. The Newton solver for computing the stabilizing solution of the associated Nash-Riccati equations has been established. Moreover, a convergent linearized iterative method depending on a negative constant is introduced for each information structure. The linearized iteration has a linear convergence rate, however there are cases where the iteration is faster than Newton's method. Numerical experiments are implemented to explain the computational advantages of the introduced solvers.

Keywords: Game modelling; Nash Equilibrium; Stabilizing Solution.

2010 Mathematics Subject Classification. 15A24, 65F45.

1 Introduction

There is a correlation between the behaviour of economic agents and their profits on a market. Game theory has used to model and investigate the equilibrium of a market. The market price is defined via a dynamic system equation. Typical applications of game models are in different branches in economics [10, 13] and specially in modelling the energy markets [17], gas network optimization [1].

The Nash equilibrium theory is an effective instrument for the analysis of the equilibrium states in game models. We analyze the problem of computation the optimal strategies to the Nash equilibrium in linear quadratic differential games. Considering a linear dynamics upon the quadratic cost, the problem lead us to solve the coupled Riccati-like differential equations.

Nash equilibrium (or optimal) strategies for differential games are studied in many papers and applications. Nash equilibrium strategies depending on a special solution of coupled algebraic Riccati equations [10] - [13].

The Nash equilibrium and its applicability in the machine learning classification via support vector machines was investigated recently in many papers, for example [5, 14]. It is important to find the corresponding equilibrium fast and effective.

We consider a dynamic system of the type

$$\dot{x} = Ax + B_1u_1 + B_2u_2\,, \quad x(0) = x_0\,. \tag{1}$$

In Equation (1) the state vector is denoted by x, the initial vector is $x_0 \in \mathbb{R}^{n\times 1}$, and matrices A, B_1, B_2 belong to $\mathbb{R}^{n\times n}, \mathbb{R}^{n\times m_1}, \mathbb{R}^{n\times m_2}$, where $\mathbb{R}^{p\times q}$ denotes a set of $p\times q$ matrices with real entries. Control vectors are u_1, u_2. Each player has to choose its control in order to increase its profit. If for all nonnegative vectors x_0, u_1, u_2 the state function $x(t)$ takes only nonnegative values, then system (1) is a positive one. Moreover, system (1) is positive if and only if matrices B_1 and B_2 are nonnegative ones and the matrix (-A) is a Z-matrix [2].

We consider an infinite time horizon game model for a positive system in two cases: (a) with an open loop information design and (b) with a feedback information one. The Newton method and its computer realization for computing the

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Nash equilibrium for the same problem was presented and analyzed in [3]. The Newton algorithm solves a Lyapunov matrix equation at each iteration step via Kronecker product, which approach increase double the size of computational problem [3]. In this paper we explore a problem to find Nash equilibrium strategies for a two-player infinite horizon linear quadratic differential game in these two cases. We propose new faster iterations to determine the stabilizing solution of the corresponding Nash-Riccati equations. Based on the stabilizing solution the optimal controls of each player are established. The computational algorithm of the new iteration needs to compute only two matrix inverses of each iteration step. Numerical experiments are implemented to explain the computational advantages of the introduced solvers.

1.1 A feedback design game model

The theory of the Nash equilibrium in a feedback design was established in [15,16] and computationally investigated in [2,3,8]. The goal of each player is to maximize the corresponding cost function. Cost functionals J_1, J_2 for players are defined

$$J_i(F_1,F_2,x_0) = +\int_0^\infty x^T \left(Q_i + \sum_{j=1}^{2} F_j^T R_{ij} F_j \right) x\, dt\,, \tag{2}$$

for $i = 1,2$. Matrices Q_i and R_{ij} are symmetric ones with $Q_i \in \mathbb{R}^{n\times n}$ and $R_{ij} \in \mathbb{R}^{m_j\times m_j}$ and $i,j = 1,2$. The following additional requirements are assumed:

(a) Q_1, Q_2, R_{12}, and R_{21} are symmetric and nonnegative matrices;

(b) R_{ii}^{-1} is nonpositive, $i = 1,2$.

Moreover, to compute a feedback Nash equilibrium point one has to solve the couple of Nash-Riccati equations [2]:

$$0 = \mathscr{R}_1(X_1,X_2) := -A^T X_1 - X_1 A - Q_1 + X_1 S_1 X_1 + X_1 S_2 X_2 + X_2 S_2 X_1 - X_2 S_{12} X_2\,, \tag{3}$$
$$0 = \mathscr{R}_2(X_1,X_2) := -A^T X_2 - X_2 A - Q_2 + X_2 S_2 X_2 + X_2 S_1 X_1 + X_1 S_1 X_2 - X_1 S_{21} X_1\,, \tag{4}$$

where the matrix coefficients are computed via:

(a) $S_i = B_i R_{ii}^{-1} B_i^T\,,\ S_i = S_i^T \leq 0\,, i = 1,2;$

(b) $S_{12} = B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T\,,$
$S_{12} = S_{12}^T \geq 0,\ \ (R_{12} = R_{12}^T);$

(c) $S_{21} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T\,,$
$S_{21} = S_{21}^T \geq 0,\ R_{12} = R_{12}^T, R_{21} = R_{21}^T\,.$

We derive a faster iteration to calculate an $n\times n$ stabilizing solution $(\tilde{X}_1, \tilde{X}_2)$ of (3)-(4). The closed loop matrix $A - S_1\tilde{X}_1 - S_2\tilde{X}_2$ of system (1) is a stable one. Thus, the feedback Nash equilibrium is defined by $\tilde{F}_j = -R_{jj}^{-1} B_j^T \tilde{X}_j\,, j = 1,2$ and optimal functional's value is $J_j(\tilde{F}_1, \tilde{F}_2, x_0) = x_0^T \tilde{X}_j x_0\,, j = 1,2$ [15,16].

1.2 An open loop design game model

In addition, we define the cost functionals J_1, J_2 for the players in a game with an open loop design

$$J_i(u_1,u_2,x_0) = +\int_0^\infty \left(x^T Q_i x + \sum_{j=1}^{2} u_j^T R_{ij} u_j \right) dt\,. \tag{5}$$

The matrix coefficients in (5) are the same as (2). Players choose their own strategies u_1, u_2 based on the information for the initial state x_0 [2]. The Nash equilibrium point of the game is a solution of the couple Nash-Riccati equations:

$$0 = \mathscr{L}_1(X_1,X_2) := -A^T X_1 - X_1 A - Q_1 + X_1 S_1 X_1 + X_1 S_2 X_2\,, \tag{6}$$
$$0 = \mathscr{L}_2(X_1,X_2) := -A^T X_2 - X_2 A - Q_2 + X_2 S_2 X_2 + X_2 S_1 X_1\,. \tag{7}$$

A solution (X_1^*, X_2^*) has a property the closed loop matrix $(A - S_1 X_1^* - S_2 X_2^*)$ is stable. Moreover, the Nash optimal strategy (u_1^*, u_2^*) in the game is given by $u_j^* = -R_{jj}^{-1} B_j^T X_j^* x^*, j = 1,2$ and x^* solves the closed loop equation $\dot{x} = (A - S_1 X_1^* - S_2 X_2^*)x,\ x(0) = x_0$.

1.3 Notations and facts

A matrix $Q = (q_{ij})$ is nonnegative one in the element wise ordering if $q_{ij} \geq 0$. A real square matrix A is called a Z-matrix if there exists a real number σ and real nonnegative matrix Q, such that $A = \sigma I - Q$. A square Z-matrix has nonpositive off-diagonal elements. If $\sigma > \rho(Q)$, the matrix A is a nonsingular M-matrix. Note $\rho(Q)$ is the spectral radius of Q.

The described two player linear-quadratic differential game is applied to positive differential system (1). We need some properties for nonnegative matrices and especially for M-matrices.

According to theory of nonnegative matrices the following allegations are equivalent for a given Z-matrix (-A):
(a) $(-A)$ is a nonsingular M-matrix;
(b) A is stable .

Lemma 1.*[4]. For a Z-matrix A, the following items are equivalent:*
(a) A is a nonsingular M-matrix;
(b) $A^{-1} \geq 0$;
(c) $Au > 0$ for some vector $u > 0$;
(d) All eigenvalues of A have positive real parts.

Lemma 2.*[6]. Let $A = (a_{ij}) \in \mathbb{R}^{m\times m}$ be an M-matrix. If the elements of $B = (b_{ij}) \in \mathbb{R}^{m\times m}$ satisfy the relations $b_{ii} \geq a_{ii}$, $a_{ij} \leq b_{ij} \leq 0$, $i \neq j$, $i, j = 1,\ldots,m$ then B also is an M-matrix.*

The paper is organized as follows. In section 2, we consider the linearized process to modify the Newton method to compute the feedback Nash equilibrium. The convergence proof is derived. In section 3, we slightly modify the introduced iteration to a game with an open loop design. In section 4, we present some numerical illustrations of the proposed iteration. Finally, we finish the paper with some conclusions.

2 Linearized iteration applied to a feedback equilibrium

We discuss how to compute the feedback Nash equilibrium. The Newton iteration is defined and investigated in [2,8] $(i = 1,2)$:

$$-A^{(k)^T} X_i^{(k+1)} - X_i^{(k+1)} A^{(k)} + \sum_{j\neq i} \left[W_{ij}^{(k)} X_j^{(k+1)} + X_j^{(k+1)} W_{ij}^{(k)^T} \right] = Q_i^{(k)} , \tag{8}$$

where

$$\begin{aligned} A^{(k)} &= A - S_1 X_1^{(k)} - S_2 X_2^{(k)} , \\ W_{12}^{(k)} &= X_1^{(k)} S_2 - X_2^{(k)} S_{12} , \\ W_{21}^{(k)} &= X_2^{(k)} S_1 - X_1^{(k)} S_{21} , \\ Q_i^{(k)} &= Q_i + X_i^{(k)} S_i X_i^{(k)} + \sum_{j\neq i} [X_i^{(k)} S_j X_j^{(k)} + X_j^{(k)} S_j X_i^{(k)}] . \end{aligned} \tag{9}$$

The linearized process was effectively applied to construct iterative methods for solving the algebraic Riccati equation associated with M-matrices [7,9].

At each step of Newton iteration (8) it is necessary to find a solution of a Lyapunov matrix equation. We propose a linearized modification for the Newton method. We take $X_1^{(0)} = X_2^{(0)} = 0$, and negative constant γ, and construct two matrix sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 via:

$$\begin{aligned} F_1^{(p)} &= \gamma I_n + A - S_1 X_1^{(p)} - S_2 X_2^{(p)} , \\ T_1^{(p)} &= \gamma I_n - A^T + X_1^{(p)} S_1 + X_2^{(p)} S_2 , \\ Y_i^{(p)} F_1^{(p)} &= T_1^{(p)} X_i^{(p)} - Q_i(X_1^{(p)}, X_2^{(p)}) \end{aligned} \tag{10}$$

$$\begin{aligned} F_2^{(p)} &= \gamma I_n + A^T - Y_1^{(p)} S_1 - Y_2^{(p)} S_2 , \\ T_2^{(p)} &= \gamma I_n - A + S_1 Y_1^{(p)} + S_2 Y_2^{(p)} \\ F_2^{(p)} X_i^{(p+1)} &= Y_i^{(p)} T_2^{(p)} - Q_i(Y_1^{(p)}, Y_p^{(k)}) \end{aligned} \tag{11}$$

where

$$Q_i(Z_i, Z_j) = Q_i + Z_i S_i Z_i + Z_j S_{ij} Z_j \,,$$

with $(i, j = 1,2; j \neq i)$.

We remark that the standard properties for the matrices of the above matrix sequences in the following Lemma:

Lemma 3. *The matrix sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 are obtained applying iteration (10) - (11) with initial zero matrices $X_1^{(0)} = 0\,, X_2^{(0)} = 0$, and $\gamma < 0$. Then, the following equalities are satisfied for $p = 0, \ldots, \infty$:*

$$\begin{aligned}
&(i)\ (Y_i^{(p)} - X_i^{(p)})F_1^{(p)} = (X_i^{(p)} - Y_i^{(p-1)})\,(\gamma I_n - A + S_1 X_1^{(p)} + S_2 X_2^{(p)}) \\
&+ (X_j^{(p)} - Y_j^{(p-1)})S_j X_i^{(p)} + Y_i^{(p)} S_j (X_j^{(p)} - Y_j^{(p-1)}) \\
&- (X_j^{(p)} - Y_j^{(p-1)})S_{ij} X_i^{(p)} - Y_j^{(p)} S_{ij} (X_j^{(p)} - Y_j^{(p-1)}) \,,
\end{aligned}$$

$$\begin{aligned}
&(ii) F_2^{(p)}(X_i^{(p+1)} - Y_i^{(p)}) = (\gamma I_n - A^T + Y_1^{(p)} S_1 + Y_2^{(p)} S_2)(Y_i^{(p)} - X_i^{(p)}) \\
&+ Y_i^{(p)} S_j (Y_j^{(p)} - X_j^{(p)}) + (Y_j^{(p)} - X_j^{(p)}) \\
&\times S_j X_i^{(p)} + (X_j^{(p)} - Y_j^{(p)}) S_{ij} X_i^{(p)} + Y_j^{(p)} S_{ij} (X_j^{(p)} - Y_j^{(p)}) \,.
\end{aligned}$$

Moreover, if the couple $(\tilde{X}_1, \tilde{X}_2)$ is an exact solution of (3)-(4) the identities can be verified (i=1,2):

$$\begin{aligned}
&(iii)\ (Y_i^{(p)} - \tilde{X}_i)F_1^{(p)} = (\gamma I_n - A^T)(X_i^{(p)} - \tilde{X}_i) + \tilde{X}_i S_i (X_i^{(p)} - \tilde{X}_i) \\
&+ \tilde{X}_i S_j (X_j^{(p)} - \tilde{X}_j) + (X_j^{(p)} - \tilde{X}_j) S_j \tilde{X}_i + \tilde{X}_j S_j (X_i^{(p)} - \tilde{X}_i) \\
&+ (\tilde{X}_j - X_j^{(p)}) S_{ij} \tilde{X}_j + X_j^{(p)} S_{ij} (\tilde{X}_j - X_j^{(p)}) \,, \\
&(iv)\ F_2^{(p)}(X_i^{(p+1)} - \tilde{X}_i) = +(Y_i^{(p)} - \tilde{X}_i)(\gamma I_n - A) \\
&+ (Y_i^{(p)} - \tilde{X}_i) S_i \tilde{X}_i + (Y_j^{(p)} - \tilde{X}_j) S_j \tilde{X}_j \\
&+ (Y_i^{(p)} - \tilde{X}_i) S_j \tilde{X}_j + \tilde{X}_i S_j (Y_j^{(p)} - \tilde{X}_j) \\
&+ (\tilde{X}_j - Y_j^{(p)}) S_{ij} \tilde{X}_j + Y_j^{(p)} S_{ij} (\tilde{X}_j - Y_j^{(p)}) \,.
\end{aligned}$$

Based on the proved Lemma, we confirm the convergence of the proposed iteration (10) - (11) in the following Theorem:

Theorem 1. *Assume matrices A, S_1, S_2, and Q_1, Q_2 are coefficients of the set of matrix equations $\mathscr{R}_j(X_1, X_2) = 0$, j=1,2. There exists a negative $\gamma < 0$, such that $-(\gamma I_n + A)$ is an M-matrix and $\gamma I_n - A \leq 0$.*

The sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 obtained via (10) - (11) satisfy the properties:

(i) *$\tilde{X}_i \geq X_i^{(p+1)} \geq Y_i^{(p)} \geq X_i^{(p)}$ for $p = 0, 1, \ldots$, i=1,2 for any exact nonnegative solution $\tilde{X}_1, \tilde{X}_2$ of $\mathscr{R}_i(X_1, X_2) = 0$, i=1,2. ;*

(ii) *The matrices $(-F_1^{(p)})$ and $(-F_2^{(p)})$ are M-matrices for any positive p.*

(iii) *The matrix sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 converge to the stabilizing nonnegative solution $\hat{X}_1, \hat{X}_2$ to couple of Nash-Riccati equations (3)-(4).*

Proof. We provide a proof by mathematical induction on the number p of the iteration step. In the beginning, we prove theorem's statements for $p = 0$. We take $X_1^{(0)} = X_2^{(0)} = 0$, and construct the couple of sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}, i = 1,2$ applying recursive equations (10) - (11) with $X_1^{(0)} = 0\,, X_2^{(0)} = 0$ and $\gamma < 0$.

For $p = 0$ we have $F_1^{(0)} = \gamma I_n + A$, i.e. $(\gamma I_n + A)^{-1} \leq 0$. This means that $(-F_1^{(0)})$ and $(-F_2^{(0)})$ are M-matrices. and $Q_1(X_1^{(0)}, X_2^{(0)}) \geq 0$. Thus $Y_j^{(0)} \geq 0; Y_j^{(0)} \geq X_j^{(0)}, \ j = 1,2$.

In the second step, we formulate the inductive hypothesis, i.e. we assume that the statements are true for the a positive value of p. We assume that $X_i^{(p)} \geq Y_i^{(p-1)} \geq X_i^{(p-1)} \geq 0$ for some integer p and i=1,2. It is true that $X_i^{(p)} - Y_i^{(p-1)} \geq 0$, and $Y_i^{(p-1)} - X_i^{(p-1)} \geq 0$. In addition, $(-F_1^{(p)})$ and $(-F_2^{(p)})$ are M-matrices.

The next is the induction step, where we prove the statements for $p+1$. We have to prove inequalities $X_i^{(p+1)} \geq Y_i^{(p)} \geq X_i^{(p)} \geq 0$ and $(-F_1^{(p+1)})$ and $(-F_2^{(p+1)})$ are M-matrices.

Applying Lemma 3 (i), we get

$$Y_i^{(p)} - X_i^{(p)} = W_i^{(p)} \, (F_1^{(p)})^{-1} \,,$$

where

$$\begin{aligned} W_i^{(p)} &:= (X_i^{(p)} - Y_i^{(p-1)})(\gamma I_n - A + S_1 X_1^{(p)} + S_2 X_2^{(p)}) \\ &+ (X_j^{(p)} - Y_j^{(p-1)}) S_j X_i^{(p)} + Y_i^{(p)} S_j (X_j^{(p)} - Y_j^{(p-1)}) - (X_j^{(p)} - Y_j^{(p-1)}) S_{ij} X_i^{(p)} \\ &- Y_j^{(p)} S_{ij} (X_i^{(p)} - Y_i^{(p-1)}) \,. \end{aligned}$$

We note the following $S_1 \leq 0, S_2 \leq 0, \gamma I_n - A \leq 0, S_{12} \geq 0, S_{21} \geq 0$. Thus $W_i^{(p)} \leq 0$. Therefore $Y_i^{(p)} - X_i^{(p)} \geq 0$, because $(F_1^{(p)})^{-1} \leq 0$ for $i = 1,2$.

Further on, according to Lemma 3 (ii) we have

$$X_i^{(p+1)} - Y_i^{(p)} = (F_2^{(p)})^{-1} \, H^{(p)} \,,$$

where

$$\begin{aligned} H^{(p)} &:= (\gamma I_n - A^T + Y_1^{(p)} S_1 + Y_2^{(p)} S_2)\,(Y_i^{(p)} - X_i^{(p)}) + Y_i^{(p)} S_j (Y_j^{(p)} - X_j^{(p)}) \\ &+ (Y_j^{(p)} - X_j^{(p)}) S_j X_i^{(p)} + (X_j^{(p)} - Y_j^{(p)}) S_{ij} X_i^{(p)} + Y_j^{(p)} S_{ij} (X_i^{(p)} - Y_i^{(p)}) \,. \end{aligned}$$

With similar arguments we arrive at the conclusion $X_j^{(p+1)} - Y_j^{(p)} \geq 0 \,,\; j = 1,2$.

We compute (i=1,2) $X_i^{(p+1)}$ via (11) and $Y_i^{(p+1)}$ via (10). Consider the matrices $F_1^{(p+1)} = \gamma I_n + A - S_1 X_1^{(p+1)} - S_2 X_2^{(p+1)}$ and $F_2^{(p+1)} = \gamma I_n + A^T - Y_1^{(p+1)} S_1 - Y_2^{(p+1)} S_2$. According to Lemma 2 and properties $X_i^{(p+1)} \geq X_i^{(p)}$ and $Y_i^{(p+1)} \geq X_i^{(p+1)}, i = 1,2$ we derive the conclusion $(-F_1^{(p+1)})$ and $(-F_2^{(p+1)})$ are M-matrices and therefore $(F_1^{(p+1)})^{-1} \leq 0$ and $(F_2^{(p+1)})^{-1} \leq 0$.

Thus, the sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}, i = 1,2$ are monotone increasing. We have to prove that they are bonded above. Consider any exact nonnegative solution $(\tilde{X}_1, \tilde{X}_2)$ of $\mathscr{R}_j(X_1, X_2) = 0$, j=1,2. We shall prove that the solution is an upper bound of the matrix sequences.

For $p = 0$, we have $\tilde{X}_i \geq X_i^{(0)} = 0$, and according to Lemma 3 (iii)

$$(Y_i^{(0)} - \tilde{X}_i) F_1^{(0)} = -(\gamma I_n - A^T)\tilde{X}_i - \tilde{X}_i S_i \tilde{X}_i - \tilde{X}_i S_j \tilde{X}_j - \tilde{X}_j S_j \tilde{X}_i - \tilde{X}_j S_j \tilde{X}_i + \tilde{X}_j S_{ij} \tilde{X}_j \; \geq 0 \,,$$

we infer $Y_i^{(0)} - \tilde{X}_i \leq 0, i = 1,2$.

Moreover, for $p > 0$ we have

$$\begin{aligned} (Y_i^{(p)} - \tilde{X}_i) F_1^{(p)} &= (\gamma I_n - A^T)(X_i^{(p)} - \tilde{X}_i) + \tilde{X}_i S_i (X_i^{(p)} - \tilde{X}_i) + \tilde{X}_i S_j (X_j^{(p)} - \tilde{X}_j) \\ &+ (X_j^{(p)} - \tilde{X}_j) S_j \tilde{X}_i + \tilde{X}_j S_j (X_i^{(p)} - \tilde{X}_i) + (\tilde{X}_j - X_j^{(p)}) S_{ij} \tilde{X}_j + X_j^{(p)} S_{ij} (\tilde{X}_j - X_j^{(p)}) \; \geq \; 0 \,, \end{aligned}$$

we have $Y_i^{(p)} - \tilde{X}_i \leq 0, i = 1,2$.

We evaluate the matrix difference $X_i^{(p+1)} - \tilde{X}_i, i = 1,2$. Applying Lemma 3 (iv) we obtain:

$$\begin{aligned} F_2^{(p)} (X_i^{(p+1)} - \tilde{X}_i) &= (Y_i^{(p)} - \tilde{X}_i)(\gamma I_n - A) + (Y_i^{(p)} - \tilde{X}_i) S_i \tilde{X}_i + (Y_j^{(p)} - \tilde{X}_j) S_j \tilde{X}_j \\ &+ (Y_i^{(p)} - \tilde{X}_i) S_j \tilde{X}_j + \tilde{X}_i S_j (Y_j^{(p)} - \tilde{X}_j) \\ &+ (\tilde{X}_j - Y_j^{(p)}) S_{ij} \tilde{X}_j + Y_j^{(p)} S_{ij} (\tilde{X}_j - Y_j^{(p)}) \; \geq \; 0 \,. \end{aligned}$$

Thus, $X_j^{(p+1)} - \tilde{X}_j \leq 0, j = 1,2$.

The matrix sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 of nonnegative matrices converge to the couple of nonnegative matrices $(\hat{X}_1, \hat{X}_2)$. By taking the limits in (10) - (11) it follows that the couple of matrices is a solution to Nash-Riccati equations

(3)-(4). Moreover, the limit matrix has the property $\hat{X}_i \leq \tilde{X}_i, i = 1,2$ (in the element wise ordering). The matrix $-A + S_1\hat{X}_1 + S_2\hat{X}_2$ is an M-matrix because $(-F_1^{(p)})$ is an M-matrix for all positive p. Therefore, the matrix $A - S_1\hat{X}_1 + S_2\hat{X}_2$ is stable. The solution $(\hat{X}_1, \hat{X}_2)$ is a stabilizing one.

Corollary 1*The stabilizing solution $(\hat{X}_1, \hat{X}_2)$ of Nash-Riccati equations (3)-(4) derived in Theorem 1 is the minimal one to (3)-(4).*

3 Linearized iteration applied to a open loop design

In this section, we change iteration formula (10) - (11) to obtain a new iteration to compute the stabilizing solution of the set of Nash-Riccati equations in case of a game with open loop design. In formula (10) - (11), we change the matrices $Q_i(Z_i, Z_j)$, $i, j = 1,2; j \neq i$ as follows:

$$Q_i(Z_i, Z_j) = Q_i + Z_i S_i Z_i + Z_j S_j Z_i\,, \tag{12}$$

with $i, j = 1,2; j \neq i$.

Applying Theorem 1, we derive a proof for the convergence of iteration (10) - (11) in the next Theorem:

Theorem 2.*Assume matrices A, S_1, S_2 , and Q_1, Q_2 are coefficients of the set of matrix equations $\mathscr{L}_i(X_1, X_2) = 0$, i=1,2 defined with (6) - (7). There exists negative $\gamma < 0$, such that $-(\gamma I_n + A)$ is an M-matrix and $\gamma I_n - A \leq 0$.*

The sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 constructed by (10) - (11) with $Q_i(Z_i, Z_j)$ defined in (12) fulfill the properties:

(i) *$\tilde{X}_i \geq X_i^{(p+1)} \geq Y_i^{(p)} \geq X_i^{(p)}$ for $p = 0,1,\ldots$, i=1,2 for any exact nonnegative solution $\tilde{X}_1, \tilde{X}_2$ of $\mathscr{L}_i(X_1, X_2) = 0$, i=1,2. ;*

(ii) *The matrices $(-F_1^{(p)})$ and $(-F_2^{(p)})$ are M-matrices for any positive p.*

(iii) *The matrix sequences $\{X_i^{(p)}, Y_i^{(p)}\}_{p=0}^{\infty}$, i=1,2 converge to the stabilizing nonnegative solution $(\hat{X}_1, \hat{X}_2)$ to couple of Nash-Riccati equations $\mathscr{L}_i(X_1, X_2) = 0$, i=1,2. In fact the matrix $A - S_1\hat{X}_1 - S_2\hat{X}_2$ is stable.*

4 Results

In this section, we apply the proposed iterations to compute the stabilizing solution of the couple Nash-Riccati equations which help to find the Nash equilibrium point for the games with feedback information structure and the open loop information stricture. Experiments are provided with different matrix coefficients of Nash-Riccati equations (3)-(4) and (6)-(7). In addition, we present the comparative analysis between Newton method (8) and proposed linearized iterations in the considered two cases. All experiments are executed with MATLAB R2018b on a Laptop with 1.50 GHz Intel(R) Core(TM) and 8 GB RAM, running on Windows 10. The stop criterion for each iteration is $max\left(\|\mathscr{R}_1(X_1^{(k)}, X_2^{(k)})\|_2, \|\mathscr{R}_2(X_1^{(k)}, X_2^{(k)})\|_2\right) \leq tol$ or $max\left(\|\mathscr{L}_1(X_1^{(k)}, X_2^{(k)})\|_2, \|\mathscr{L}_2(X_1^{(k)}, X_2^{(k)})\|_2\right) \leq tol$, where $\|.\|_2$ is the spectral matrix norm and $tol = 0.1e-10$.

Moreover, the property of symmetry for matrices S_1, S_2 give us possibility to improve the computational scheme of iteration (10)-(11) in order to decrease the computations for each iteration step and accelerate the algorithm based on (10)-(11).

*Example 1.*Consider the matrix coefficients of system (1) and cost functions J_1, J_2:

$$A = \begin{pmatrix} -4 & 1 & 1 & 0.5 \\ 1 & -5 & 0.8 & 1 \\ 1 & 1 & -4 & 1 \\ 0.9 & 1 & 2 & -6 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 5 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0.8 & 1 & 0 & 0.2 \\ 0.3 & 1 & 1 & 0 \\ 0.6 & 0 & 0 & 1 \end{pmatrix},$$

Q_1 = diag [5; 0; 0.5; 3] , Q_2 = diag [50; 4; 5; 0] , $R_{11} = -90 \in \mathbf{R}^{1\times 1}$; $R_{21} = 200 \in \mathbf{R}^{1\times 1}$, R_{12} = diag [400; 200; 500; 300] , and

$$R_{22} = \begin{pmatrix} -400 & 0 & 0 & -10 \\ 0 & -100 & 0 & 0 \\ 0 & 0 & -200 & 0 \\ -10 & 0 & 0 & -400 \end{pmatrix}.$$

We compute the matrix coefficients $S_1 \leq 0, S_2 \leq 0, S_{12} \geq 0$, and $S_{21} \geq 0$.

	The proposed iteration (10)-(11)	
γ	It	CPU time seconds
-10	219	0.55
-5	118	0.30
-1	73	0.18
-1.25	42	0.105
-0.5		no result

Table 1: Results for Example 2 with $tol = .1e-10$.

To compute the stabilizing solution of the couple of Nash-Riccati equations (3)-(4) we apply linearized iteration (10) - (11). After 118 iteration steps with $\gamma = -5$ we obtain the stabilizing solution $(\hat{X}_1, \hat{X}_2)$. The matrices are nonnegative and symmetric:

$$\hat{X}_1 = \begin{pmatrix} 1.1055 & 0.3416 & 0.4615 & 0.2485 \\ 0.3416 & 0.1767 & 0.2514 & 0.1449 \\ 0.4615 & 0.2514 & 0.4074 & 0.2194 \\ 0.2485 & 0.1449 & 0.2194 & 0.3415 \end{pmatrix}, \quad \hat{X}_2 = \begin{pmatrix} 8.8338 & 2.0633 & 2.6256 & 1.2793 \\ 2.0633 & 1.2650 & 1.2269 & 0.5980 \\ 2.6256 & 1.2269 & 2.1363 & 0.8011 \\ 1.2793 & 0.5980 & 0.8011 & 0.3741 \end{pmatrix}.$$

The closed loop matrix $A - S_1\hat{X}_1 - S_2\hat{X}_2$ has the eigenvalues
$-1.1274, -4.7929, -5.6682, -6.8395$.

To compute the stabilizing solution of the couple of Nash-Riccati equations (6)-(7) we apply linearized iteration (10) - (11) with the matrices $Q_i(Z_i, Z_j)$, $i, j = 1,2; j \neq i$ defined by (12). Matrices $\hat{X}_1$, and $\hat{X}_2$ are nonnegative and nonsymmetric:

$$\hat{X}_1 = \begin{pmatrix} 0.7775 & 0.1567 & 0.2067 & 0.1260 \\ 0.1569 & 0.0678 & 0.1007 & 0.0726 \\ 0.2079 & 0.1011 & 0.1990 & 0.1195 \\ 0.1279 & 0.0734 & 0.1202 & 0.2938 \end{pmatrix}, \quad \hat{X}_2 = \begin{pmatrix} 7.5325 & 1.4131 & 1.7341 & 0.8439 \\ 1.4048 & 0.9198 & 0.7526 & 0.3661 \\ 1.7156 & 0.7499 & 1.4809 & 0.4807 \\ 0.8248 & 0.3617 & 0.4765 & 0.2154 \end{pmatrix}.$$

The closed loop matrix has the eigenvalues $-1.3124, -4.8023, -5.6708, -6.8395$.

*Example 2.*Consider the same matrix coefficients as in Example 1. We compare the Newton iteration and the proposed linearized iteration to compute the stabilizing solution of (3)-(4).

The Newton method computes the solution for 6 iteration steps and CPU time of 0.21 seconds for 100 runs. Results from experiments with proposed iteration are given in Table 1. The execution CPU time for 100 runs is given. The convergence of the proposed method is proved in Theorem 1. The proposed iteration executes smaller number of iteration steps (It=42) for $\gamma = -1.25$. For this value of γ the proposed method is faster than Newton method which has a quadratic convergence rate. In addition, the method does not converge for $\gamma = -0.5$. Weakness of the method that one has to find a properly value of a which gives speed of the method. In addition, we check the conditions of Theorem 1 for choosing values of γ.

*Example 3.*Define the matrix coefficients of system (1) and cost functions J_1, J_2 as follows (n=8).

$$A_0 = \begin{pmatrix} -24 & 0 & 0 & 2 \\ 20 & -25 & 0 & 0 \\ 0 & 16 & -25 & 0 \\ 1.5 & 0 & 18 & -24 \end{pmatrix}, B_{10} = \begin{pmatrix} 0.7 \\ 0.9 \\ 0.9 \\ 0.8 \end{pmatrix}, \quad B_{20} = \begin{pmatrix} 2.8 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 1.5 \\ 0 & 0 & 3 & 8 \end{pmatrix},$$

$$A = diag[A_0, A_0], \quad B_1 = diag[B_{10}, B_{10}], \quad B_2 = diag[B_{20}, B_{20}],$$
$$R_{11} = -1.9 \in \mathbf{R}^{1\times 1}, \quad R_{21} = 20 \in \mathbf{R}^{1\times 1},$$
$$v = diag[40, 30, 20, 10, 40, 60, 70, 80], \quad R_{12} = diag[v],$$
$$R_{22} = diag[-150, -1, ..., -1, -120] \in \mathbf{R}^{n\times n},$$
$$Q_1 = diag[4, 1, ..., 1, 1.5] \in \mathbf{R}^{n\times n},$$
$$Q_2 = 0.5\, Q_1 .$$

	The proposed iteration (10)-(11)	
γ	It	CPU time seconds
-20	101	0.32
-16	88	0.297
-15	85	0.292
-12	75	0.254

Table 2: Results for Example 3 with $tol = .1e-10$.

We compare the Newton iteration and the proposed linearized iteration to compute the stabilizing solution to set of matrix equations (3)-(4). Results are given in Table 2 The execution CPU time for 100 runs is given. The Newton method computes the stabilizing solution of (3)-(4) for 7 iteration steps and CPU time of 0.97 seconds for 100 runs.

*Example 4.*Define the matrix coefficients of system (1) and cost functions J_1, J_2 as follows (n=16) using notations from Example 3;
$A = 2diag[A_0, A_0, A_0, A_0]$,
$B_1 = diag[B_{10}, B_{10}, B_{10}, B_{10}]$,
$B_2 = diag[B_{20}, B_{20}, B_{20}, B_{20}]$, $R_{12} = diag[v, v]$.

The Newton method computes the stabilizing solution of (3)-(4) for 4 iteration steps and CPU time of 1.774 seconds for 10 runs. Moreover, new iteration (10)-(11) finds the stabilizing solution for 22 iteration steps, CPU time of 0.047 seconds for 10 runs and $\gamma = -20$.

5 Conclusion

The computation of the stabilizing solution of the Nash-Riccati equations is important for applications. In this paper, we applied a linearized process to modify Newton's method to compute the stabilizing solution for a set of Nash-Riccati equations. Moreover, we have proposed fast iterative methods to find this solution. Here, we were presented a convergence proof to effective iteration scheme (10)- (11). The computational simplicity of the algorithm leads to the efficiency of the proposed iteration and it makes the new iteration an alternative method for computing the stabilizing solution. Related discussions are expected to lead to new computational algorithms to similar problems. Based on the considered examples we may conclude that the proposed iteration is an effective solver for these examples. As a future research the linearized process may be extended to construct a new iteration to find the Nash equilibrium strategies of an N-player infinite horizon linear quadratic differential game.

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Uncertainty Inequalities for Continuous Laguerre Wavelet Transform

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Abstract: This paper deals with the Continuous Laguerre Wavelet Transform CLWT, and we prove several versions of the uncertainty inequalities. More precisely, we get the analogue of Heisenberg inequality for CLWT. Moreover, dealing with concentration in time and frequency, we find an L^p local type uncertainty principle. Finally, we provide the analogue of Benedicks Amrein Berthier's type theorem in the case of CLWT.

Keywords: Continuous Laguerre wavelet transform; Heisenberg inequality; uncertainty principle; Time frequency concentration theorems; local uncertainty principle.

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1 Introduction

The theory of wavelets and continuous wavelet transforms has garnered increased interest due to the limitations of Fourier transform in providing complete information about a signal. In particular, Fourier transform can not be a suitable tool for non stationary signals, in which frequency changes with respect to time. Hence appears the importance of the wavelet and the continuous wavelet transform CWT. For an overview of CWT, we refer the reader to [5,27]. Motivated by the works of [31,14,1], we consider in this paper, time-frequency localization problems in the case of continuous Laguerre wavelet transform CLWT. The interest of studing Laguerre transform comes from Heisenberg group which replace the euclidean space in quantum mechanics. Roughly speaking, Fourier Laguerre transform is non other than the Fourier transform of radial functions in this occurence. Studying the uncertainty principle for $\mathscr{F}_L$ was subject of several works by the authors and many more, one can cite for instance [9,10,20,22,26]. However studying the uncertainty principle for CLWT still less aborded. Note that the harmonic analysis associated to CLWT was initiated in [23], where the Plancherel and the inversion formulas were established for CLWT. Recently Mejjaoli and Trimèche in [16,15] considered such problems in the case of two-wavelets in Laguerre occurence. In this paper, we improve the litterature by giving uncertainty inequalities for CLWT.

The uncertainty principle is one of the most interesting result which gives us an overview on the positioning of a function and its Fourier transform. This principle states, in quantum mechanics, that an observer cannot determine simultaneously the values of position and momentum of a quantum particule with precision. A precise quantitative formulation of the uncertainty principle, usually called Heisenberg inequality [11,30] is stated for $f \in L^2(\mathbb{R})$, as follows:

$$\int_{\mathbb{R}} x^2 |f(x)|^2 dx \ . \ \int_{\mathbb{R}} \xi^2 \left|\widehat{f}(\xi)\right|^2 d\xi \geq \ \frac{1}{4}\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2, \tag{1}$$

where $\widehat{f}$ is the Fourier transform, given for suitable functions by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\xi x} dx.$$

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Another version of the uncertainty principle concerns with concentration of f and its Fourier transform. We reference two results: the first one was studied by Faris [17] and Price [18,19] in the classical Fourier setting, known as the local uncertainty principle. The second one goes to Benedicks and Amrein-Berthier. Benedicks [3] first introduced this theorem, stating that if a function f has a subset S of finite measure as its support, and its Fourier transform $\hat{f}$ has a subset Σ of finite measure as its support, then f must be the null function. A stronger formulation of this principle was provided by Amrein and Berthier in [2] for the classical Fourier occurence. In this paper we prove the analogue of all previous uncertainty principle theorems when considering the CLWT.

Our paper is structured as follows.

In section 2, we start by giving some useful background evoking Laguerre hypergroup $\mathbb{K}$ and Fourier Laguerre transform $\mathscr{F}_L$. Section 3 summarizes key facts about basic Laguerre wavelet theory. Section 4 is devoted to our main results. First, we prove Heisenberg-type uncertainty inequalities, analogous of inequality (1), considering the product of dispersions with both position and scale as variables, for the CLWT. Second, we prove two theorems dealing with concentration in the support of a given function and its CLWT. The first is a local uncertainty principle and the second deals with a Benedicks-Amrein-Berthier's uncertainty principle.

2 Laguerre hypergroup and Fourier Laguerre transform

Laguerre hypergroup emerges as the fundamental manifold of the radial function space in the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$, where the multiplication operator is given by

$$(z_1,t_1).(z_2,t_2) = (z_1+z_2, t_1+t_2-Im(z_1z_2)).$$

A function f on $\mathbb{H}^n$ is considered radial if it remains invariant under the action of the unitary group $\mathscr{U}(n)$ via $u.(z,t) = (u.z,t)$. For additional details we refere the reader to [6,28,29]. Let $\alpha \geq 0$. The Laguerre hypergroup $\mathbb{K} = [0,+\infty)\times\mathbb{R}$ is equiped with the convolution product $*_\alpha$. This product is defined for two bounded Radon measures μ_1 and μ_2 on $\mathbb{K}$ as:

$$< \mu_1 *_\alpha \mu_2, f >= \int_{\mathbb{K}\times\mathbb{K}} T^\alpha_{x,t} f(y,s)\, d\mu_1 d\mu_2,$$

where $T^\alpha_{x,t}$ is the generalized translation operator on $\mathbb{K}$ given, for $\alpha = 0$, by

$$T^\alpha_{x,t} f(y,s) = \frac{1}{2\pi}\int_0^{2\pi} f(\sqrt{x^2+y^2+2xy\cos\theta}, t+s+xy\sin\theta)\, d\theta \tag{2}$$

and, for $\alpha > 0$, by

$$T^\alpha_{x,t} f(y,s) = \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 f(\sqrt{x^2+y^2+2xyr\cos\theta}, t+s+xyr\sin\theta) r(1-r^2)^{\alpha-1} dr d\theta. \tag{3}$$

Remark that if μ_1 and μ_2 are equal to Dirac measure at (x,t) and $(y,s)\in\mathbb{K}$ then

$$(\delta_{(x,t)} *_\alpha \delta_{(y,s)})(f) = T^\alpha_{x,t} f(y,s).$$

We find in [23] that $(\mathbb{K}, *_\alpha)$ has a commutative hypergroup structure in the sense of Jewett. The involution is defined by the homeomorphism $i(x,t) = (x,-t)$ and the Haar measure is given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}}{\pi\Gamma(\alpha+1)} dx dt. \tag{4}$$

$e = (0,0)$ is the unit element of $(\mathbb{K}, *_\alpha)$ since $\delta_{(x,t)} *_\alpha \delta_{(0,0)} = \delta_{(0,0)} *_\alpha \delta_{(x,t)} = \delta_{(x,t)}$. In the case of Laguerre hypergroup, the dual space, the space of all bounded functions $\chi : \mathbb{K}\to\mathbb{C}$ satisfying for $(x,t)\in\mathbb{K}$, $\tilde{\chi}(x,t) = \overline{\chi(x,-t)} = \chi(x,t)$, is described by

$$\{\varphi_{\lambda,m};\, (\lambda,m)\in\mathbb{R}^*\times\mathbb{N}\}\cup\{\varphi_\rho;\, \rho\geq 0\},$$

where

$$\varphi_\rho = j_\alpha(\rho x) \qquad \text{and} \qquad \varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathscr{L}^{(\alpha)}_m(|\lambda|x^2). \tag{5}$$

Note that j_α is the normalized Bessel function of order α and $\mathscr{L}^{(\alpha)}_m$ is the Laguerre function given on $\mathbb{R}_+$ by

$$\mathscr{L}^{(\alpha)}_m(x) = e^{-\frac{x}{2}}\frac{L^\alpha_m(x)}{L^\alpha_m(0)}, \tag{6}$$

where L_m^α is the Laguerre polynomial of order α and degree m,

$$L_m^\alpha(x) = \sum_{k=0}^{m} (-1)^k \frac{\Gamma(m+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{1}{k!(m-k)!} x^k. \tag{7}$$

Topologically, the dual space can be identified to the Heisenberg fan, the set

$$\bigcup_{m\in\mathbb{N}} \{(\lambda,\mu)\in\mathbb{R}^2; \mu = |\lambda|(2m+\alpha+1)\} \bigcup \{(0,\mu)\in\mathbb{R}^2; \mu\geq 0\}.$$

The subset $\{(0,\mu)\in\mathbb{R}^2;\mu\geq 0\}$ is usually disregarded since it has zero Plancherel measure. Therefore, it is natural to concentrate on the characters $\varphi_{\lambda,m}$. For $(\lambda,m)\in\mathbb{R}\times\mathbb{N}$, $\varphi_{\lambda,m}$ is the unique solution to the problem

$$\begin{cases} D_1 u = i\lambda\, u, \\ D_2\, u = -4|\lambda|(m+\dfrac{\alpha+1}{2})u, \end{cases} \tag{8}$$

with the initial condition

$$u(0,0)=1,\ \frac{\partial u}{\partial x}(0,t)=0 \qquad \text{for all } t\in\mathbb{R},$$

where, for all $\alpha\geq 0$,

$$\begin{cases} D_1 = \dfrac{\partial}{\partial t} \\ D_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2\dfrac{\partial^2}{\partial t^2}. \end{cases} \tag{9}$$

For $(\lambda,m)\in\hat{\mathbb{K}}=\mathbb{R}\times\mathbb{N}$, the function $\varphi_{\lambda,m}$ satisfies, for all $(x,t),(y,s)\in\mathbb{K}$,

$$\varphi_{\lambda,m}(x,t)\,\varphi_{\lambda,m}(y,s) = T_{x,t}^\alpha\,\varphi_{\lambda,m}(y,s). \tag{10}$$

Furthermore, the Laguerre kernel is bounded function, and we have

$$\forall(\lambda,m)\in\hat{\mathbb{K}}, \quad \sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1.$$

Denote $L^p(\mathbb{K}) = L^p(\mathbb{K},dm_\alpha)$ the space of measurable functions f satisfying

$$\|f\|_{p,m_\alpha} = \left(\int_{\mathbb{K}} |f(x,t)|^p dm_\alpha(x,t)\right)^{\frac{1}{p}} < +\infty.$$

The Fourier Laguerre transform of a function f in $L^1(\mathbb{K})$ is defined by

$$\mathscr{F}_L f(\lambda,m) = \int_{\mathbb{K}} f(x,t)\,\varphi_{-\lambda,m}(x,t) dm_\alpha(x,t). \tag{11}$$

The $\mathscr{F}_L$ is bounded operator from $L^1(\mathbb{K})$ to $L^\infty(\hat{\mathbb{K}})$ and it satisfies $\|\mathscr{F}_L f\|_\infty \leq \|f\|_{1,m_\alpha}$. Moreover, the Fourier Laguerre transform can be inverted by

$$\mathscr{F}_L^{-1} f(x,t) = \int_{\hat{\mathbb{K}}} f(\lambda,m)\,\varphi_{\lambda,m}(x,t) d\gamma_\alpha(\lambda,m), \tag{12}$$

where $d\gamma_\alpha$ is the unique positive Radon measure on $\hat{\mathbb{K}}$ for which the Fourier Laguerre transform becomes an L^2-isometry. This measure is given by

$$d\gamma_\alpha(\lambda,m) = L_m^\alpha(0)\delta_m \otimes |\lambda|^{\alpha+1} d\lambda. \tag{13}$$

To simplify we will denote, when needed, $d\gamma_\alpha$ to state $d\gamma_\alpha(\lambda,m)$. $\mathscr{F}_L$ transform satisfies the following Plancherel Formula

$$\|\mathscr{F}_L f\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}, \tag{14}$$

where

$$\|g\|_{p,\gamma_\alpha} = \left(\int_{\hat{\mathbb{K}}} |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{p}} < +\infty.$$

By Riesz Thorin interpolation, we can expand the definition of $\mathscr{F}_L f$ on $L^p(\mathbb{K})$ for $1 \le p \le 2$. Consequently, we obtain the Hausdorff-Young inequality, for $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|\mathscr{F}_L f\|_{p',\gamma_\alpha} \le \|f\|_{p,m_\alpha}. \tag{15}$$

If $f \in L^p(\mathbb{K})$ then, for all $(x,t) \in \mathbb{K}$, $T^\alpha_{x,t} f \in L^p(\mathbb{K})$ and verifies

$$\|T^\alpha_{x,t} f\|_{p,m_\alpha} \le \|f\|_{p,m_\alpha}. \tag{16}$$

Moreover

$$\mathscr{F}_L(T^\alpha_{x,t} f)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathscr{F}_L f(\lambda,m). \tag{17}$$

The generalized convolution product of two functions f and g in $L^1(\mathbb{K})$ is defined by

$$f \star_\alpha g(x,t) = \int_{\mathbb{K}} T^\alpha_{x,t} f(y,s).g(y,-s)dm_\alpha(y,s), \quad (x,t) \in \mathbb{K}. \tag{18}$$

Young's inequality allows to extend the definition of $\star_\alpha$ to $L^p(\mathbb{K}) \times L^q(\mathbb{K})$, where $p,q,r \ge 1$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. For $f \in L^p(\mathbb{K})$ and $g \in L^q(\mathbb{K})$, where $1 \le p,q,r \le 2$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, we get

$$\|f \star_\alpha g\|_{r,m_\alpha} \le \|f\|_{p,m_\alpha}\|g\|_{q,m_\alpha}, \tag{19}$$

and

$$\mathscr{F}_L(f \star_\alpha g) = \mathscr{F}_L(f)\mathscr{F}_L(g). \tag{20}$$

3 Basic Laguerre wavelet theory

In this section, we gather some background related to CLWT. First and foremost, we shall adapt the definition of the dilation operator in order to get formulas that can be compared to the classical Fourier Wavelets. We consider as in [21, 22] the dilated of $(x,t) \in \mathbb{K}$ defined by $\delta_r(x,t) = (rx, r^2 t)$. For $f_r(x,t) = r^{-(2\alpha+4)} f(\delta_{\frac{1}{r}}(x,t))$, we have

$$\int_{\mathbb{K}} f_r(x,t)\, dm_\alpha(x,t) = \int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t). \tag{21}$$

We define, for $a > 0$, the **dilation operator** Δ_a by

$$\Delta_a \psi(x,t) = \frac{1}{a^{\alpha+2}} \psi(\frac{x}{a}, \frac{t}{a^2}) = \frac{1}{a^{\alpha+2}} \psi(\delta_{\frac{1}{a}}(x,t)). \tag{22}$$

We can easily deduce the following properties.

Proposition 1.*Let $a > 0$, we have*

1.For all $a,b > 0$ $\quad \Delta_a \Delta_b = \Delta_{ab}$.
2.For all $\psi \in L^2(\mathbb{K})$, the function $\Delta_a(\psi)$ belongs to $L^2(\mathbb{K})$ and satisfies

$$\|\Delta_a \psi\|_{2,m_\alpha} = \|\psi\|_{2,m_\alpha}. \tag{23}$$

3.For all $\psi \in L^2(\mathbb{K})$, the Fourier Laguerre of $\Delta_a(\psi)$ is well defined and we have

$$\mathscr{F}_L \Delta_a \psi = \hat{\Delta}_{\frac{1}{a}} \mathscr{F}_L \psi, \tag{24}$$

where

$$\hat{\Delta}_a f(\lambda,m) = a^{-(\alpha+2)} f(\delta'_{\frac{1}{a}}(\lambda,m)),$$

and $\delta'_r(\lambda,m) = (r^2\lambda, m)$ is the dilated of $(\lambda,m) \in \mathbb{R} \times \mathbb{N}$.
4.Let $h,g \in L^2(\mathbb{K})$, we have

$$< \Delta_a h, g >_{L^2(\mathbb{K})} = < h, \Delta_{\frac{1}{a}} g >_{L^2(\mathbb{K})}.$$

5.*For all* $a>0$ *and* $(x,t)\in\mathbb{K}$, $\Delta_a T^{\alpha}_{x,t} = T_{\delta_a}(x,t)\Delta_a$, *where* $T^{\alpha}_{x,t}$ *is the translation operator associated to Laguerre hypergroup given by (2) and (3).*

*Proof.*1. For all $a,b>0$,

$$\Delta_a\Delta_b\psi(x,t)=\Delta_a\left(\frac{1}{b^{\alpha+2}}\psi\left(\frac{x}{b},\frac{t}{b^2}\right)\right)=\frac{1}{(ab)^{\alpha+2}}\psi\left(\frac{x}{ab},\frac{t}{(ab)^2}\right)=\Delta_{ab}\psi(x,t).$$

2. The result is obvious by considering the substitutions $y=\frac{x}{a}$ and $u=\frac{t}{a^2}$.
3. Considering $y=\frac{x}{a}$ and $u=\frac{t}{a^2}$ in (11), we get

$$\mathscr{F}_L\Delta_a f(\lambda,m)=\int_{\mathbb{K}} f(y,u)\,\varphi_{-\lambda,m}(ay,a^2u)\,a^{\alpha+2}\,dm_\alpha(y,u).$$

Now using (5), we observe that $a^{\alpha+2}\varphi_{-\lambda,m}(ay,a^2u)=\varphi_{-a^2\lambda,m}(y,u)$, which gives the wanted result.
4. By the same change of variables, we obtain

$$<\Delta_a h,g>=a^{\alpha+2}\int_{\mathbb{K}} h(y,u)g(ay,a^2u)dm_\alpha(y,u).$$

Hence, the result holds from (22).
5. The last point follows by remarking, in (2) and (3), that

$$f\left(\sqrt{x^2+\left(\frac{y}{a}\right)^2+2x\frac{y}{a}r\cos\theta},\frac{t}{a^2}+s+x\frac{y}{a}r\sin\theta\right)=f\left(\frac{\sqrt{(ax)^2+y^2+2(ax)yr\cos\theta}}{a},\frac{t+a^2s+(ax)yr\sin\theta}{a^2}\right).$$

Definition 1.*Let* $\psi\in L^2(\mathbb{K})$*. We say that* ψ *is an admissible Laguerre wavelet on* $\mathbb{K}$ *if there exists a constant* c_ψ *satisfying, for all* $m\in\mathbb{N}$ *and* $\lambda\in\mathbb{R}$,

$$0<c_\psi=\int_0^{+\infty}|\mathscr{F}_L\psi(\delta'_a(\lambda,m)|^2\frac{da}{a}<+\infty. \tag{25}$$

According to [23], such admissible wavelet in Laguerre hypergroup exists. For instance, we cite the following function in $L^2(\mathbb{K})$: $\psi=\mathscr{F}_L^{-1}(\Theta)$, where

$$\forall(\lambda,m)\in\hat{\mathbb{K}},\quad \Theta(\lambda,m)=\lambda(m+\frac{\alpha+1}{2})\,e^{-\lambda^2(m+\frac{\alpha+1}{2})^2}. \tag{26}$$

Now, let ψ be a Laguerre wavelet on $\mathbb{K}$ in $L^2(\mathbb{K})$. We consider the family $\psi^{a,x,t}$, of Laguerre wavelets on $\mathbb{K}$, defined by

$$\forall(x',t')\in\mathbb{K},\quad \psi^{a,x,t}(x',t')=T^{\alpha}_{x,t}(\Delta_a\psi(x',-t')). \tag{27}$$

By virtue of (16) and (23), we get immediately, for all $a>0$ and $(x,t)\in\mathbb{K}$,

$$\|\psi^{a,x,t}\|_{2,m_\alpha}\le\|\psi\|_{2,m_\alpha}. \tag{28}$$

Definition 2.*The continuous Laguerre wavelet transform CLWT,* W^L_ψ *is defined for a regular function* f *on* $\mathbb{K}$ *by*

$$\forall(a,x,t)\in(0,+\infty)\times\mathbb{K},\quad W^L_\psi f(a,x,t)=\int_{\mathbb{K}} f(x',t')\overline{\psi^{a,x,t}(x',t')}dm_\alpha(x',t'). \tag{29}$$

We can also write

$$W^L_\psi f(a,x,t)=<f,\psi^{a,x,t}>_{L^2(\mathbb{K})}=<f,T^{\alpha}_{x,t}\Delta_a\psi>_{L^2(\mathbb{K})}. \tag{30}$$

Moreover, relation (29) can be written as:

$$W^L_\psi f(a,x,t)=f\star_\alpha\overline{\Delta_a\psi}(x,t). \tag{31}$$

By Young's inequality, the CLWT can be defined for a function $f \in L^p(\mathbb{K})$, where $p \in [1,+\infty]$, and an admissible wavelet $\psi \in L^{p'}(\mathbb{K})$, where $p' = \frac{p}{p-1}$. Consequently, for all $(a,x,t) \in (0,+\infty)\times\mathbb{K}$,

$$|W^L_\psi f(a,x,t)| \leq a^{\frac{2\alpha+4}{p'}-(\alpha+2)} \|\psi\|_{p',m_\alpha}\ \|f\|_{p,m_\alpha}. \tag{32}$$

Let $\mathbb{U} = (0,+\infty)\times\mathbb{K}$. For $p \geq 1$, we equiped this space by the "affine" measure

$$d\nu_\alpha(a,x,t) = \frac{da\, dm_\alpha(x,t)}{a^{2\alpha+5}}. \tag{33}$$

Denote by $L^p(\mathbb{U})$ the space of measurable functions f on $\mathbb{U}$ that satisfies

$$\|f\|_{p,\nu_\alpha} = \left(\int_0^{+\infty}\int_{\mathbb{K}} |f(a,x,t)|^p d\nu_\alpha(a,x,t)\right)^{\frac{1}{p}} < +\infty. \tag{34}$$

According to (31) we assert that if ψ is an admissible Laguerre wavelet on $\mathbb{K}$, and $f \in L^2(\mathbb{K})$ then the following Plancherel's formula for CLWT holds.

$$\|W^L_\psi f\|^2_{2,\nu_\alpha} = c_\psi \|f\|^2_{2,m_\alpha}. \tag{35}$$

Furthermore, we can deduce the following Parseval's relation for f and g in $L^2(\mathbb{K})$,

$$c_\psi < f,g >_{L^2(\mathbb{K})} = \int_0^{+\infty}\int_{\mathbb{K}} W^L_\psi f(a,x,t)\overline{W^L_\psi g(a,x,t)}\, d\nu_\alpha(a,x,t). \tag{36}$$

According to (32) and (35), we derive from Riesz Thorin interpolation theorem that the definition of CLWT can be extended to $L^p(\mathbb{K})$ when $1 < p < 2$. We get that $W^L_\psi f$ belongs to $L^{p'}(\mathbb{U})$, where $\frac{1}{p}+\frac{1}{p'} = 1$, and

$$\|W^L_\psi f\|_{p',\nu_\alpha} \leq c_\psi^{\frac{1}{p'}} \left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{p'}} \|f\|_{p,m_\alpha}. \tag{37}$$

4 Main results : CLWT uncertainty inequalities

We shall introduce the following notations. For all $(x,t) \in \mathbb{K}$, the homogeneous norm on $\mathbb{K}$ is given by

$$|(x,t)| = |(x,t)|_{\mathbb{K}} = (x^4+4t^2)^{\frac{1}{4}}. \tag{38}$$

$\mathbb{R}\times\mathbb{N}$ is equiped with the quasinorm defined, for all $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$, by

$$|(\lambda,m)| = 4|\lambda|(m+\frac{\alpha+1}{2}). \tag{39}$$

4.1 Heisenberg type inequalites for CLWT

From [9,26], the Heinsenberg inequality for $\mathscr{F}_L$ states that for $b \geq 1$ and $f \in L^2(\mathbb{K})$,

$$\|\ |(x,t)|^b f\|_{2,m_\alpha} . \|\ |(\lambda,m)|^{\frac{b}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha} \geq C\, \|f\|^2_{2,m_\alpha}. \tag{40}$$

In the case of CLWT, Heisenberg type inequality dealing with dispersion on position (x,t), is given by the following theorem.

Theorem 1. *Let ψ be an admissible Laguerre Wavelet on $\mathbb{K}$ and $b \geq 1$. Then for all $f \in L^2(\mathbb{K})$,*

$$\||(x,t)|^b W^L_\psi f\|_{2,\nu_\alpha} . \||(\lambda,m)|^{\frac{b}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha} \geq C\sqrt{c_\psi}\|f\|^2_{2,m_\alpha}, \tag{41}$$

where C is the same constant given in (40).

Proof. By virtue of relations (31) and (20), we have

$$|\mathscr{F}_L W_\psi^L f(\lambda,m)|^2 = |\mathscr{F}_L f(\lambda,m)|^2 |\mathscr{F}_L \Delta_a \psi(\lambda,m)|^2.$$

Relation (24) and the admissible condition (25) lead to

$$\int_0^{+\infty} |\mathscr{F}_L \Delta_a \psi(\lambda,m)|^2 \frac{da}{a^{2\alpha+5}} = c_\psi. \tag{42}$$

Therefore, using Fubini's theorem, we get

$$\int_0^{+\infty}\int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L W_\psi^L f(\lambda,m)|^2 d\gamma_\alpha \frac{da}{a^{2\alpha+5}} = c_\psi \int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha.$$

On the other hand, since f belongs to $L^2(\mathbb{K})$ then we deduce that the function $W_\psi^L f(a,.,.)$ belongs to $L^2(\mathbb{K})$. Applying Heinsenberg type inequality (40) to $W_\psi^L f(a,.,.)$, we get, for all $a \in (0,+\infty)$,

$$\left(\int_{\mathbb{K}} |(x,t)|^{2b} |W_\psi^L f(a,x,t)|^2 dm_\alpha(x,t)\right)^{\frac{1}{2}} \left(\int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L W_\psi^L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{2}} \geq C\int_{\mathbb{K}} |W_\psi^L f(a,x,t)|^2 dm_\alpha(x,t).$$

Integrating with respect to $\frac{da}{a^{2\alpha+5}}$, the left hand side is given by

$$\sqrt{c_\psi}\left(\int_{\mathbb{U}} |(x,t)|^{2b} |W_\psi^L f(a,x,t)|^2 d\nu_\alpha(a,x,t)\right)^{\frac{1}{2}} \left(\int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{2}},$$

and the right hand side is written as multiple of

$$\int_0^{+\infty}\int_{\mathbb{K}} |W_\psi^L f(a,x,t)|^2 dm_\alpha(x,t) \frac{da}{a^{2\alpha+5}}.$$

Using Plancherel formula, this integral equals to

$$X = \int_0^{+\infty}\int_{\hat{\mathbb{K}}} |\mathscr{F}_L W_\psi^L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \frac{da}{a^{2\alpha+5}}.$$

Therefore, relation (42) leads to

$$\begin{aligned} X &= \int_0^{+\infty}\int_{\hat{\mathbb{K}}} |\mathscr{F}_L f(\lambda,m)|^2 |\mathscr{F}_L \Delta_a \psi(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \frac{da}{a^{2\alpha+5}} \\ &= c_\psi \int_{\hat{\mathbb{K}}} |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \\ &= c_\psi \|f\|_{2,m_\alpha}^2. \end{aligned}$$

Consequently

$$\sqrt{c_\psi}\left(\int_{\mathbb{U}} |(x,t)|^{2b} |W_\psi^L f(a,x,t)|^2 d\nu_\alpha(a,x,t)\right)^{\frac{1}{2}} \left(\int_{\hat{\mathbb{K}}} |(\lambda,m)|^b |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha\right)^{\frac{1}{2}} \geq C c_\psi \|f\|_{2,m_\alpha}^2.$$

which allows to deduce inequality (41).

As an application, we proceed in similar way as in [1], we deduce the following result:

Corollary 1. *For all $s,\beta \geq 1$ and for all $f \in L^2(\mathbb{K})$, we have*

$$\||(x,t)|^s W_\psi^L f\|_{2,\nu_\alpha}^{\beta} . \||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^{s} \geq C(\sqrt{c_\psi})^{1-\beta(s-1)} \|f\|_{2,m_\alpha}^{s+\beta}. \tag{43}$$

*Proof.*Let $s,\beta > 1$. For $f \in L^2(\mathbb{K})$, assume that

$$\||(x,t)|^s \mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^\beta,\ \||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^s < +\infty.$$

Applying Hölder's inequality, we have

$$\||(x,t)|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha} \leq \||(x,t)|^s \mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^{1/s} \|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^{1/s'}$$

and

$$\||(\lambda,m)|^{\frac{1}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha} \leq \||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^{1/\beta} \|\mathscr{F}_L f\|_{2,\gamma_\alpha}^{1/\beta'}.$$

Therefore

$$\||(x,t)|^s \mathbf{W}_\psi^L f\|_{2,\nu_\alpha} \geq \frac{\||(x,t)|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^s}{\|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^{s-1}}$$

and

$$\||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha} \geq \frac{\||(\lambda,m)|^{\frac{1}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^\beta}{\|\mathscr{F}_L f\|_{2,\gamma_\alpha}^{\beta-1}}.$$

Using Theorem 1, we derive that

$$\||(x,t)|^s \mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^\beta \||(\lambda,m)|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^s \geq \frac{C\sqrt{c_h}\|f\|_{2,m_\alpha}^{2\beta s}}{\|\mathbf{W}_\psi^L f\|_{2,\nu_\alpha}^{\beta(s-1)} \|\mathscr{F}_L f\|_{2,\gamma_\alpha}^{s(\beta-1)}}.$$

Plancherel formula and relation (37) allow to deduce the wanted result.

Lemma 1.*Let $\beta \in \mathbb{R}$. We consider ψ, an admissible Laguerre wavelet, satisfying*

$$\forall (\lambda,m) \in \hat{\mathbb{K}}, \quad \mathscr{F}_L \psi(\lambda,m) = \phi(|\lambda|). \tag{44}$$

If f belongs to $L^2(\mathbb{K})$ then

$$\|a^\beta W_\psi^L f\|_{2,\nu_\alpha}^2 = \mathscr{M}(|\tilde{\psi}|^2)(2\beta).\||\lambda|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^2, \tag{45}$$

where $\tilde{\phi}(\lambda) = \phi(\lambda^2)$ and $\mathscr{M}$ is the Mellin transform defined by

$$\mathscr{M} f(x) = \int_0^{+\infty} t^x f(t) \frac{dt}{t}.$$

Proof.

$$\begin{aligned} \|a^\beta W_\psi^L f\|_{2,\nu_\alpha}^2 &= \int_0^{+\infty} a^{-2\beta} \int_{\hat{\mathbb{K}}} |\mathscr{F}_L f(\lambda,m)|^2 |\mathscr{F}_L \Delta_a \psi(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \frac{da}{a^{2\alpha+5}} \\ &= \sum_{m=0}^{+\infty} L_m^\alpha \int_{\mathbb{R}} |\mathscr{F}_L f(\lambda,m)|^2 \Psi(\lambda) d\gamma_\alpha(\lambda,m), \end{aligned}$$

where

$$\Psi(\lambda) = \int_0^{+\infty} a^{2\beta} |\mathscr{F}_L \delta'_a \psi(\lambda,m)|^2 \frac{da}{a}.$$

Making a change of variable, we have

$$\Psi(\lambda) = |\lambda|^\beta \int_0^{+\infty} u^{2\beta} |\tilde{\phi}(u)|^2 \frac{du}{u} = |\lambda|^\beta \mathscr{M}(|\tilde{\phi}|^2)(2\beta).$$

Thus

$$\|a^{-\beta} W_\psi^L f\|_{2,\nu_\alpha}^2 = \mathscr{M}(|\tilde{\phi}|^2)(2\beta) \sum_{m=0}^{+\infty} L_m^\alpha \int_{\mathbb{R}} |\lambda|^\beta |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha(\lambda,m).$$

This gives the wanted result.

Theorem 2.*Let $s,\beta \geq 1$ and h an admissible Laguerre wavelet verifying (44). Then, for all f belonging to $L^2(\mathbb{K})$, we have*

$$\|a^\beta W_\psi^L f\|_{2,\nu_\alpha}^s \||(x,t)|^s W_\psi^L f\|_{2,\nu_\alpha}^\beta \geq C c_\psi \mathscr{M}(|\tilde{\phi}|^2)(2\beta).\||\lambda|^{\frac{\beta}{2}} \mathscr{F}_L f\|_{2,\gamma_\alpha}^2 \|f\|_{2,m_\alpha}^2. \tag{46}$$

*Proof.*Theorem 2 holds from Lemma 1 and Corollary 1.

4.2 L^p Local uncertainty principles for CLWT

This section is devoted to uncertainty principles of concentration type for CLWT in the L^p theory.

Theorem 3. *If $1 < p \leq 2$, $q = \frac{p}{p-1}$ and , then for all nonzero $f \in L^p(\mathbb{K})$ and for all measurable subset $T \subset \mathbb{U}$ such that $0 < \nu_\alpha(T) < +\infty$, we have*
(a) If $0 < s < \frac{2\alpha+4}{q}$,

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha} \leq \ C_1(s,q,\alpha)\ c_\psi^{\frac{1}{q}-\frac{s}{2\alpha+4}}\ \nu_\alpha(T)^{\frac{s}{2\alpha+4}}\, \||(x,t)|^s f\|_{p,m_\alpha}, \tag{47}$$

where $C_1(s,q,\alpha)$ is a constant that depends on s, q and α.
(b) If $s > \frac{2\alpha+4}{q}$,

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha} \leq \ C_2(s,q,\alpha)\ \ \nu_\alpha(T)^{\frac{1}{q}}\, \|f\|_{p,m_\alpha}^{\frac{p}{q}(1-\frac{2\alpha+4}{sq})}\, \|\, |(x,t)|^s\ f\|_{p,m_\alpha}^{\frac{p}{q}\frac{(2\alpha+4)}{sq}}\ , \tag{48}$$

where $C_2(s,q,\alpha)$ is a constant that depends on s, q and α.
(c) If $s = \frac{2\alpha+4}{q}$,

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha} \leq C_3(q,\alpha) c_\psi^{\frac{1}{q}-\frac{2}{q(2\alpha+4)}}\ \nu_\alpha(T)^{\frac{2}{q(2\alpha+4)}}\, \|f\|_{p,\alpha}^{1-\frac{1}{\alpha+2}} \left\|\ |(x,t)|^{\frac{2\alpha+4}{q}}\ f\right\|_{p,\alpha}^{\frac{1}{\alpha+2}}, \tag{49}$$

where $C_3(q,\alpha) = C_1(\frac{2}{q},q,\alpha)(\alpha+2)\ (\alpha+1)^{\frac{1}{\alpha+2}-1}$.

Proof.(a) For all $r > 0$, we define $B_r = \{(x,t) \in \mathbb{K}\ \ ;\ \ |(x,t)| \leq r\}$. Denote by χ_{B_r} and $\chi_{B_r^c}$ the characteristic functions. Let $f \in L^p_\alpha(\mathbb{K}), 1 < p \leq 2$ and $q = \frac{p}{p-1}$. It follows using Minkowski's inequality,

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha} \leq \left\|\chi_T W_\psi^L(f\chi_{B_r})\right\|_{q,\nu_\alpha} + \left\|\chi_T W_\psi^L(f\chi_{B_r^c})\right\|_{q,\nu_\alpha}.$$

Therefore

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha} \leq \nu_\alpha(T)^{\frac{1}{q}} \left\|W_\psi^L(f\chi_{B_r})\right\|_{\infty,\nu_\alpha} + \left\|W_\psi^L(f\chi_{B_r^c})\right\|_{q,\nu_\alpha}. \tag{50}$$

Using relation (32), we get

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha} \leq \nu_\alpha(T)^{\frac{1}{q}} a^{-(\alpha+2)} \|\psi\|_{\infty,m_\alpha} \|f\chi_{B_r}\|_{1,m_\alpha} + \left\|W_\psi^L(f\chi_{B_r^c})\right\|_{q,\nu_\alpha}. \tag{51}$$

Let $0 < s < \frac{2\alpha+4}{q}$. By Hölder's inequality, we obtain

$$\|f\chi_{B_r}\|_{1,m_\alpha} \leq \|\ |(x,t)|^s\ f\|_{p,m_\alpha}\ \left\|\ |(x,t)|^{-s}\chi_{B_r}\right\|_{q,m_\alpha}. \tag{52}$$

Considering (38), let's examine polar coordinates in the Laguerre hypergroup structure:

$$\begin{cases} x = \rho\cos(\theta)^{\frac{1}{2}} \\ t = \frac{\rho^2}{2}\sin(\theta) \end{cases}, \quad \textit{where } \rho = |(x,t)|_{\mathbb{K}}.$$

The Jacobian is given by:

$$\begin{vmatrix} \cos(\theta)^{\frac{1}{2}} & \rho\sin(\theta) \\ -\frac{\rho}{2}\sin(\theta)\cos(\theta)^{-\frac{1}{2}} & \frac{\rho^2}{2}\cos(\theta) \end{vmatrix} = \frac{\rho^2}{2}\cos(\theta)^{-\frac{1}{2}}$$

and

$$\|\ |(x,t)|^{-s}\chi_{B_r}\|_{q,m_\alpha}^q = \frac{1}{2\pi\Gamma(\alpha+1)}\int_0^r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho^{-sq+2\alpha+3}\cos(\theta)^\alpha d\rho d\theta = A(s,q,\alpha)^q.$$

Therefore, we have

$$\|f\chi_{B_r}\|_{1,m_\alpha} \leq\ A(s,q,\alpha)\ \ r^{\frac{2\alpha+4}{q}-s}\ \|\ |(x,t)|^s\ f\|_{p,m_\alpha}\ , \tag{53}$$

where $A(s,q,\alpha)=\left(\frac{B(\frac{\alpha+1}{2},\frac{1}{2})}{2\pi\Gamma(\alpha+1)(2\alpha+4-sq)}\right)^{\frac{1}{q}}$, B is the beta function.
On the other hand, by relation (32), we obtain

$$\begin{aligned}\left\|W_\psi^L(f\chi_{B_r^c})\right\|_{q,\nu_\alpha} &\le c_\psi^{\frac{1}{q}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}}\|f\chi_{B_r^c}\|_{p,m_\alpha}\\ &\le c_\psi^{\frac{1}{q}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}}\|\,|(x,t)|^s f\|_{p,m_\alpha}\left\|\,|(x,t)|^{-s}\chi_{B_r^c}\right\|_{\infty,m_\alpha}.\end{aligned}$$

Hence

$$\left\|W_\psi^L(f\chi_{B_r^c})\right\|_{q,\nu_\alpha}\le c_\psi^{\frac{1}{q}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}} r^{-s}\|\,|(x,t)|^s f\|_{p,m_\alpha}. \tag{54}$$

Combining the relations (51),(53) and (54), we deduce that

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha}\le g_{\alpha,s}(r)\ \ \|\,|(x,t)|^s f\|_{p,m_\alpha}, \tag{55}$$

where $g_{\alpha,s}$is the function defined on $(0,+\infty)$ by

$$g_{\alpha,s}(r)=c_\psi^{\frac{1}{q}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}} r^{-s}+A(s,q,\alpha)\ a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\ \ \nu_\alpha(T)^{\frac{1}{q}} r^{\frac{2\alpha+4}{q}-s}.$$

By minimization of the right-hand side of the relation (55) over $r>0$, we get

$$\left\|\chi_T W_\psi^L f\right\|_{q,\nu_\alpha}\le\ \ C_1(s,q,\alpha)\, c_\psi^{\frac{1}{q}-\frac{s}{2\alpha+4}}\ \nu_\alpha(T)^{\frac{s}{2\alpha+4}}\,\||(x,t)|^s f\|_{p,m_\alpha},$$

where

$$C_1(s,q,\alpha)=\left(\frac{2\alpha+4}{2\alpha+4-sq}\right)\left(\frac{2\alpha+4}{sq}-1\right)^{\frac{sq}{2\alpha+4}}\left(a^{-(\alpha+2)}\|\psi\|_{\infty,m_\alpha}\right)^{1-\frac{2}{q}+\frac{2s}{2\alpha+4}} A(s,q,\alpha)^{\frac{sq}{2\alpha+4}}.$$

(b) The inequality (48) holds if $\||(x,t)|^s f\|_{p,m_\alpha}=+\infty$. Assume that $\||(x,t)|^s\ f\|_{p,m_\alpha}<+\infty$.
From the hypothesis $s>3\alpha+2$, we derive that the function

$$(x,t)\longmapsto(1+\ |(x,t)|^{ps})^{-\frac{1}{p}}$$

belongs to $L^q(\mathbb{K})$. Hölder's inequality leads to

$$\|f\|_{1,m_\alpha}\le\left\|(1+\ |(x,t)|^{ps})^{\frac{1}{p}} f\right\|_{p,m_\alpha}\left\|(1+\ |(x,t)|^{ps}\ )^{-\frac{1}{p}}\right\|_{q,m_\alpha}. \tag{56}$$

Since

$$\left\|(1+\ |(x,t)|^{ps})^{\frac{1}{p}} f\right\|_{p,m_\alpha}^p=\|f\|_{p,m_\alpha}^p+\|\,|(x,t)|^s f\|_{p,m_\alpha}^p$$

then

$$\|f\|_{1,m_\alpha}\le N(s,q)\ \left(\|f\|_{p,\alpha}^p+\|\,|(x,t)|^s f\|_{p,m_\alpha}^p\right)^{\frac{1}{p}}. \tag{57}$$

where

$$N(s,q,\alpha)=\left\|(1+\ |(x,t)|^{ps})^{-\frac{1}{p}}\right\|_{q,m_\alpha}.$$

Using polar coordinates in the Laguerre hypergroup structure, we obtain

$$N(s,q,\alpha)=\left(\frac{B(\frac{\alpha+1}{2},\frac{1}{2})B(\frac{q}{p}-\frac{2(\alpha+2)}{sp},\frac{2(\alpha+2)}{sp})}{2\pi\ sp\Gamma(\alpha+1)}\right)^{\frac{1}{q}}.$$

For $r>0$, we consider $f_r(x,t)=r^{-(2\alpha+4)}f(\frac{x}{r},\frac{t}{r^2})$. Then we have

$$\|f_r\|_{1,m_\alpha}=\|f\|_{1,m_\alpha}, \tag{58}$$

$$\|f_r\|^p_{p,m_\alpha} = r^{\frac{-(2\alpha+4)p}{q}} \|f\|^p_{p,m_\alpha}, \tag{59}$$

and

$$\| \, |(x,t)|^s \, f_r\|^p_{p,m_\alpha} = r^{p(s-\frac{2\alpha+4}{q})} \| \, |(x,t)|^s \, f\|^p_{p,m_\alpha}. \tag{60}$$

Considering f_r in relation (57), we conclude that for all $r > 0$, we get

$$\|f\|^q_{1,m_\alpha} \leq N(s,q,\alpha)^q \left(r^{\frac{-(2\alpha+4)p}{q}} \|f\|^p_{p,m_\alpha} + r^{p(s-\frac{2\alpha+4}{q})} \| \, |(x,t)|^s \, f\|^p_{p,m_\alpha} \right)^{\frac{q}{p}}.$$

By minimizing the right-hand side of this inequality, we deduce

$$\|f\|^q_{1,m_\alpha} \leq N(s,q,\alpha)^q (\frac{sq}{2\alpha+4} - 1)^{\frac{2\alpha+4}{sq}} \left(\frac{sq}{sq-(2\alpha+4)} \right) \|f\|^{p(1-\frac{2\alpha+4}{sq})}_{p,m_\alpha} \||(x,t)|^s f\|^{p\frac{(2\alpha+4)}{sq}}_{p,m_\alpha}. \tag{61}$$

Then, according to relation (61), the function f belongs to $L^1(\mathbb{K})$, and we have

$$\begin{aligned} \left\|\chi_T W^L_\psi f\right\|^q_{q,\nu_\alpha} &\leq \nu_\alpha(T) \left\|W^L_\psi f\right\|^q_{\infty,\nu_\alpha} \\ &\leq \nu_\alpha(T) \left(a^{-(\alpha+2)} \|\psi\|_{\infty,m_\alpha} \right)^q \|f\|^q_{1,m_\alpha}. \end{aligned}$$

Using the relation (61), we get

$$\left\|\chi_T W^L_\psi f\right\|^q_{q,\nu_\alpha} \leq \nu_\alpha(T) \, C^q_2(s,q,\alpha) \, \|f\|^{p(1-\frac{2\alpha+4}{sq})}_{p,m_\alpha} \| \, |(x,t)|^s \, f\|^{p\frac{(2\alpha+4)}{sq}}_{p,m_\alpha},$$

where

$$C^q_2(s,q,\alpha) = \left(a^{-(\alpha+2)} \|\psi\|_{\infty,m_\alpha} \right)^q N(s,q,\alpha)^q \left(\frac{sq}{2\alpha+4} - 1 \right)^{\frac{2\alpha+4}{sq}} \left(\frac{sq}{sq-(2\alpha+4)} \right).$$

(c) Consider $s = \frac{2}{q}(\alpha+2)$. Using the fact that for $\varepsilon > 0$,

$$\frac{|(x,t)|^{\frac{2}{q}}}{\varepsilon^{\frac{2}{q}}} \leq 1 + \frac{|(x,t)|^{\frac{2(\alpha+2)}{q}}}{\varepsilon^{\frac{2(\alpha+2)}{q}}},$$

it follows that

$$\left\| \, |(x,t)|^{\frac{2}{q}} \, f \right\|_{p,\alpha} \leq \varepsilon^{\frac{2}{q}} \|f\|_{p,\alpha} + \varepsilon^{\frac{2}{q}-\frac{2}{q}(\alpha+2)} \left\| \, |(x,t)|^{\frac{2}{q}(\alpha+2)} \, f \right\|_{p,\alpha}.$$

Optimizing in ε, we get:

$$\left\| \, |(x,t)|^{\frac{2}{q}} \, f \right\|_{p,\alpha} \leq (\alpha+2) \, (\alpha+1)^{\frac{1}{\alpha+2}-1} \|f\|^{1-\frac{1}{\alpha+2}}_{p,\alpha} \left\| \, |(x,t)|^{\frac{2\alpha+4}{q}} \, f \right\|^{\frac{1}{\alpha+2}}_{p,\alpha}.$$

Together with (47) for $s = \frac{2}{q} < \frac{2\alpha+4}{q}$, we get the wanted result.

Theorem 4. *Let s,p be two real numbers such that $0 < s < 2\alpha+4$ and $p \geq 1$. Then, for every function $f \in L^p(\mathbb{K})$ and for every measurable subset $T \subset \mathbb{U}$ such that $0 < \nu_\alpha(T) < +\infty$, we have*

$$\left\|\chi_T W^L_\psi f\right\|_{p,\nu_\alpha} \leq C \, \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\| |(\frac{1}{a},x,t)|^s W^L_\psi f \right\|^\kappa_{2,\nu_\alpha} \left(\|h\|_{2,m_\alpha} \|f\|_{2,m_\alpha} \right)^{1-\kappa}, \tag{62}$$

where $\kappa = \dfrac{2(2\alpha+4)-s}{(p+1)(2\alpha+4)}$ and C is a constant that depends on s, p and α.
Note here that

$$\left|(\frac{1}{a},x,t)\right| = (\frac{1}{a^4} + x^4 + 4t^2)^{\frac{1}{4}}.$$

*Proof.*One can assume that $\|f\|_{2,m_\alpha} = 1$, and $\|\psi\|_{2,m_\alpha} = 1$. The general formula follows by making the substitution $f := \frac{f}{\|f\|_{2,m_\alpha}}$ and $\psi := \frac{\psi}{\|\psi\|_{2,m_\alpha}}$.
For all $r > 0$, we put $V_r = \{(a,x,t) \in (0,+\infty)\times\mathbb{K} \;\;;\;\; |(\frac{1}{a},x,t)| \leq r\}$. Let $0 < s < 2\alpha+4$. By Hölder's inequality, we obtain

$$\left\|\chi_T W_\psi^L f\right\|_{p,\nu_\alpha} \leq \left\|\chi_{T\cap V_r} W_\psi^L f\right\|_{p,\nu_\alpha} + \left\|\chi_{T\cap V_r^c} W_\psi^L f\right\|_{p,\nu_\alpha}.$$

Let $0 < s < 2\alpha+4$. Using Hölder's inequality and relation (32), we obtain

$$\begin{aligned}\left\|\chi_{T\cap V_r} W_\psi^L f\right\|_{p,\nu_\alpha} &\leq \left\|W_\psi^L(f)\right\|_{\infty,\nu_\alpha}^{\frac{p}{p+1}} \left(\int_{\mathbb{U}} \chi_T(a,x,t)\chi_{V_r}(a,x,t)|W_\psi^L f(a,x,t)|^{\frac{p}{p+1}} d\nu_\alpha\right)^{\frac{1}{p}} \\ &\leq \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\|\chi_{V_r} W_\psi^L f\right\|_{1,\nu_\alpha}^{\frac{1}{p+1}} \\ &\leq \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\||(\frac{1}{a},x,t)|^s W_\psi^L f\right\|_{2,\nu_\alpha}^{\frac{1}{p+1}} \left\||(\frac{1}{a},x,t)|^{-s}\chi_{V_r}\right\|_{2,\nu_\alpha}^{\frac{1}{p+1}}.\end{aligned}$$

Making the change of variables $\begin{cases} u = \frac{1}{a^2} \\ v = x^2 \\ w = 2t \end{cases}$, we get

$$\left\||(\frac{1}{a},x,t)|^{-s}\chi_{V_r}\right\|_{2,\nu_\alpha}^2 = \int_{\mathbb{U}} (u^2+v^2+w^2)^{-\frac{2s}{4}} \chi_{V_r} \frac{u^{\alpha+1}v^\alpha}{8\pi\Gamma(\alpha+1)} du\,dv\,dw.$$

Applying polar coordinates in $\mathbb{R}^3$, we find

$$\left\||(\frac{1}{a},x,t)|^{-s}\chi_{V_r}\right\|_{2,\nu_\alpha}^2 = \int_0^r \rho^{-2s} \int_0^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\rho\cos(\theta)\cos(\varphi))^{\alpha+1}(\rho\sin(\theta)\cos(\varphi))^\alpha}{8\pi\Gamma(\alpha+1)} \rho^2\cos(\varphi) d\rho\, d\theta\, d\varphi.$$

By a simple calculation, we get

$$\||(\frac{1}{a},x,t)|^{-s}\chi_{V_r}\|_{2,\nu_\alpha} = A_1(s,\alpha) r^{2\alpha+4-s},$$

where $A_1(s,\alpha) = \left(\frac{B(\frac{\alpha+1}{2},\frac{\alpha}{2}+1)B(\alpha+\frac{3}{2},\frac{1}{2})}{16\pi\;(2\alpha+4-s)\Gamma(\alpha+1)}\right)^{\frac{1}{2}}$. Thus we obtain

$$\left\|\chi_{T\cap V_r} W_\psi^L f\right\|_{p,\nu_\alpha} \leq \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\||(\frac{1}{a},x,t)|^s W_\psi^L f\right\|_{2,\nu_\alpha}^{\frac{1}{p+1}} C_1^{p+1} s^{\frac{2\alpha+4-s}{p+1}}. \tag{63}$$

On the other hand, using Hölder's inequality and relation (32), we conclude that

$$\begin{aligned}\left\|\chi_{T\cap V_r^c} W_\psi^L f\right\|_{p,\nu_\alpha} &\leq \left\|W_\psi^L f\right\|_{\infty,\nu_\alpha}^{\frac{p-1}{p+1}} \left(\int_{\mathbb{U}} \chi_{T\cap V_r^c}(a,x,t)|W_\psi^L f(a,x,t)|^{\frac{2p}{p+1}} d\nu_\alpha(a,x,t)\right)^{\frac{1}{p}} \\ &\leq \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left(\int_{\mathbb{U}} \chi_{V_r^c}(a,x,t)|W_\psi^L f(a,x,t)|^2 d\nu_\alpha(a,x,t)\right)^{\frac{1}{p+1}} \\ &\leq \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\||(\frac{1}{a},x,t)|^s W_\psi^L f\right\|_{2,\nu_\alpha}^{\frac{2}{p+1}} r^{\frac{-s}{p+1}}.\end{aligned}$$

Hence

$$\left\|\chi_T W_\psi^L f\right\|_{p,\nu_\alpha} \leq h_{\alpha,s}(r)\;\nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\||(\frac{1}{a},x,t)|^s W_\psi^L f\right\|_{2,\nu_\alpha}^{\frac{1}{p+1}}, \tag{64}$$

where $h_{\alpha,s}$ is the function defined on $(0,+\infty)$ by

$$h_{\alpha,s}(r) = A_1(s,\alpha)^{p+1} s^{\frac{2\alpha+4-s}{p+1}} + \left\| |(\frac{1}{a},x,t)|^s W_\psi^L f \right\|_{2,\nu_\alpha}^{\frac{1}{p+1}} r^{\frac{-s}{p+1}}.$$

By minimizing the right-hand side of the inequality (64) with respect to $r > 0$, we obtain

$$\left\| \chi_T W_\psi^L f \right\|_{p,\nu_\alpha} \leq C(s,p,\alpha) \;\; \nu_\alpha(T)^{\frac{1}{p(p+1)}} \left\| |(\frac{1}{a},x,t)|^s W_\psi^L f \right\|_{2,\nu_\alpha}^{\frac{2}{p+1} - \frac{s}{(p+1)(2\alpha+4)}},$$

where

$$C(s,p,\alpha) = (\frac{2\alpha+4}{2\alpha+4-s}) \, (\frac{2\alpha+4-s}{s})^{\frac{s}{2\alpha+4}} A_1(s,\alpha)^{\frac{s}{p(2\alpha+4)}}.$$

4.3 Benedicks-Amrein-Berthier's uncertainty principle for CLWT

A strong formulation of Benedicks-Amrein-Berthier's result for the Laguerre Fourier transform was established by the second author in [20]. This result asserts that, for $S \subset \mathbb{K}$, $\Sigma \subset \hat{\mathbb{K}}$ a pair of measurable subsets of finite measures $\mu_\alpha(S)$, $\hat{\mu}_\alpha(\Sigma) < +\infty$, we can find a constant $C(S,\Sigma)$ such that, for all $f \in L^2(\mathbb{K})$,

$$\|f\|_{2,m_\alpha}^2 \leq C(S,\Sigma) \left(\int_{\mathbb{K}\setminus S} |f(x,t)|^2 dm_\alpha(x,t) + \int_{\hat{\mathbb{K}}\setminus\Sigma} |\mathscr{F}_L f|^2 d\gamma_\alpha(\lambda,m) \right). \tag{65}$$

The constant $C(S,\Sigma)$ is called the annihilating constant, and (S,Σ) is termed a strong annihilating pair. In the context of CLWT, we obtain the following result.

Theorem 5.*Consider two measurable subsets $S \subset \mathbb{K}$, $\Sigma \subset \hat{\mathbb{K}}$ with finite measures $\mu_\alpha(S)$, $\hat{\mu}_\alpha(\Sigma) < +\infty$. Let ψ be a Laguerre wavelet on $\mathbb{K}$ in $L^2(\mathbb{K})$. For an arbitrary function $f \in L^2(\mathbb{K})$,the following uncertainty inequality holds.*

$$\frac{c_\psi \|f\|_{2,m_\alpha}^2}{C(S,\Sigma)} \leq \int_0^{+\infty} \int_{\mathbb{K}\setminus S} |W_\psi^L f(a,x,t)|^2 d\nu_\alpha(a,x,t) + c_\psi \int_{\hat{\mathbb{K}}\setminus\Sigma} |\mathscr{F}_L f(\lambda,m)|^2 d\gamma_\alpha, \tag{66}$$

where $C(S,\Sigma)$ is the annihilating constant given in (65).

*Proof.*We have, for all $a > 0$, $W_\psi^L f(a,.,.) \in L^2(\mathbb{K})$ whenever $f \in L^2(\mathbb{K})$. This allows using (65) to get

$$\|W_\psi^L f\|_{2,m_\alpha}^2 \leq C(S,\Sigma) \left(\int_{\mathbb{K}\setminus S} |W_\psi^L f(x,t)|^2 dm_\alpha(x,t) + \int_{\hat{\mathbb{K}}\setminus\Sigma} |\mathscr{F}_L W_\psi^L f|^2 d\gamma_\alpha(\lambda,m) \right).$$

Integrating both sides with respect to $\frac{da}{a^{2\alpha+5}}$, we proceed similarly to the proof of Theorem 1. Consequently, (66) holds using relations (35) and (42).

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